Normalisation strategies revisited

Maxime Lucas

September 26, 2017

Motivation (Homotopy)

Let $(\mathcal{E}, \otimes, I)$ be a closed monoidal category equipped with a (cofibrantly generated) model structure, and let P be an $\mathcal{E}$-operad.

$$P\text{-}\mathbf{Alg} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{E}$$

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Theorem (Berger - Moerdijk, '07)

Suppose that:

- $\mathcal{E}$ *is a* ***monoidal model category****.*
- *(Some other conditions that we won't go into...)*

Then it is possible to transfer the model structure from $\mathcal{E}$ *to* P*-algebras.*

Motivation (Homotopy)

As a consequence, there is a *cofibrant replacement functor* $Q : P\text{-}\mathbf{Alg} \to P\text{-}\mathbf{Alg}$.

Example

- Boardman-Vogt resolution
- Bar-Cobar construction

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Example

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Question

Is it possible to use rewriting in order to compute efficiently cofibrant replacements?

Motivation (Rewriting)

Theorem (Squier's Existence Theorem)

Let Σ be a convergent monoidal 1-polygraph, and M the monoid it presents. Then it is possible to extend Σ into a monoidal $(2,1)$-polygraph such that:

- *Elements of Σ_2 correspond to critical pairs.*
- *The (strict) monoidal 2-groupoid generated by Σ forms a coherent presentation of M.*

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Theorem (Existence Theorem generalised, Guiraud-Malbos, '12)

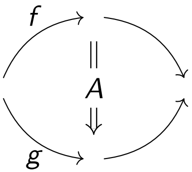
Under the same hypothesis, it is possible to extend Σ into an monoidal $(\omega, 1)$-polygraph such that

- *Elements of Σ_n correspond to critical n-fold branchings.*
- *The (strict) monoidal ω-groupoid generated by Σ forms a polygraphic resolution of M.*

Motivation (Rewriting)

Theorem (Squier's Detection Theorem)

Let Σ be a terminating monoidal $(2,1)$-polygraph, and let M be the monoid it presents. Suppose that for any critical pair (f,g) there exists a 2-cell $A \in \Sigma_2^{m(1)}$ of the form



Then $\Sigma^{m(0)}$ forms a coherent presentation of M.

Goals of this talk

- Extend the Detection Theorem to higher dimensions.

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- Extend the Detection Theorem to higher dimensions.
- Understand the rewriting Theorems as computing *efficient* cofibrant replacement within the framework of Berger-Moerdijk's Theorem.

What is the homotopical setting?

We are looking for a model structure on monoid objects in ω-groupoids:

$$\mathbf{Mon}(\omega\text{-}\mathbf{Gpd}) \underset{}{\overset{}{\rightleftarrows}}_{\bot} \omega\text{-}\mathbf{Gpd}$$

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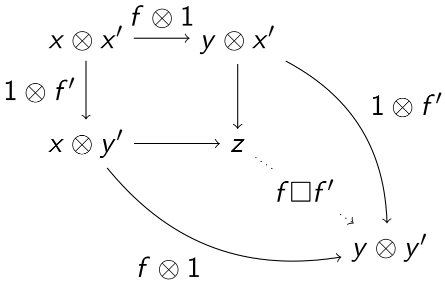
*Can we transfer the model structure of ω-**Gpd** through this adjunction?*
*Is ω-**Gpd** equipped with a structure of monoidal model category?*

Monoidal model category

Definition

A monoidal model category is a closed monoidal category $(\mathcal{E}, \otimes, I)$ equipped with a model structure such that:

- For any cofibrations f, f', $f \Box f'$ is a cofibration:



- If f or f' is a trivial cofibration, then so is $f \Box f'$
- (Some condition on I which is trivial if I is cofibrant)

The monoidal model structure of ω-**Gpd**

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$(\omega\text{-}\mathbf{Gpd}, \times, I)$ *is* ***not*** *a monoidal model category.*

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Remark

If $(\mathcal{E}, \otimes, I)$ is a monoidal model category, then the product of two cofibrant objects is cofibrant.

