A Complificial Compendium

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CIRM, Luminy
Simplicial sets are lovely objects about which algebraic topologists know a lot. If something is described as a simplicial set, it is ready to be absorbed into topology. Or, in other words, no matter which definition of weak \(\omega\)-category eventually becomes dominant, it will be valuable to know its simplicial nerve.

Ross Street (1987)
Algebra of Oriented Simplices
What is a Complicial Set?

Nerves of (strict) $\omega$-Categories $\rightarrow$ Marked Simplicial Sets

Kan-type Fillers for Admissible Horns
Simplices as strict $\omega$-categories

$O : \Delta \rightarrow \omega\text{-Cat}$

$O^0 := \circ$

$O^1 := \begin{array}{c} \circ \\ \rightarrow \\ \circ \end{array}$

$O^2 := \begin{array}{c} \circ \\ \rightarrow \circ \\ \rightarrow \circ \\ \rightarrow \circ \end{array}$

$O^3 := \begin{array}{c} \circ \\ \rightarrow \circ \\ \rightarrow \circ \\ \rightarrow \circ \\ \rightarrow \circ \\ \rightarrow \circ \end{array}$

Orientals functor
The Maclane Pentagon

\[ O^4 := \]

\[ \quad \]
The Street Nerve

\[ \omega\text{-CAT} \overset{\mathbb{R}_S}{\underset{\mathbb{N}_S}{\leftrightarrow}} \text{SSet} \]

\[ \mathbb{N}_S(C)_n := \text{Hom}_{\omega\text{-CAT}}(O^n, C) \]

Questions

Our experience of the classical nerve construction for categories leads us to wonder:

1) Is Street's nerve fully faithful?

2) How can we characterise its essential image?

Apply Kan's construction to the functor \( O \)

\( \omega \)-categorical realisation
Marked Simplicial Sets

(Roberts 1977)

A marked simplicial set consists of:

- A simplicial set $X$ accompanied by
- A subset $mX \subseteq X$ of simplices of arbitrary dimension

subject to the stipulation that

- Every degenerate simplex of $X$ is a member of $mX$.

A map $f : (X, mX) \to (Y, mY)$ of marked simplicial sets is a simplicial map $f : X \to Y$ that preserves marks.

$mSSet := \text{category (quasi-tops) of marked simplicial sets and mark preserving simplicial maps}$
Marked Nerves

A simplex $x : O^n \rightarrow \mathcal{C}$ of the nerve $N_s(\mathcal{C})$ is said to be thin if it maps the unique non-identity $n$-cell of $O^n$ to an identity $n$-cell in $\mathcal{C}$.

The set of thin simplices is a marking of the nerve $N_s(\mathcal{C})$ called the Roberts marking.

This extends Street's nerve to an adjunction:

$$\text{W-Cat} \xleftarrow{N_s} \xrightarrow{\text{msSet}} \text{msSet}$$
A Keystone Result

The Street Nerve Functor $N_s: \omega\text{-Cat} \to mSSet$ Is Full + Faithful.

- Proved by Verity (2007-ish).

Proof relies upon:

(i) Kan-like Horn Filler Properties of Street nerves.

(ii) The introduction of a Gray tensor product of marked simplicial sets.

(iii) APath category construction for marked simplicial sets that satisfy the conditions in (i).

(iv) An ordinal subdivision analysis of $\omega$-categorical realisations of Gray tensors of simplices.
LAX GRAY TENSOR

$X$ AND $Y$ MARKED SIMPLICAL SETS.

$\times Y$ THE LAX GRAY TENSOR HAS

- UNDERLYING SIMPLICAL SET $X \times Y$,
- $n$-SIMPLEX $(x, y)$ MARKED IF FOR ALL $i + j = n$
  EITHER $x \cdot \ll_{i,j}^0$ MARKED IN $X$ OR $y \cdot \ll_{i,j}^1$ MARKED IN $Y$.

\[ \ll_{i,j}^0 : [i] \to [n], k \mapsto k \]
\[ \ll_{i,j}^1 : [j] \to [n], k \mapsto k+i \]

PSEUDO-GRAY = PRODUCT.

