A syntactic approach
to
polynomial functors, polynomial monads and opetopes

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A polynomial functor $P$ from a set of colors $I$ to a set of colors $J$ (notation $P : I \to J$) is given by the following data:

- A set $B^P$ of operations and a mapping $t^P : B^P \to J$ (the target color map),
- and for each $b \in B$, a (finite) set $A^P(b)$ (the arity and a map $s^P_b : A^P(b) \to I$ (the source colors map).

These data can be recorded as **typing judgements**:

$$(A^P(b), s^P_b) \vdash_P b : t^P(b)$$

Think of $(A^P(b), s^P_b)$ as a set of declarations $x : s^P_b(x) \ (x \in A^P(b))$. 

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**Polynomial functors**

**Polynomial monads**

**Opetopes**
Note the difference between names and decorations: the latter can be repeated, while the former ones are in bijection with the number of wires going into the operation.
Polynomial functors (standard presentation)

Polynomial functors were introduced by Gambino and Kock (following ideas of Joyal) (2013), as a triple of maps

\[ I \leftarrow^s A \rightarrow^p B \rightarrow^t J \]

We recover a polynomial functor in standard presentation from our presentation by taking

\[ A = \sum_{b \in B} A(b) \quad s(b, x) = s_b(x) \quad p(x, b) = b \]

Conversely, take \( A(b) = p^{-1}(b) \) and \( s_b = s |_{A(b)} \).
Morphisms of polynomial functors

Let $P$ and $P'$ be two polynomial functors from $I$ to $J$. A morphism $F$ of polynomial functors is given by the following data:

- a function $f : B \to B'$ such that $t' \circ f = t$,

- and for each $b$ a bijection
  \[ f_b : A^P(b) \to A^{P'}(f(b)) \]
  such that $s'_{f(b)} \circ f_b = s_b$.

The (vertical) composition of morphisms is defined in the obvious way:

\[ (g', (g_{b'})_{b' \in B'}) \circ (f, (f_b)_{b \in B}) = (g \circ f, (g_{f(b)} \circ f_b)_{b \in B}) \]
Composition of polynomial functors

Let $P : I \to J$, $Q : J \to K$. We set $Q \circ P = R$, where $B^R$ is the set of all formal “explicit simultaneous substitutions” that are well-typed according to the following rule:

\[
(X, s) \vdash^Q g : k \quad (\forall x \in X, \ (Y_x, s_x) \vdash^P f_x : s(x))
\]

\[
(\sum_{x \in X} Y_x, [s_x]_{x \in X}) \vdash^R g\{x \leftarrow f_x \mid x \in X\} : k
\]

where $[s_x]_{x \in X}$ is the copairing of all $s_x$'s.
Composition of polynomial functors (pictorially)
Identity polynomial functor

\( \text{id}_I : I \to I \) is defined by setting, for every element \( i \in I \):

\[
(\{\ast\}, \ast \mapsto i) \vdash i : i
\]

Polynomial functors are the 1-morphisms of a bicategory

(composition is associative only up to isomorphism).
Polynomial monad (simultaneous multiplication)

A polynomial monad is given by a polynomial endofunctor $P : I \to I$, together with a multiplication from $P \circ P$ to $P$, and a unit from $\text{Id}_I$ to $P$. Notation for (simultaneous) multiplication:

\[ f \circ \{ g_x \mid x \in A(f) \} \quad \text{or} \quad f \circ \{ x \leftarrow g_x \mid x \in A(f) \} \]

\[
\chi_f\{x\leftarrow g_x\mid x\in A(f)\} : \sum_{x\in A(f)} A(g_x) \to A(f \circ \{ g_x \mid x \in A(f) \})
\]

(the bijection $\chi_f\{x\leftarrow g_x\mid x\in X\}$ indicates how the inputs of $f \circ \{ g_x \mid x \in A(f) \}$ match those of the the $g_x$'s).

Notation for the unit: $\eta(i) = \text{id}_i \quad (A_i, s_i) \vdash \text{id}_i : i$, where $A_i$ is a singleton whose unique element is mapped by $s_i$ to $i$.

The multiplication and unit have to satisfy the equalities of a monoid, and the associated bijections must satisfy some coherence conditions.
Polynomial monads are a version of (set) operads that are

- non-symmetric
- non-skeletal (inputs are named, rather than numbered)
- coloured

Note that the mechanics of polynomial functors dictates that the renaming of wires after composition be specified as part of the data defining the structure.