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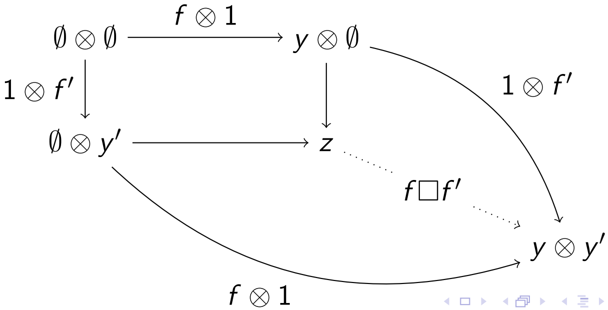

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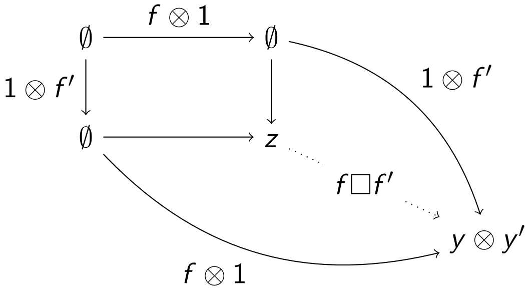

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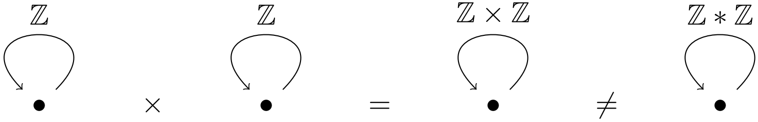

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Conjecture

$(\omega\text{-}\mathbf{Gpd}, \otimes, I)$ *is a monoidal model category, where $\otimes$ is the Gray tensor product.*

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$(\omega\text{-}\mathbf{Gpd}, \otimes, I)$ *is a monoidal model category, where $\otimes$ is the Gray tensor product.*
$(\omega\text{-}\mathbf{Cat}, \otimes, I)$ *is a monoidal model category, where $\otimes$ is the lax Gray tensor product.*

The model and monoidal structures of ω-**Cat**

Proposition (L.)

*If f and f' are two cofibrations in ω-**Cat**, then so is $f \Box f'$.*

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*If f and f' are two cofibrations in ω-**Cat**, then so is $f \square f'$.*

Lemma

*The model structure on ω-**Cat** is cofibrantly generated, with generating cofibrations:*

$$j_n : \square_n \to \blacksquare_n$$

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Lemma

$$j_n \Box j_m = j_{n+m}$$

The model and monoidal structures of ω-**Cat**

Proposition (L.)

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Corollary (Ara-Maltsiniotis, Hadzihasanovic, L.)

The Gray tensor product of two ω-polygraph is still an ω-polygraph.

Back to rewriting

Idea

- *Use "Gray monoids": monoid objects in* $(\omega\text{-}\mathbf{Cat}, \otimes)$ *instead of cartesian monoids.*
- *(For combinatorial reasons): work with cubical* ω*-categories instead of globular ones.*

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Theorem (Batanin, Garner, Shulman)

Let $\hat{\mathcal{I}}$ *be the category of presheaves over an inverse category (e.g. the globular sets, semi-simplicial sets, pre-cubical sets...)*
Any monad T *on* $\hat{\mathcal{I}}$ *induces a notion of* T*-polygraph generating* T*-algebras.*

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Proposition (L.)

Gray monoids are monadic over cubical sets. So there is an associated notion of Gray polygraph.

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Find a sufficient condition so that $\pi : \Sigma^{G(0)} \to M$ is a weak equivalence of Gray monoids.

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The setup (II)

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$$\Sigma^{G(0)} \underset{\mathbf{NF}}{\overset{\pi}{\rightleftarrows}} M$$

We have $\pi \circ \mathbf{NF} = 1_M$.

Problem

Find a sufficient condition so that there exists a natural transformation $S : 1_{\Sigma^{G(0)}} \Rightarrow \pi \circ \mathbf{NF}$.

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We define first a natural transformation $S : 1_{\sum G(1)} \Rightarrow \pi \circ \mathbf{NF}$.

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Idea (General idea of rewriting)

We define first a natural transformation $S : 1_{\Sigma^{G(1)}} \Rightarrow \pi \circ \mathbf{NF}$.

