Segal-type models of higher categories

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Categories in Homotopy Theory and Rewriting

CIRM

Three prototype examples in dimension 2

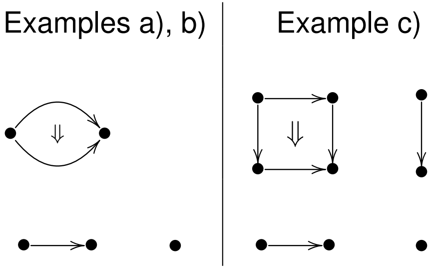
a) *Objects*: categories
1-Morphisms: functors
2-Morphisms: natural transformations.

b) *Objects*: points of a space X
1-Morphisms: paths in X
2-Morphisms: 2-tracks.

c) Let $\mathcal{C}$ be a category with pullbacks.

Objects: objects of $\mathcal{C}$
Vertical 1*-morphisms*: morphisms of $\mathcal{C}$
Horizontal 1*-morphisms*: spans $A \leftarrow J \rightarrow B$ in $\mathcal{C}$
Squares: commuting diagrams of spans in $\mathcal{C}$.

Pictorial representations



- Cells in dimensions 1 compose in the directions of the arrows.
- Cells in dimensions 2 compose vertically and horizontally.
- There are identity cells in dimensions 1 and 2.

Main difference between examples a) and b):

a) All compositions are associative and unital. This is a strict 2-category.

b) Composition of paths is associative and unital only up to homotopy; given paths

$$\underset{a}{\bullet} \xrightarrow{\ f\ } \underset{b}{\bullet} \xrightarrow{\ g\ } \underset{c}{\bullet} \xrightarrow{\ h\ } \underset{d}{\bullet}$$

there is a homotopy

$$a \bullet \underset{h\circ(g\circ f)}{\overset{(h\circ g)\circ f}{\rightrightarrows}} \bullet\, d \qquad (h\circ g)\circ f \simeq h\circ(g\circ f)$$

The structure we obtain is a weak 2-category.

Three motivating examples, cont.

- Main difference between examples a), b) versus c)
  - In examples a), b) the 0-cells forms a set. These are examples of globular structures.
  - In examples c) there is an extra dimension. This is an example of a cubical structure.
- All three examples are truncated higher structures.
- There are higher structures with infinitely many higher cells, called infinity structures.

Types of higher structures

a) *n*-Truncated globular structures

strict *n*-categories $\subset$ weak *n*-categories
(several models)

b) *n*-Truncated cubical structures

Cat^n (*n*-Fold categories); weak versions

c) Infinity structures

strict ω-categories $\subset$ weak ω-categories
(complicial sets)
$\cup$
(∞, n)-categories
(several models)

Strict n-categories

Idea of strict n-category: in a strict n-category there are cells in dimension $0, \dots, n$, identity cells and compositions which are associative and unital. Each k-cell has source and target which are $(k-1)$-cells, $1 \leq k \leq n$.

Strict n-categories are defined by iterated enrichment:

$$1\text{-}Cat = Cat, \qquad n\text{-}Cat = ((n-1)\text{-}Cat)\text{-}Cat$$

When all cells have inverses, we obtain a strict n-groupoid

Strict *n*-groupoids and *n*-types

- An *n*-type is a topological space whose homotopy groups vanish in dimension higher than n.
- n-types are the building blocks of spaces via the Postnikov decomposition.
- Fact: Groupoids are algebraic models of 1-types.
- Are strict n-groupoids an algebraic model n-types ?

Strict n-groupoids and n-types, cont.

- Fact: Strict n-groupoids do not model n-types when $n > 2$.

This was one of the motivations for the development of weak n-categories: in the weak n-groupoid case it gives an algebraic model of n-types (homotopy hypothesis).

Weak n-categories

Idea of weak n-category: in a weak n-category there are cells in dimension $0, \dots, n$, identity cells and compositions which are associative and unital up to an invertible cell in the next dimension, in a coherent way.

- In dimensions $n = 2, 3$ it is possible to give an explicit definition of the axioms with the notions of bicategory and tricategory.
- For general n there are several different models of weak n-categories and weak n-groupoids.