Polynomial monads are exactly the version of multicategories given by Hermida, Makkai and Power.
Simultaneous multiplication pictorially

\[ b \circ \{ b_x \mid x \in A(b) \} \]
Equivalently, we can specify a multiplication of two operations along one chosen input of the first one.
Formally, one specifies an operation mapping each triple \((f, x, g)\) such that \(x \in A(f)\) and \(t(g) = s_f(x)\) to a triple \((f \circ_x g, \phi, \psi)\) such that

\[
(Z, s_Z) \vdash f \circ_x g : t(f) \quad \phi : A(f) \setminus \{x\} \to Z \quad \psi : A(g) \to Z,
\]

\(\phi\) and \(\psi\) commute with source functions, and

\([\phi, \psi] : A(f) \setminus \{x\} + A(g) \to Z\) is a bijection.
Individual multiplication pictorially

\[ b' \quad \ldots \ldots \quad x \quad b \quad \mapsto \quad \ldots \ldots \quad b \circ_x b' \]
We shall be working with an equivalent intermediate notion of multiplication, neither simultaneous nor individual, parameterised by any subset $Y$ of $A(f)$:

$$
f \circ \{ y \leftarrow g_y \mid y \in Y \}$$

$$
\chi f \{ y \leftarrow g_y \mid y \in Y \} : (A(f) \setminus Y) + \sum_{y \in Y} A(g_y) \to A(f \circ \{ g_y \mid y \in Y \})
$$

In the degenerate case where $Y$ is empty, we set $f \circ \{} = f$ and $\chi f \{ Y \} = id$. 
Intermediate multiplication pictorially

\[ b \circ \{b_u, b_v\} \]

... \( b_u \) \( x \) \( u \) \( v \) \( y \) ...
Free polynomial monad (trees)

Let $P : I \to I$. We define a new polynomial endofunctor $P^*$ on $I$. The operations of $P : I \to I$ are trees

- whose nodes are decorated by operations of $P$, and
- whose edges are decorated in $I$.

Syntactically, they are those terms from the raw syntax

$$T ::= \langle i \rangle \mid T$$

$$T ::= b\{y \leftarrow T_y \mid x \in Y\} \quad \text{(where $Y$ is a subset of $A(b)$)}$$

that are accepted by the following typing system:

$$\frac{(\{*\}, * \to i) \vdash^P \langle i \rangle : i}{(X, s) \vdash^P b : i \quad Y \subseteq X \quad (\forall y \in Y \ (L_y, s_y) \vdash^P T_y : s(y))}$$

$$\frac{((X \setminus Y) + \sum_{y \in Y} L_y, [s\mid_{X \setminus Y}, [s_y]_{y \in Y}]) \vdash^P b\{y \leftarrow T_y \mid y \in Y\} : i}{\text{where the symbol } \mid \text{ denotes restriction.}}$$
The typing system builds the arities of our trees by induction.

Fact: In $P^*$, the arity of a tree $T$ is the set of the occurrences of its leaves, which are each unambiguously identified by following the sequence of the names of the edges leading to it. We call such sequences addresses.
We set \( \text{id}_i = \langle i \rangle \)

We define \( S \circ_i T \) recursively, as follows:

\[
(L, s) \vdash T : i \\
\langle i \rangle \circ_* T = T
\]

\[
z \in Y \quad S'_z = S_z \circ_i T \quad \forall y \neq z \quad S'_y = S_y
\]

\[
\text{body}\{y \leftarrow S_y \mid y \in Y\} \circ_{(z,l)} T = \text{body}\{y \leftarrow S'_y \mid y \in Y\}
\]

\[
z \notin Y \quad S'_z = T
\]

\[
\text{body}\{y \leftarrow S_y \mid y \in Y\} \circ_{(y,l)} T = \text{body}\{y \leftarrow S'_y \mid y \in Y \cup \{z\}\}
\]

\( S \circ_i T \) is the tree obtained by

\textit{grafting} \( T \) on the leaf of address \( l \) of \( S \).

Note that in the process, the leaves of \( T \) are all renamed in \( S \circ_i T \).
The star multiplication (pictorially)
A consequence of the freeness of $P^*$

$P \mapsto P^*$ is left adjoint to the forgetful functor from polynomial monads to polynomial endofunctors (over a fixed set $I$ of colours).

