CwA's of equivalences
Equivalences of CwA's

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Uniqueness of interpretation

$\mathbf{T}$: dependent type theory with e.g. Id, Σ, Π.

$\mathbf{C}_0, \mathbf{C}_1$: models of $\mathbf{T}$, with same underlying category, two different implementations of the constructors. E.g. simplicial sets, two different choices of path-objects.

$\vdash_{\mathbf{T}} A$ some (possibly complex) type.

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In same question for IHOL, this is 2-categorical universal property of syntax.

Concretely: $\mathbf{C}^{\cong}$ also a model of IHOL, with structures of $\mathbf{C}_0$, $\mathbf{C}_1$ on source/target of iso.

Categories with Attributes, Contextual Categories

Definition

Category with attributes: category $\mathbf{C}$, presheaf $\mathrm{Ty} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, and cartesian functor

$$\begin{array}{ccc} \int_{\mathbf{C}} \mathrm{Ty} & \xrightarrow{\ \chi\ } & \mathbf{C}^{\to} \\ & \searrow \quad \swarrow_{\mathrm{cod}} & \\ & \mathbf{C} & \end{array}$$

and distinguished object $\diamond \in \mathbf{C}$.

- "types"/"fibrations": $A \in \mathrm{Ty}(\Gamma)$, $\chi_A : \Gamma.A \twoheadrightarrow \Gamma$
- "terms": sections $a : \Gamma \longrightarrow \Gamma.A$

Definition

CwA $\mathbf{C}$ is contextual if every object of $\mathbf{C}$ uniquely expressible as iterated comprehension $\diamond.A_1.\cdots.A_n$.

Contextual categories $\mathbf{Cxl}$ coreflective in $\mathbf{CwA}$.

Logical structure on CwA's

Type theory $\mathbf{T}$ with some logical constructors $(\mathrm{Id}, \Sigma, \Pi, \ldots)$ corresponds to CwA's with extra structure ("Id-structure", …).

Theorem

For e.g. $\mathbf{T} = (\mathrm{Id}, \Sigma, \Pi)$*, syntactic category* $\mathbf{C}_{\mathbf{T}}$ *is the initial CwA with* Id-*,* Σ-*,* Π-*structure, and is moreover contextual.*

CwA's with $\mathbf{T}$-structure give strictly algebraic notion of models of $\mathbf{T}$.

Classes of maps

Definition

A map $F : \mathbf{C} \longrightarrow \mathbf{D}$ of contextual cats (resp. CwA's) with (at least) Id-types is:

- (local) equivalence ($\mathcal{W}$) if types lift along F up to equivalence, and terms lift up to propositional equality;
- trivial fibration ($\mathcal{TF}$) if types and terms lift on the nose;
- fibration ($\mathcal{F}$) if F has "path-lifting" for equivalences and propositional equalities.

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Get awfs's $(\mathcal{C}, \mathcal{TF}) \longrightarrow (\mathcal{A}, \mathcal{F})$ on $\mathbf{Cxl}_{\mathrm{Id},\ldots}$, $\mathbf{CwA}_{\mathrm{Id},\ldots}$. Intuition:

- Maps in $\mathcal{C}$ built up by freely adjoining types, terms. (In particular: cofibrant CwA's are contextual.)
- Maps in $\mathcal{A}$, by adjoining types/terms that are equivalent/propositionally equal to existing ones.

Assembly

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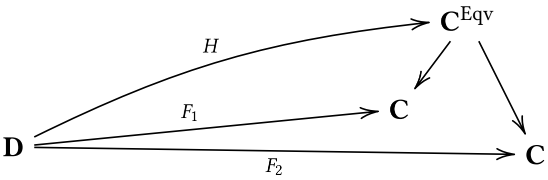
Key tool: something like "path objects" in $\mathbf{Cxl}_{\mathrm{Id},\ldots}$.