Definition

- An *internal category* in a category $\mathcal{C}$ with pullbacks consists of a diagram in $\mathcal{C}$

$$C_1 \times_{C_0} C_1 \xrightarrow{c} C_1 \underset{s}{\overset{d_0, d_1}{\rightrightarrows}} C_0$$

  where these maps satisfies the axiom of a category.

- Denote by Cat $\mathcal{C}$ the category of internal categories and internal functors.

- An *internal groupoid* in $\mathcal{C}$ is an internal category with all morphisms invertible.

n-Fold categories.

Definition

n-fold categories are defined inductively as

$$\mathrm{Cat}^1 = \mathrm{Cat}$$

$$\mathrm{Cat}^n = \mathrm{Cat}(\mathrm{Cat}^{n-1})$$

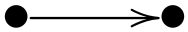
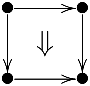
Example: double categories

- Let $X \in \mathrm{Cat}(\mathrm{Cat})$

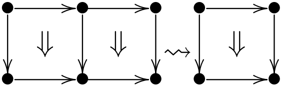
  $X_0 \in \mathrm{Cat}$ has

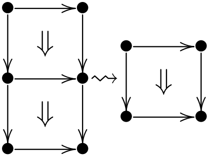
  objects • morphisms 

  $X_1 \in \mathrm{Cat}$ has

  objects  morphisms 

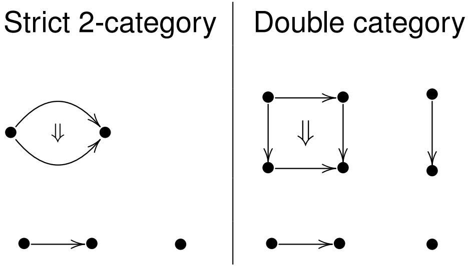
  Thus squares can be composed horizontally and vertically

  All compositions are associative and unital; interchange law.

Strict 2-categories versus double categories



Note: the picture on the right becomes the one on the left when all vertical morphisms are identities.

Strict n-categories versus n-fold categories

There is an embedding

$$n\text{-Cat} \hookrightarrow \text{Cat}^n.$$

A strict n-category $X \in n$-Cat is a n-fold category in which certain substructures are discrete (that is just sets).

- This discreteness condition is called the globularity condition.
- The sets underlying these discrete substructures are the sets of cells in the strict n-category.

A motivating question

The category n-Cat is too small to model weak n-category while Cat^n is too large. Is there an intermediate category

$$n\text{-Cat} \hookrightarrow ? \hookrightarrow \mathrm{Cat}^n$$

which is a model of weak n-categories?

The answer is provided by the category $\mathrm{Cat}^n_{\mathrm{wg}}$ of weakly globular n-fold categories.

Simplicial combinatorics.

- Let Δ be the simplicial category. Its objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$ and its morphisms are non decreasing monotone functions.

- The functor category $[\Delta^{op}, \mathcal{C}]$ is the category of simplicial objects and simplicial maps in $\mathcal{C}$.

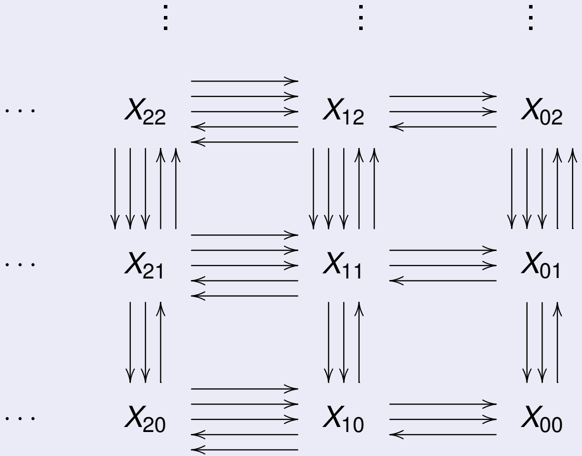
$$X \in [\Delta^{op}, \mathcal{C}] \qquad \cdots X_3 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longleftarrow\\\longleftarrow} X_1 \substack{\longrightarrow\\\longrightarrow\\\longleftarrow} X_0$$

- Let $\Delta^{n^{op}} = \Delta^{op} \times \overset{n}{\cdots} \times \Delta^{op}$.