$\Delta^3 \otimes \Delta^4$
Admissible Simplices

1) $\Delta^n$ \text{ THE USUAL STANDARD SIMPLEX IN WHICH ONLY THE DEGENERATE SIMPLICES ARE MARKED.} 

2) $\Delta^{n,k}$ \text{ (for } k=0, \ldots, n) \text{ CONSTRUCTED FROM } \Delta^n \text{ BY ALSO MARKING ALL FACES THAT HAVE THE INTERGERS } \{k-1, k, k+1\} \cap [n] \text{ AS VERTICES.}

3) $R_s(\Delta^{3,2}) = \frac{1}{2} \cdot \Delta^{3,2}$

4) $\Lambda^{n,k}$ \text{ (for } k=0, \ldots, n) \text{ THE USUAL } (n,k)-\text{HORN WITH MARKING INHERITED FROM } \Delta^{n,k}.$
**SLOGAN** THE INNER ADMISSIBLE SIMPLEX $\Delta^{n,k}$ DESCRIBES HOW ITS $(n-1)$-FACE $S^k$ MAY BE OBTAINED AS A COMPOSITE OF ITS $(n-1)$-FACES $S^{k-1}, S^{k+1}$ ALONG THEIR COMMON $(n-2)$-FACE.

**SOME OTHER MARKED SIMPLICIES**

1. $\#\Delta^{n}$ CONSTRUCTED FROM THE STANDARD $n$-SIMPLEX $\Delta^n$ BY MARKING ITS UNIQUE NON-DEGENERATE $n$-DIMENSIONAL FACE $s_n : [n] \rightarrow [n]$.

2. $\#\Delta^{n,k}$ CONSTRUCTED FROM THE ADMISSIBLE $n$-SIMPLEX $\Delta^{n,k}$ BY MARKING ITS $(n-1)$-DIMENSIONAL FACES $s^{i}_n : [n-1] \rightarrow [n]$ FOR $i \in \{k-1, k+1\} \cap [n]$.

3. $\#\Delta^{n,k}$ CONSTRUCTED FROM THE $n$-SIMPLEX $\#\Delta^{n,k}$ BY ALSO MARKING THE $(n-1)$-DIMENSIONAL FACE $s^k_n : [n-1] \rightarrow [n]$.
The Roberts Characterisation

(Prelude to Complicial Sets)

Suppose that $A$ is a marked simplicial set, we say that it is:

Pre-complicial $\equiv$

Right Lifting Property

\[
\begin{array}{ccc}
\eta \Delta^{n,k} & \xrightarrow{a} & A \\
\downarrow & & \downarrow \\
\#\Delta^{n,k} & \xrightarrow{\varepsilon!} & A
\end{array}
\]

$(\forall n \geq 2, k=0, \ldots, n)$

“Marked simplices are closed under simplicial composition”

Strictly complicial $\equiv$

Pre-complicial + Unique RLP

\[
\begin{array}{ccc}
\wedge \Delta^{n,k} & \xrightarrow{a} & A \\
\downarrow & & \downarrow \\
\Delta^{n,k} & \xrightarrow{\varepsilon!} & A
\end{array}
\]

$(\forall n \geq 1, k=0, \ldots, n)$

“All composable simplices have a simplicial composite”

+ All marked $1$-simplices are degenerate.
Complicial Sets

Street’s insight (1987): By weakening the lifting property in Roberts’ characterisation we might hope to construct a model of \((\infty, \infty)\)-categories!

\[
\text{Strictly complicial} \equiv \text{pre-complicial} + \text{unique RLP}
\]

\[\begin{align*}
\Delta^n & \to A \\
\sum_{k=0}^{n} & \cong A
\end{align*}\]

"All composable simplices have weak a \times simplicial composite”

(\forall n \geq 1, k = 0, \ldots, n)

+ All marked 1-simplices are degenerate.

Now we think of marked simplices a being equivalences rather than identities.
Simplicial Bordisms (more formally)

A simplicial bordism is a functor:

\[ \Delta_5 \downarrow \mathcal{F} \xrightarrow{M} \text{PL-Mane} \]

Subcategory of face maps in \( \Delta \)

(PL) Manifolds and regular embeddings

Subject to the conditions that:

1) For each object \( \gamma : \{r\} \rightarrow \mathcal{F} \) of \( \Delta_5 \downarrow \mathcal{F} \) we have that for all \( i \in \{r\} \) either:
   * \( M(\gamma \delta^i) = \emptyset \) or
   * \( \dim(M(\gamma \delta^i)) = \dim(M(\gamma)) - 1 \) and \( M(\gamma \delta^i) \subseteq \partial M(\gamma) \).