Bonus:
$\Sigma^{G(1)}$ is free as an $(\omega, 1)$-category. So it will be enough to define S on the generators.

Natural transformation between cubical categories

$\Sigma^{G(1)}$ is free as an $(\omega, 1)$-category on an $(\omega, 1)$-polygraph Γ

- $\Gamma_0 = \Sigma_0^{G(1)}$: all the words on Σ_0. For $u \in \Gamma_0$, we want to define:

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- $\Gamma_1 = \{ufv | u, v \in \Gamma_0^* \text{ and } f \in \Sigma_1\}$. For $ufv \in \Gamma_1$, we want to define:

$$\begin{array}{ccc} x & \xrightarrow{\quad ufv \quad} & y \\ {\scriptstyle S(x)}\downarrow & S(ufv) & \downarrow{\scriptstyle S(y)} \\ \hat{x} & = = = = & \hat{x} \end{array}$$

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- $\Gamma_1 = \{ufv | u, v \in \Gamma_0 \text{ and } f \in \Sigma_1\}$: all the rewriting steps. For $ufv \in \Gamma_1$, we want to define:

$$\begin{array}{ccc} x & \xrightarrow{\quad ufv \quad} & y \\ {\scriptstyle S(x)}\downarrow & S(ufv) & \downarrow{\scriptstyle S(y)} \\ \hat{x} & = & \hat{x} \end{array}$$

No compatibility with the product is required!

Let's try! (I)

For $u \in \Gamma_0$ which is not a normal form, we fix $\tau_u : u \to v$ a rewriting step of source u.

$$S(u) := u \xrightarrow{\quad \tau_u \quad} v \xrightarrow{\quad S(v) \quad} \hat{v} = \hat{u}$$

Let's try! (II)

$$S(u) := u \xrightarrow{\quad \tau_u \quad} v \xrightarrow{\quad S(v) \quad} \hat{v} = \hat{u}$$

For $ufv : x \to y \in \Gamma_1$, we are looking to fill the following diagram:

$$\begin{array}{ccc} x & \xrightarrow{\quad ufv \quad} & y \\ {\scriptstyle S(x)} \big\downarrow & S(ufv) & \big\downarrow {\scriptstyle S(y)} \\ \hat{x} & = = = = = & \hat{x} \end{array}$$

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For $ufv : x \to y \in \Gamma_1$, we are looking to fill the following square:

$$\begin{array}{ccc}
x & \xrightarrow{ufv} & y \\
{\scriptstyle \tau_x}\downarrow & & \\
x' & & \downarrow{\scriptstyle S(y)} \\
{\scriptstyle S(x')}\downarrow & & \\
\hat{x} & = & \hat{x}
\end{array}$$

Let's try! (III)

We suppose given a cell $\Phi(\tau_x, ufv)$ of suitable shape:

$$
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\begin{array}{ccccc}
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\| & & \| & & \downarrow S(y) \\
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\tau_x \downarrow & \Phi(\tau_x, ufv) & \downarrow g & T_1 S(g) & \| \\
x' & \xrightarrow{\quad h \quad} & z & \xrightarrow{\quad S(z) \quad} & \hat{x} \\
S(x') \downarrow & S(h) & \downarrow S(z) & & \| \\
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$$

How to extend Φ to higher dimensions?

More generally, we want $\Phi : \mathbf{LocBr}(\Sigma_0, \Sigma_1)_n \to \Sigma_n^{G(1)}$.

Question

How to express the fact that $\Phi(\bar{f})$ *has to have a "suitable shape"?*

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Define $\partial_1(f,g) = g$ and $\partial_2(f,g) = f$, $\Phi(h) = h$ for all $h \in \Gamma_1$.
Then we had:

$$\partial_1^- \Phi(f,g) = \Phi \partial_1^-(f,g) \qquad \partial_2^- \Phi(f,g) = \Phi \partial_2^-(f,g)$$

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- Define $\partial_i(f_1, \ldots, f_n) = (f_1, \ldots, \hat{f}_i, \ldots, f_n)$. The define a semi-simplicial set structure on $\mathbf{LocBr}(\Sigma_0, \Sigma_1)$.
- The operations ∂_i^- induce a structure of semi-simplicial set on $\Sigma^{G(1)}$.