- Multi-simplicial objects in $\mathcal{C}$ are functors $[\Delta^{n^{op}}, \mathcal{C}]$.

Example: Bisimplicial object

$$X \in [\Delta^{2^{op}}, \mathcal{C}]$$



Nerves and multinerves

- There is a fully faithful nerve functor

$$N : \mathrm{Cat}\,\mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

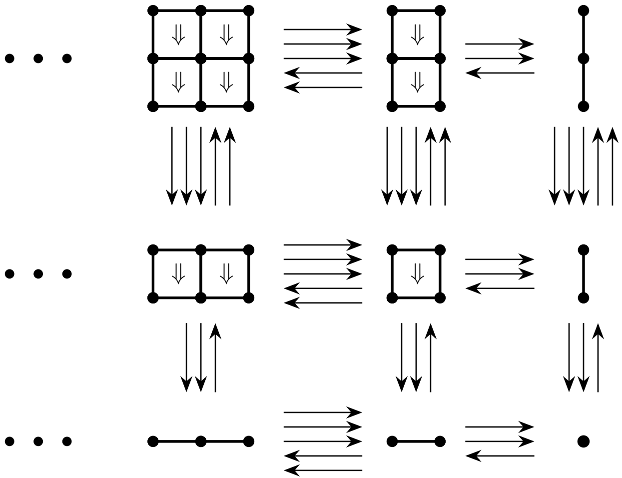
$X \in \mathrm{Cat}\,\mathcal{C}$

$$NX \quad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \rightrightarrows X_1 \times_{X_0} X_1 \rightrightarrows X_1 \rightrightarrows X_0$$

By iterating the nerve construction, we obtain fully faithful multinerve functors

$$N_{(n)} : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n^{op}}, \mathrm{Set}], \quad J_n : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n-1^{op}}, \mathrm{Cat}]$$

Example: the double nerve of a double category



Higher categories via multi-simplicial objects.

Multi-simplicial objects are a good environment for the definition of higher categorical structures because there are natural candidates for the compositions given by the Segal maps.

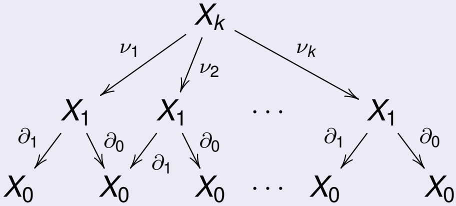
Our structures are based on $[{\Delta^{n-1}}^{op}, \mathrm{Cat}]$. These can be used to model higher categories by imposing additional conditions to encode:

i) The sets of cells in dimension 0 up to n.
ii) The behavior of the compositions.
iii) The higher categorical equivalences.

Segal maps.

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks. Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j-1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

Segal maps and internal categories

- Recall the nerve functor

$$N : \mathrm{Cat}\,\mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\,\mathcal{C}$

$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \; X_1 \times_{X_0} X_1 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} \; X_1 \; \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \; X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are isomorphisms.

Multi-simplicial description of Cat^n and n-Cat

- For each $X \in [\Delta^{n^{op}}, \mathcal{C}]$ there are Segal maps in each of the n simplicial directions.

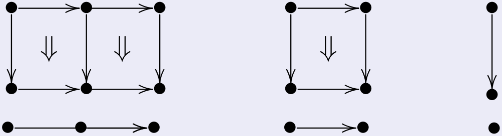
Using these Segal maps we can describe the image of the multinerves

$$J_n : \mathsf{Cat}^n \hookrightarrow [\Delta^{n-1^{op}}, \mathsf{Cat}], \qquad J_n : n\text{-}\mathsf{Cat} \hookrightarrow [\Delta^{n-1^{op}}, \mathsf{Cat}]$$

$$N_{(n)} : \mathsf{Cat}^n \hookrightarrow [\Delta^{n^{op}}, \mathsf{Set}], \qquad N_{(n)} : n\text{-}\mathsf{Cat} \hookrightarrow [\Delta^{n^{op}}, \mathsf{Set}]$$