If $P$ is already a polynomial monad, then the freeness of $P^*$ induces

- an interpretation morphism $(T \mapsto \llbracket T \rrbracket^{P^*}) : P^* \to P$:

  $$\llbracket \langle i \rangle \rrbracket^{P^*} = \text{id}_i$$
  $$\llbracket b\{y \leftarrow \mathcal{T}_y \mid y \in Y\} \rrbracket^{P^*} = b \circ P\{\llbracket \mathcal{T}_y \rrbracket^{P^*} \mid y \in Y\}$$

- with associated bijections $\rho_T$, each from the set of leaves of $T$ to the arity of $\llbracket T \rrbracket^{P^*}$:

  $$\rho_{b\{y \leftarrow \mathcal{T}_y \mid y \in Y\}}(yl) = \chi^{P}_{b\{y \leftarrow \llbracket \mathcal{T}_y \rrbracket \mid y \in Y\}}(y, \rho_{\mathcal{T}_y}(l))$$
  $$\rho_{b\{y \leftarrow \mathcal{T}_y \mid y \in Y\}}(z) = \chi^{P}_{b\{y \leftarrow \llbracket \mathcal{T}_y \rrbracket \mid y \in Y\}}(z)$$
Another monad on trees: the $+$ construction (Baez-Dolan)

Here we follow (Kock, Joyal, Batanin, and Mascari 2010).

We again suppose that $P$ is a polynomial monad on some $I$. Then $\text{Tr}(P) = B^P$ gives rise to another polynomial monad $P^+$, not on $I$, but on $B = B^P$:

- $B^{P^+} = B^P$
- The arity of a tree is not its set of leaves anymore, but its set of nodes. Formally:

  $$\begin{align*}
  \mathcal{A}^{P^+}(\langle i \rangle) &= \emptyset \\
  \mathcal{A}^{P^+}(b\{y \leftarrow T_y | y \in Y\}) &= \{\epsilon\} + \sum_{y \in Y} \mathcal{A}^{P^+}(T_y) \\
  s_{B \{y \leftarrow T_y | y \in Y\}}^{P^+}(\epsilon) &= b \\
  s_{B \{y \leftarrow T_y | y \in Y\}}^{P^+}(z\ n) &= s_{T_z}^{P^+}(n) \quad (z \in Y)
  \end{align*}$$

- The target colour of $T$ is $[[T]]^{P^*}$
Contrasting the $\star$ and the $+$ constructions

- For a polynomial functor on $I$, $P^\star$ is a polynomial monad on $I$,
- For a polynomial monad on $I$, $P^\star$ is a polynomial monad on $B^P$.

We have

$$B^{P^\star} = \text{Tr}(P) = B^{P^+}$$

but arities are different:

- **leaves** (colored in $I$)
- versus **nodes** (colored in $B^P$)
The $+$ multiplication (pictorially)
The + composition

$$[T] = b$$

$$b\{x \leftarrow S_y \mid y \in Y\} \circ_{c}^{P+} T = T \circ^{P*} \{l \leftarrow S_{\rho_T^{P*}(l)} \mid \rho_T^{P*}(l) \in Y\}$$

$$z \in Y \quad S'_z = S_z \circ_{n}^{P^+} T \quad (\forall y \neq z \quad S'_y = S_y)$$

$$b\{y \leftarrow S_y \mid y \in Y\} \circ_{z \ n}^{P+} T = b\{y \leftarrow S'_y \mid y \in Y\}$$

$$\chi_{S\{n_0 \leftarrow T\}}^{P^+}(n.1) = n$$

$$(n_0 \text{ is not a prefix of } n)$$

$$\chi_{S\{n_0 \leftarrow T\}}^{P^+}((n_0 \times n_1).1) = n_0 \left( (\rho_T^{P^+})^{-1}(x) \right) n_1$$

$$\chi_{S\{n_0 \leftarrow T\}}^{P^+}((n_0 \times n_1).1) = n_0 n$$
An iterative architecture (with its bookkeeping)

From a polynomial monad \( P \) (with \( \text{id}^P, \circ^P, \chi^P \)):

- **Build the \( P^* \) structure:**
  \( \circ^{P^*} \) (using just the polynomial functor structure of \( P \))
  \( \Box \) needs \( \text{id}^P \) and \( \circ^P \)
  \( \rho^{P^*} \) needs \( \chi^P \)

- **Build the \( P^+ \) structure:**
  \( \circ \) needs \( \circ^{P^*} \) and \( \rho^{P^*} \)
  \( \chi^{P^+} \) needs \( \rho^{P^*} \)
Opetopes are defined by iteration of the $+$ construction.