2) For each \( x \in M \) there exists a unique face map \( \gamma : \{r\} \rightarrow \mathcal{F} \) such that \( x \in \text{interior}(M(\gamma)) \).
The Kan complex of bordisms

Let $B_n$ denote the semi-simplicial set with:

1. $n$-simplices: The simplicial bordisms $M: Δ^+_n \to PL-Man_e$

2. Action of a face operator $ι: [r] \to [n]$ defined to carry a simplicial bordism $M$ to $M \circ ι$ given by:

$$(M \circ ι)(β) := M(ι β)$$

Prop: The semi-simplicial set $B_n$ admits fillers for all horns:

So we may construct actions of degeneracy operators which make it into a genuine Kan complex.
A Proof Sketch

"GLUING"

N, M n-manifolds, P (n-1)-manifold

\[ P \hookrightarrow N \]

\[ \text{m} \hookrightarrow \text{M} \sqcup \text{N} \]

IF P \hookrightarrow N AND P \hookrightarrow M ARE EMBEDDINGS INTO DM AND D\text{N} THEN M \sqcup \text{N} IS AN n-MANIFOLD.

WE MAY APPLY THIS RESULT INDUCTION TO SHOW THAT THE (n-1)-FACES OF A HORN h: \[ \Lambda^{n,k} \rightarrow \text{B}_d \] CAN BE GLUED TO GIVE A SIMPLICIAL BORDISM WHOSE BOUNDARY COINCIDES WITH THAT OF THE MISSING FACE OF THAT HORN.

"STITCHING IN"

SUPPOSE THAT H IS THE MANIFOLD CONSTRUCTED BY GLUING TOGETHER THE (n-1)-FACES OF THE HORN h: \[ \Lambda^{n,k} \rightarrow \text{B}_d \].

WE MAY ATTACH THE CYLINDRICAL MANIFOLD H \times [0,1] TO THE HORN BY ITS END H \times \{0\}. THIS GIVES US AN n-SIMPLEX h: \[ \Delta^n \rightarrow \text{B}_d \] WHOSE RESTRICTION ALONG \[ \Lambda^{n,k} \rightarrow \Delta^n \] IS THE ORIGINAL HORN. ITS FACE h \cdot S^k IS (H \times S^k) \cup (\partial H \times I).

THIS IS ISOMORPHIC TO H BY THE COLLARING THEOREM FOR PL-MANIFOLDS.
COLLAPSING

As a Kan complex $B_\mathfrak{A}$ classifies bordisms upto bordism. We'd rather like it to classify bordisms upto trivial bordism. We shall use our marking technology to keep track of the trivial bordisms.

**Defn** If $X$ and $Y$ are polytopes then there is an elementary collapse from $X$ to $Y$, denoted $X \Downarrow Y$, if if there is a ball $B^c$ in $X$ with

- $Y \cap B^c$ is a face $B^{c-1}$ of $\partial B^c$
- $X = Y \cup B^c$

And $X$ collapses to $Y$, written $X \Downarrow Y$, if $X \Downarrow X_1 \Downarrow X_2 \ldots \Downarrow Y$.

If $M$ is a simplicial bordism we define

$$\partial M = UM(\delta^i) \quad \partial M = UM(\delta^i)$$

if even if odd

We shall mark $M$ in $B_\mathfrak{A}$ if there is a collapse $M \Downarrow \partial M$. 
\(B_a\) is a complicial set

Or more precisely, \(B_a\) with the marking defined above is a complicial set.

Comments about the proof of this fact:

1) We may apply shelling theory to show that if \(M\) is a marked simplex in \(B_a\), then \(\partial^rM = \partial^rM\) and \(M = \partial^rM \times [0,1]\). In particular, \(M \not\subseteq \partial^rM\).

2) To show that \(B_a\) has fillers for admissible horns, it is enough to show that the specific filler that we constructed above is marked.