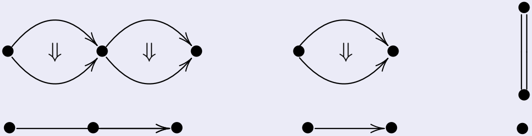
Example: $n = 2$

- Double category $X \in \mathrm{Cat}(\mathrm{Cat}) \xhookrightarrow{J_2} [\Delta^{op}, \mathrm{Cat}]$



$$\cdots \quad X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

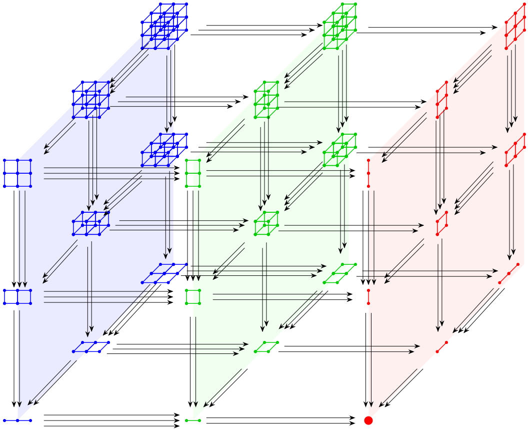
- Strict 2-category $X \in 2\text{-}\mathrm{Cat} \xhookrightarrow{J_2} [\Delta^{op}, \mathrm{Cat}]$



$$\cdots \quad X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

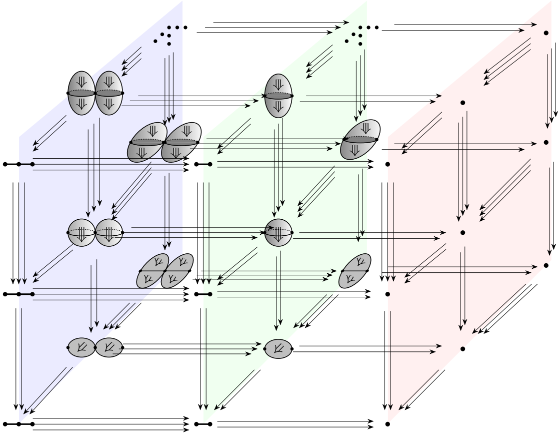
Example: $n = 3$

$$X \in \ \mathsf{Cat}^3 \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \mathsf{Set}]$$



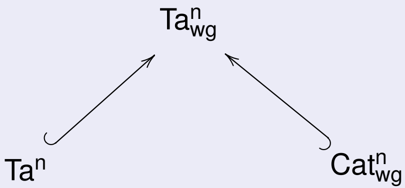
Example: $n = 3$, cont.

$$3\text{-Cat} \overset{N_{(3)}}{\hookrightarrow} [\Delta^{3^{op}}, \mathrm{Set}]$$



Segal-type models.

- We discuss three Segal-type models of weak n-categories, collectively denoted Seg^n



- We have $\mathrm{Seg}^n \subset [\Delta^{n-1^{op}}, \mathrm{Cat}]$.

- Recall that in a weak n-category we want to have k-cells with source and target being $(k-1)$-cells for $1 \leq k \leq n$.
- $\mathrm{Seg}^{\mathrm{n}}$ is built by induction on n starting with $\mathrm{Seg}^{1} = \mathrm{Cat}$.

For each $n > 1$:

$$\mathrm{Seg}^{\mathrm{n}} \hookrightarrow [\Delta^{op}, \mathrm{Seg}^{\mathrm{n-1}}]$$

Encoding the sets of cells.