- **Basis** = identity polynomial functor on a singleton set.
  
  $$B^0 = \{\diamond\} \quad B^1 = \{\blacksquare\}$$

  $B^0$ is just a set, and $B^1$ features a polynomial monad $O^1$ on $B^0$:

  $$\left(\{\ast\}, \ast \mapsto \diamond \right) \vdash \blacksquare : \diamond$$

  The monad structure is given by setting $\blacksquare \circ \ast \blacksquare = \blacksquare$.
  
  - **Induction**: We set
    
    $$O^n = (O^{n-1})^+$$
The unique 1-opetope

\[ O^1 \]

\[ B^0 = \{ \diamond \} \]

\[ B^4 = \{ \quad \} \]

\[ \text{id} = \quad \]

\[ 0 \quad 0 = \quad \]

\[ \quad \]
2-opetopes (sequences of arrows)

2-opetopes describe shapes of generating 2-cells, from a sequence of arrows to an arrow:

\[
\begin{align*}
&\left[\epsilon\right]^1 \cdot \bullet \\
&\left[\ast\right]^1 \cdot \bullet \\
&\left[\ast\ast\right]^1 \cdot \bullet
\end{align*}
\]
Syntax versus Pictogrammes
The monad structure on 2-opetopes

\[
\begin{align*}
\{ [\epsilon]^1 \cdot \blacksquare, \\
[\ast]^1 \cdot \blacksquare, \\
[\ast\ast]^1 \cdot \blacksquare \} \circ [\ast]^1 &= \{ [\epsilon]^1 \cdot \blacksquare, \\
[\ast]^1 \cdot \blacksquare, \\
[\ast\ast]^1 \cdot \blacksquare, \\
[\ast \ast \ast]^1 \cdot \blacksquare \}
\end{align*}
\]
The shape of a 3-opetope

Diagram showing the structure of a 3-opetope.
3-opetoposes (shortened as pasting diagrams)

The information of a 3-opetope can be captured by a pasting diagram:
3-opetopes (example 1)

Think of ■ as an arrow, and of ♦ as a point (and then of ⟨♦⟩ as a degeneracy).
Example 1 in pictogramme form
3-opetopes (example 2)

\[
\left\{
\begin{array}{c}
[\epsilon]^2 \cdot \left\{
\begin{array}{c}
[\epsilon]^1 \cdot \blacksquare
\end{array}
\right\}
\\
[[\epsilon]^1]^2 \cdot \left\{
\begin{array}{c}
[\epsilon]^1 \cdot \blacksquare
\end{array}
\right\}
\end{array}
\right.
\]
3-opetopes (example 3)

\[
\begin{align*}
&\{[\epsilon]^2 \cdot [\epsilon]^1 \cdot \blacksquare, \\
&[[**]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, \\
&[[*]^1]^2 \cdot \langle \Diamond \rangle, \\
&[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, \\
&[[\epsilon][\epsilon^{-1}]][2 \cdot \{[\epsilon]^1 \cdot \blacksquare \}
\end{align*}
\]
A composition of 3-opetopes
An example of a 4-opetope (drawing)
An example of a 4-opetope (encoding)

\[
\begin{align*}
[\epsilon]^3 & \cdot \begin{cases} 
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet, [\ast\ast]^1 \cdot \bullet, [\ast\ast\ast]^1 \cdot \bullet \} \\
[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} \\
[[\ast]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet, [\ast\ast]^1 \cdot \bullet \} \\
[[\ast\ast]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet \} \\
[[\ast\ast\ast]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} 
\end{cases} \\
[[\epsilon]^2]^3 & \cdot \begin{cases} 
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} \\
[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet, [\ast\ast]^1 \cdot \bullet \} \\
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \bullet \} 
\end{cases} \\
[[[\epsilon]^1]^2]^3 & \cdot \begin{cases} 
[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet \} \\
[[\epsilon]^1[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} 
\end{cases} \\
[[[\ast]^1]^2]^3 & \cdot \begin{cases} 
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} \\
[[\ast]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} 
\end{cases} \\
[[\epsilon]^2[[\epsilon]^1]^2]^3 & \cdot \begin{cases} 
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} \\
[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \bullet, [\ast]^1 \cdot \bullet \} 
\end{cases}
\end{align*}
\]
An example of a 4-opetope (pictogram)
An example of a 4-opetope (from pictogram to syntax)
A composition of 4-opetopes
What next

This is work in progress. We want to test the usefulness and readability of this notation (opetopes are notoriously difficult to work with).

Experience will tell!

This work seems closest to the work of Hermida, Makkai and Power, who lacked the language of polynomial monads.