I.e.: a fibration $\mathbf{C}^{\mathrm{Eqv}} \twoheadrightarrow \mathbf{C} \times \mathbf{C}$, representing "homotopy"/"natural equivalence" between functors into $\mathbf{C}$.



So: objects/types of $\mathbf{C}^{\mathrm{Eqv}}$ should be pairs of objs/types from $\mathbf{C}$, connected by an equivalence.

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Definition

$\mathbf{C}^{\mathrm{Eqv}}$: the CwA of Reedy span-equivalences in $\mathbf{C}$.

Reedy diagrams, by example

A 2-globular object in a category **C**:

$$A_2 \underset{t}{\overset{s}{\rightrightarrows}} A_1 \underset{t}{\overset{s}{\rightrightarrows}} A_0 \,, \qquad ss = st, \;\; ts = tt$$

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A Reedy 2-globular object in a fibration category $\mathbf{C}$:

$$A_0; \qquad A_1 \twoheadrightarrow A_0 \times A_0; \qquad A_2 \twoheadrightarrow A_1 \times_{A_0 \times A_0} A_1$$

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A Reedy 2-globular type in a CwA $\mathbf{C}$:

$$\vdash A_0 \;\mathsf{type} \qquad\qquad x_0, y_0{:}A_0 \vdash A_1(x_0, y_0) \;\mathsf{type}$$

$$x_0, y_0{:}A_0,\; x_1, y_1{:}A_1(x_0, y_0) \vdash A_2(x_0, y_0, x_1, y_1) \;\mathsf{type}$$

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Inverse category: the kind of category on which this construction makes sense; e.g. the (n-)globular and (n-)semi-simplicial categories.

Reedy span-equivalences

Proposition (following Shulman, Tonnelli)

$\mathbf{C}$ *a CwA,* $\mathcal{I}$ *an inverse cat. Have a "Reedy" CwA structure on* $\mathbf{C}^{\mathcal{I}}$*, with types corresponding to "Reedy fibrations".*
Given $\mathrm{Id}, \Sigma, \Pi, \Pi_{\mathrm{ext}}, \ldots$ *on* $\mathbf{C}$*, can lift them to* $\mathbf{C}^{\mathcal{I}}$*.*

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A Reedy span $B \twoheadrightarrow A_0 \times A_1$ is an equivalence if its legs $B \longrightarrow A_i$ are each equivalences.

Proposition

Reedy span-equivalences form a sub-CwA $\mathbf{C}^{\mathrm{Eqv}}$ of $\mathbf{C}^{\mathrm{span}}$.
If $\mathbf{C}$ has Id, Σ, *or* Π_{ext}, *then $\mathbf{C}^{\mathrm{Eqv}}$ is closed under these in $\mathbf{C}^{\mathrm{span}}$.*

Proof.

Closure under constructors: amounts to showing constructors preserve equivs. (Hence why need Π_{ext}; can't lift Π alone.) □

Wrapping up

$\mathbf{C}^{\mathrm{Eqv}} \twoheadrightarrow \mathbf{C} \times \mathbf{C}$ not quite path object: no refl, trans, generally. But:

Proposition

For $\mathbf{D}$ cofibrant, $\mathbf{C}^{\mathrm{Eqv}}$ induces an equiv. rel. on $\mathbf{CwA}_{\mathrm{Id},\ldots}(\mathbf{D}, \mathbf{C})$.

A left semi model structure: almost a Quillen model structure, except $\mathcal{C} \cap \mathcal{W} = {}^{\boxslash}\mathcal{F}$ holds only under cofibrant domains.