We encode in two ways the sets of cells of $X \in \mathrm{Seg}^{\mathrm{n}}$

i) Globularity condition:

$X_0, \quad X_{\overset{r}{1\dots10\text{-}}} \qquad 1 \leq r < n-1 \qquad \text{discrete}$

ii) Weak globularity condition:

$X_0, \quad X_{\overset{r}{1\dots10\text{-}}} \qquad 1 \leq r < n-1 \qquad \text{homotopically discrete}$

Let $X \in \mathrm{Seg}^{\mathrm{n}} \subset [\Delta^{op}, \mathrm{Seg}^{\mathrm{n-1}}]$ to be such that X_0 satisfies i) or ii).

Homotopically discrete n-fold categories.

- Homotopically discrete n-fold categories are an iteration of the notion of internal equivalence relation.

Lemma

Given $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}}$, there is a map $\gamma : X \to X^d$ where X^d is discrete, which is a suitable equivalence.

The n^{th} truncation functor.

- There is a functor

$$p^{(n)} : \mathrm{Seg^n} \rightarrow \mathrm{Seg^{n-1}}$$

which divides out by the highest dimensional invertible cells.

The functor $p^{(n)}$ is used to define inductively the notion of *n*-equivalence in $\mathrm{Seg^n}$.

Hom $(n-1)$-categories

- Let $X \in \mathrm{Seg}^{\mathrm{n}}$, so $X_0 \in \mathrm{Cat}^{\mathrm{n}-1}_{\mathrm{hd}}$, $\gamma : X_0 \to X_0^d$.
- For each $a, b \in X_0^d$, let $X(a,b) \in \mathrm{Seg}^{\mathrm{n}-1}$ be the fiber at (a, b) of

$$X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0 \xrightarrow{\gamma\times\gamma} X_0^d \times X_0^d \ .$$

Think of $X(a,b)$ as hom $(n-1)$-categories.

Definition

Define *n-equivalences* in Seg^n by induction on n.

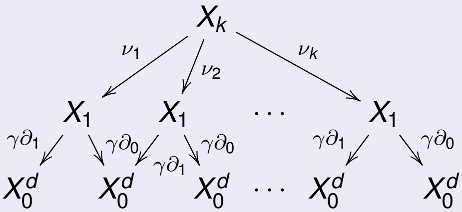
1-equivalences are equivalences of categories.

$f : X \to Y$ in Seg^n is a n-equivalence if and only if

a) $f(a,b) : X(a,b) \to Y(fa, fb)$ is a $(n-1)$-equivalence in Seg^{n-1} for all $a, b \in X_0^d$.

b) $p^{(n)} f$ is a $(n-1)$-equivalence in Seg^{n-1}.

Given $X \in \mathrm{Seg}^n \subset [\Delta^{op}, \mathrm{Seg}^{n-1}]$, consider the commuting diagram



where $k \geq 2$, $\nu_j = X(r_j)$, $r_j(0) = j - 1$, $r_j(1) = j$. This gives the induced Segal map

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

The induced Segal maps condition.

To define $X \in \mathrm{Seg^n} \subset [\Delta^{op}, \mathrm{Seg^{n-1}}]$ we require the induced Segal maps

$$X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

to be $(n-1)$-equivalences.

This condition controls the behaviour of the compositions of higher cells.

Summary of main common features of $\mathrm{Seg^n}$.

- Inductive multi-simplicial definition.
- Globularity/weak globularity condition.
- Functor $p^{(n)} : \mathrm{Seg^n} \to \mathrm{Seg^{n-1}}$ and n-equivalences.
- $(n-1)$-equivalences of the induced Segal maps

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

Three different models corresponding to different behavior of:

Induced Segal maps $\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$

Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$

	X_0	$\hat{\mu}_k$	η_k
Ta^n	discrete	$(n-1)$-eq	$(n-1)$-eq
Cat^n_{wg}	homotopically discrete	$(n-1)$-eq	isomorphisms
Ta^n_{wg}	homotopically discrete	$(n-1)$-eq	-

Model comparison results.

Theorem (P. case n>2; P. and Pronk case n=2)

There are functors

$Q_n : \mathsf{Ta}^n \to \mathsf{Cat}^n_{\mathsf{wg}}$ *rigidification functor*

$Disc_n : \mathsf{Cat}^n_{\mathsf{wg}} \to \mathsf{Ta}^n$ *discretization functor*

producing n-equivalent objects in $\mathsf{Ta}^n_{\mathsf{wg}}$.