We want Φ to be a morphism of semi-simplicial sets.

The Detection Theorem, generalised

Theorem (L.)

Let Σ be a terminating targets-only Gray $(\omega, 1)$-polygraph, and let M be the monoid presented by Σ. We suppose that there exists a morphism of simplicial monoids

$$\Phi : \mathbf{BrLoc}(\Sigma_0, \Sigma_1) \to \Sigma^{G(1)}$$

such that for all $A \in \Sigma$, $\Phi(\mathbf{br}(A)) = A$.
Then the free Gray-monoid generated by Σ is equivalent to M.

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Why do we need to define Φ on all branchings? Squier-like Theorems should only require hypothesis about critical branchings!

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Theorem (L.)

The simplicial monoid $\mathbf{BrLoc}(\Sigma_0, \Sigma_1)$ is freely generated by the critical branchings.

The reduced standard presentation of a monoid (I)

We fix a monoid M.
We define a Gray $(\omega, 1)$-polygraph Σ as follows:

$$\Sigma_n = \{(m_1|\dots|m_{n+1}) \mid m_i \neq 1\}$$

$$\partial_i^-(m_1|\dots|m_{n+1}) = (m_1|\dots|m_{i-1}) \otimes (m_i|\dots|m_{n+1})$$

$$\partial_i^+(m_1|\dots|m_{n+1}) = \begin{cases} (m_1|\dots|m_i m_{i+1}|m_{i+2}|\dots|m_{n+1}) & m_i m_{i+1} \neq 1 \\ \epsilon_1(m_3|\dots|m_{n+1}) & i=1 \quad m_1 m_2 = 1 \\ \Gamma_{i-1}^+(m_1|\dots|m_{i-1}|m_{i+2}|\dots|m_{n+1}) & 2 \leq i < n \quad m_i m_{i+1} = 1 \\ \epsilon_{n-1}(m_1|\dots|m_{n-1}) & i=n \quad m_n m_{n+1} = 1 \end{cases}$$

The reduced standard presentation of a monoid (II)

$$m_1 \otimes m_2 \xrightarrow{(m_1|m_2)} m_1 m_2$$

$$\begin{array}{ccc}
m_1 \otimes m_2 \otimes m_3 & \xrightarrow{m_1 \otimes (m_2|m_3)} & m_1 \otimes m_2 m_3 \\
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Theorem

The free Gray monoid $\Sigma^{G(0)}$ is equivalent to M.

Existence Theorem

Theorem

Let Σ be a convergent Gray 1-polygraph, and let M be the monoid it presents. There exists an extension of Σ into a Gray $(\omega, 1)$-polygraph such that:

- *The generating n-cells correspond to the critical branchings.*
- *Σ satisfies the hypothesis of the Detection Theorem. In particular, $\Sigma^{G(0)}$ is weakly equivalent to M.*

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Remark

Gray 1-polygraph $\equiv$ monoidal 1-polygraph

Future work

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Any cartesian monoid is a Gray monoid.
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Future work

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Any cartesian monoid is a Gray monoid.
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(Intuition: any group is a monoid, therefore any presentation of monoid can be seen as a presentation of group).

Future work

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Question

What is the relationship between $\Sigma^{G(0)}$ and $\Sigma^{c(0)}$? Is $\Sigma^{c(0)}$ a polygraphic resolution of M?

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Question

Possible extension to operads beyond **Mon***?*

It's not a bug, it's a feature!

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Let Σ be a terminating targets-only Gray $(\omega, 1)$-polygraph, and let M be the monoid presented by Σ. We suppose that there exists a morphism of simplicial monoids

$$\Phi : \mathbf{BrLoc}(\Sigma_0, \Sigma_1) \to \Sigma^{G(1)}$$

such that for all $A \in \Sigma$, $\Phi(\mathbf{br}(A)) = A$.
Then the free Gray-monoid generated by Σ is equivalent to M.

Theorem (L.)

The simplicial monoid $\mathbf{BrLoc}(\Sigma_0, \Sigma_1)$ is freely generated by the critical branchings.