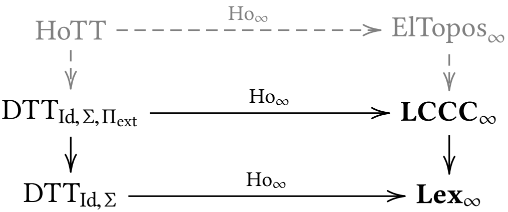
Theorem

- *$(\mathcal{W}, \mathcal{C}, \mathcal{F})$ form a left semi model structure on $\mathbf{Cxl}_{\mathrm{Id},\ldots}$.*
- *$\mathcal{N}_f : \mathbf{Cxl}_{\mathrm{Id},\ldots} \longrightarrow \mathbf{Cat}_\infty$ preserves (and reflects) equivalences, hence induces $(\infty, 1)$-functors:*

$$\begin{array}{ccc} \mathbf{Cxl}_{\mathrm{Id},\Sigma} & \xrightarrow{\mathcal{N}_f} & \mathbf{Lex}_\infty \\ \downarrow & & \downarrow \\ \mathbf{Cxl}_{\mathrm{Id},\Sigma,\Pi_{\mathrm{ext}}} & \xrightarrow{\mathcal{N}_f} & \mathbf{LCCC}_\infty \end{array}$$

Application: internal language conjectures

- **Theorem** (Kapulkin, using Szumiło's N_f). Syntax of DTT with Id, Σ (+ Π_{ext}) yields lex (resp. locally cartesian closed) quasi-categories.
- **Theorem** (Kapulkin, Lumsdaine). This construction induces ∞-functors.
- **Conjecture.** These are ∞-equivalences. (Cf. Kapulkin, Szumiło, arXiv:1709.09519.)
- **Dream.** These lift to "full HoTT", and "elementary ∞-toposes" (both still to be defined.)



Logical application: canonicality of interpretation

Other applications of span-equivalences: constructing equivalences between theories/interpretations. E.g.:

$\mathbf{T}_{\mathrm{Id},\Sigma,\Pi_{\mathrm{ext}}}$: the syntactic category, initial in $\mathbf{Cxl}_{\mathrm{Id},\Sigma,\Pi_{\mathrm{ext}}}$.

Proposition

$\mathbf{C}$ *a CwA, equipped with two (possibly different) choices of* Id, Σ, Π_{ext}. *Then the two induced interpretation functors*

$$[\![-]\!]_0, [\![-]\!]_1 : \mathbf{T}_{\mathrm{Id},\Sigma,\Pi_{\mathrm{ext}}} \longrightarrow \mathbf{C}$$

are "naturally equivalent" by Reedy span-equivalences.

Logical application: canonicality of interpretation

Proof.

Can generalise $\mathbf{C}^{\mathrm{Eqv}}$ to "equiv-comma" CwA $(F_0, F_1)^{\mathrm{Eqv}}$, for $F_i : \mathbf{D} \longrightarrow \mathbf{C}_i$ not necessarily strictly logical. Objects: span-equivs

$$F_0 C_0 \longleftarrow D \longrightarrow F_1 C_1 .$$

Logical structure on $(F_0, F_1)^{\mathrm{Eqv}}$: uses structure of $\mathbf{C}_i$ on C_i.

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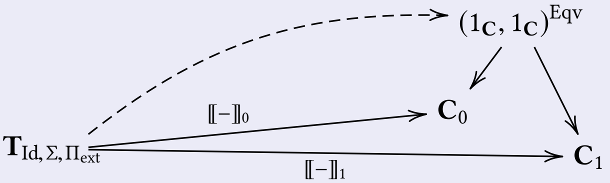
Now: take $\mathbf{C}_0$, $\mathbf{C}_1$ both as $\mathbf{C}$, with the two choices of logical structure; $\mathbf{D}$ also as $\mathbf{C}$, with either choice. Then get:



□

Summary

Technical tools

- 3 classes of maps on CwA's/contextual cats
- the CwA's $({}^{\mathrm{Eqv}}\mathbf{C})$, $({}^{\mathrm{Eqv}}F_0, F_1)$

Applications

- ∞-categorical internal language conjectures
- canonicality of interpretation
- globular ω-categories from CwA's
- giving equivalences between different type theories