Corollary

There is an equivalence of categories

$$\mathsf{Ta}^n / \sim^n \simeq \mathsf{Cat}^n_{\mathsf{wg}} / \sim^n$$

From the comparison theorem between $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ and Ta^{n} we obtain

Theorem

There is a subcategory $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}} \subset \mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ *of groupoidal weakly globular n-fold categories such that there is an equivalence of categories*

$$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^n \,\simeq\, Ho(n\text{-}types)\ .$$

There is an explicit description of the functor n-types $\to \mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$ using a construction of [Blanc and P., Alg.Geom. Topol. 2015].

Pseudo-functors.

- Recall the category $\mathrm{Ps}[\mathcal{C}, \mathrm{Cat}]$ of pseudo-functors and pseudo-natural transformations.

Theorem (Power; Lack...)

There is a strictification functor

$$St : \mathrm{Ps}[\mathcal{C}, \mathrm{Cat}] \to [\mathcal{C}, \mathrm{Cat}]$$

left adjoint to the inclusion and such that the components of the unit are equivalences in $\mathrm{Ps}[\mathcal{C}, \mathrm{Cat}]$.

Using pseudo-functors to rigidify Ta^{n}_{wg}.

We identify a subcategory

$$\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \subset \mathrm{Ps}[\Delta^{n-1^{op}}, \mathrm{Cat}]$$

of Segalic pseudo-functors such that St restricts to

$$\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{n}_{wg} \subset [\Delta^{n-1^{op}}, \mathrm{Cat}].$$

The rigidification functor factors as

$$Q_n : \mathrm{Ta}^{n}_{wg} \to \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{n}_{wg}.$$

From $\mathsf{Ta}^2_{\mathsf{wg}}$ to pseudo-functors.

- Recall $X \in \mathsf{Ta}^2_{\mathsf{wg}}$ if $X \in [\Delta^{op}, \mathsf{Cat}]$ is such that

$$X_0 \in \mathsf{Cat}_{\mathsf{hd}}, \qquad X_k \simeq X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \quad k \geq 2.$$

- Let

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1 \end{cases}$$

Then $X_k \simeq (Tr_2 X)_k$ for all k.

By transport of structure $Tr_2 X \in [ob(\Delta^{op}), \mathsf{Cat}]$ lifts to a pseudo-functor $Tr_2 X \in \mathsf{Ps}[\Delta^{op}, \mathsf{Cat}]$, which is Segalic.

Definition

Let Q_2 be the composite

$$Q_2 : \mathrm{Ta}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}$$

- The case $n > 2$ is more complex, since the induced Segal maps of $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ are $(n-1)$-equivalences but not, in general, levelwise equivalences of categories.

We identify a subcategory $\mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$ and functors

$$\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \,.$$

Definition

Define Q_n for $n > 2$ to be the composite

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

The idea of the discretization functor.

- We want a functor

$$Disc_n : \mathsf{Cat}^n_{\mathsf{wg}} \to \mathsf{Ta}^n$$

which produces an equivalent object in $\mathsf{Ta}^n_{\mathsf{wg}}$.

- The idea of $Disc_n$ is to replace the homotopically discrete sub-structures in $\mathsf{Cat}^n_{\mathsf{wg}}$ by their discretizations.

- This recovers the globularity condition, but at the expenses of the Segal maps, which from being isomorphisms become $(n-1)$-equivalences.

Summary

- Different types of higher structures.
- Multi-simplicial objects are a good environment for building models of higher categories.
- Three Segal-type models of weak n-categories. New model $\mathrm{Cat}^{n}_{\mathrm{wg}}$ based on n-fold structures and on the notion of weak globularity.
- Functors

$$Q_n : \mathrm{Ta}^{n} \rightleftarrows \mathrm{Cat}^{n}_{\mathrm{wg}} : Disc_n$$

inducing an equivalence of categories after localization.

Reference

Research Monograph:

S.Paoli, Segal-type models of higher categories, 2017, (310 pages) available at arXiv.1707.01868.

Thank you for your attention