This filler has the property that \(M \supseteq U, M(\emptyset, i \neq k)\)

But a simple argument, using the admissibility of the original horn, suffices to extend that to a collapse \(M \supseteq \partial^rM\) if \(r\) is odd or \(M \supseteq \partial^rM\) if \(r\) is even.
**Saturation**

**Question** Does \( B \) have any simplices that are morally equivalences but are not marked?

**Answer** Yes, since it is known that a cobordism can be invertible without being trivial.

**Rider** Can we describe the moral equivalences and mark them without disrupting compliciality?

**Defn** A pre-complicial set \( A \) is saturated if both it and its slices satisfy the 2-of-6 property.

Faces of dimension \( \geq 2 \) + 1-simplices \( x \) and \( y \) marked \( \Rightarrow \) all 1-simplices are marked.
The Saturation of B₀

**Defn** The saturation of a pre-complicial set is the smallest expansion of its marking that makes it both saturated and pre-complicial.

**Theorem** The saturation of a complicial set is again complicial.

**Slogan** Saturated complicial sets are a model of \((\infty,\infty)\)-categories. The saturated \(B₀\) deserves to be thought of as being an \((\infty,\infty)\)-category of bordisms.

**Question** Is the saturated marking of \(B₀\) distinct from the "all simplices" marking of \(B₀\) as a Kan complex?

**Answer** Yes, since we can show that any marked simplex \(M\) in saturated \(B₀\) is an \(h\)-cobordism from \(\emptyset\) to \(\emptyset\).
Curated Plot Points in Complexial Theory

1) Lax Gray Tensor of (pre-)complexial sets.

2) Model structure with (saturated) complexial sets as fibrant objects.

3) Homotopy coherent nerve

\[ \begin{array}{ccc}
\text{msSet} & \xrightarrow{\mathcal{C}} & \text{msSet-Cat} \\
\mathcal{L} & \downarrow & \\
\mathcal{N} & \downarrow & \mathcal{R} \\
\text{msSet} & \xleftarrow{\mathcal{P}} & [\Delta^{op}, \text{msSet}] \\
\end{array} \]

Quillen WRT complexial ms on left, complete Segal ms on right.
CASE STUDY: COMPREHENSIVE FIBRATIONS

COMPLICIAL ANALOGS OF GROTHENDIECK FIBRATIONS.

(i) AUGMENT OUR PRE-COMPLICIAL SETS WITH A SECOND MARKING, USE THIS TO TRACK THOSE SIMPCICES THAT SHOULD BE REGARDED AS BEING (CO)CARTESIAN

(ii) DEFINE A COMPREHENSIVE (CO)FIBRATION TO BE A MAP $p : E \to B$ OF AUGMENTED COMPLICIAL SETS WHICH

- IS A FIBRATION IN THE COMPLICIAL MODEL STRUCTURE AND
- PRESERVES AUXILIARY MARKINGS AND
- ADMITS LiftS OF (LEFT) RIGHT OUTER HORNS WHICH ARE ADMISSIBLE w.r.t AUXILIARY MARKINGS

EXAMPLE GIVEN A 0-SIMPLEX $a \in A$ IN A COMPLICIAL SET, THE LAX (SIMPLICIAL) SLICE $A/a$ IS AGAIN A COMPLICIAL SET, AND IT HAS A CANONICAL AUXILIARY MARKING THAT MAKES $p_a : A/a \to A$ INTO A COMPREHENSIVE FIBRATION.

THE CONTRAVARIANT REPRESENTABLE ON $a$
The Comprehension Theorem

**Thm** If $\mathcal{E}$ and $\mathcal{B}$ are enriched in complicial sets with auxiliary marking and $F : \mathcal{E} \to \mathcal{B}$ is an enriched functor such that

1. Each $F : \text{Fun}_\mathcal{E}(A, B) \to \text{Fun}_\mathcal{B}(FA, FB)$ is a comprehensive cofibration
2. For all $B \in \mathcal{E}$, $f : x \to FB \in \mathcal{B}$ there exists a cartesian lift $x_f : A \to B$ in $\mathcal{E}$.

Then the homotopy coherent nerve $NF : NE \to NB$ is a comprehensive fibration.

**Example** $\text{Comp} := \text{complcial sets with auxiliary marking}$

$\text{ComCof} := (\text{small}) \text{ comprehensive cofibrations}$

A lifting argument in $\text{coF} : \text{N(ComCof)} \to \text{N(Comp)}$ shows that any comprehensive cofibration $p : E \to B \in \text{ComCof}$ gives rise to a functor $c_p : B \to \text{N(Comp)}$ which maps each $b \in B$ to the corresponding fibre $E_b$ of $p$. 

