Symmetry & Computation

Abstracts : Talks and Posters

April 3-7th, 2018

Evelyne Hubert

Elizabeth Mansfield

Hans Munthe Kaas

Agnes Szanto

Multisymplectic geometry and covariant formalism for mechanical systems with a Lie group as configuration space: application to the Reissner beam

Joël Bensoam, Florie-Anne Baugé

Many physically important mechanical systems may be described with a Lie group G as configuration space. According to the well-known Noether's theorem, underlying symmetries of the Lie group may be used to considerably reduce the complexity of the problems. However, these reduction techniques, used without care for general problems (waves, field theory), may lead to uncomfortable infinite dimensional spaces. As an alternative, the \emph{covariant} formulation allows to consider a finite dimensional configuration space by increasing the number of independent variables. But the geometric elements needed for reduction, adapted to the specificity of covariant problems which admit Lie groups as configuration space, are difficult to apprehend in the literature (some are even missing to our knowledge). To fill this gap, this article reconsiders the historical geometric construction made by E. Cartan in this particular "covariant Lie group" context. Thus, and it is the main interest of this work, the Poincar\'e-Cartan and multi-symplectic forms are obtained for a principal G bundle. It allows to formulate the \emph{Euler-Poincar\'e equations} of motion and leads to a Noether's current form defined in the dual Lie algebra.

https://arxiv.org/abs/1708.01469

Computationnal aspects of equivariant bifurcation theory

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Abstract

Bifurcation theory is intimately linked with symmetry in physics as well as in many other areas of science. Bifurcation analysis aims at computing branching of solutions of a parameter dependent system in a neighborhood of a singular point (bifurcation point). When the system is invariant under the action of a symmetry group this induces additional algebraic structures in the problem, which need to be taken into account in order to describe the set of solutions and their dynamical properties. In this talk I will introduce basic notions about bifurcation with symmetry (so-called "equivariant bifurcation theory") and about the algebraic/geometrical tools which have been developed to carry out the computations. Examples will illustrate the methods.

Orthogonal Polynomials and Integrable Systems

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Abstract

In this talk I will discuss the relationship between orthogonal polynomials with respect to semi-classical weights, which are generalisations of the classical weights and arise in applications such as random matrices, and integrable systems, in particular the Painlevé equations and discrete Painlevé equations. It is well-known that orthogonal polynomials satisfy a three-term recurrence relation. I will show that for some semi-classical weights the coefficients in the recurrence relation can be expressed in terms of Hankel determinants, which are Wronskians, that also arise in the description of special function solutions of Painlevé equations. The determinants arise as partition functions in random matrix models and the recurrence coefficients satisfy a discrete Painlevé equation.

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Symbolic interpretation of the generating function in invariant theory

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A lot of molecules have a non–trivial symmetry point group G at their equilibrium configuration. The study of these systems leads to the problem of building objects belonging to an irreducible representation Γ_2 of G from elementary variables belonging to a representation Γ_1, reducible in general. The generated objects are often expressed in a polynomial form. The problem here is to find and characterize the set of polynomials that can appear in this polynomial expansion [1].

The generating function of Molien [2] is the right tool for this work. It can be directly computed from the matrix representation of the group action on the elementary objects and the character of the irreducible representation Γ_2. The coefficient of degree n in the series expansion of this function gives the number of linearly independent polynomials of degree n that transform according to Γ_2.

We are interested in discussing the symbolic interpretation of the generating function in terms of an integrity basis. We will begin with the well–known case of invariants of finite groups forming a free module and end with the case of covariants forming a non–free module, illustrated with the $\mathrm{SO}(2)$ group. In this case a symbolic interpretation of the generating function is possible if it is written as a sum of rational functions with different denominators [3, 4].

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TORIC REPARAMETRIZATION OF LINEAR COMPARTMENT MODELS

EMILIE DUFRESNE

Abstract. For ODE models with time series data, structural identifiability concerns determining if the parameters of a model are completely determined by continuous time series data given an input, or equivalently, from its input-output data. Linear compartment models are linear ODE models used in systems biology and pharmacokinetics and encoded in a labelled directed graph. Many such models are unidentifiable: parameters can take on an infinite number of values and yet yield the same time series or input-output data. In this work, we generalise the work of Meshkat and Sullivant using techniques of Hubert and Labahn. For a certain class of unidentifiable linear compartmental models, we show how an identifiable reparametrization can be found using scaling symmetries. More generally, even when an identifiable reparametrization is not possible, using scaling symmetries leads to reparametrizations that are one step closer to being identifiable.
(Joint with Nikki Meshkat)

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MAGNUS EXPANSION AND LIE GROUP INTEGRATORS

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Abstract. W. Magnus introduced a differential equation characterising the logarithm of the solution of linear initial value problems. The solution of this differential equation leads to a Lie series known as the Magnus expansion. It involves Bernoulli numbers, iterated Lie brackets and integrals. In this talk we discuss Magnus' expansion in the light of pre- and post-Lie algebras. The latter are examples of algebraic combinatorial structures which arise naturally from the geometry of linear connections. The interplay between algebraic and geometric aspects permits a refined comparison of Magnus methods with certain RKMK schemes in the context of Lie group integrators. The talk is based on joint work with Charles Curry (NTNU) and Brynjulf Owren (NTNU).

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Simple bespoke finite difference methods that preserve conservation laws

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Conservation laws are among the most fundamental geometric properties of a given partial differential equation. However, standard finite difference approximations rarely preserve more than a single conservation law. The fact that divergences belong to the kernel of the Euler operator can be used to construct schemes that preserve multiple conservation laws. This approach, which was introduced in [1], is limited by the complexity of the symbolic computations used to construct such schemes. We present a simpler, more efficient strategy and use it to find bespoke finite-difference schemes that preserve multiple discrete conservation laws for a nonlinear wave equation. These schemes compare well with existing methods.

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Continuous and Discrete Homotopy Operators with Applications

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Abstract

As shown in [1], the homotopy operator can be used to invert the total divergence (i.e., carry out integration by parts on jet spaces) by reducing an integration in multiple dimensions to a standard one-dimensional integration with respect to the variable that parameterizes a homotopic path. Typically, the homotopy formula is expressed in terms of (coordinate independent) differential forms and involves higher-order Euler operators [1].

We present a calculus-based formula for the continuous homotopy operator without reference to higher-order Euler operators [2, 3, 4]. This makes the homotopy operator more efficient when implemented in computer algebra systems such as *Mathematica* or *Maple*.

Likewise, the discrete homotopy operator [5, 6] allows one to invert the forward difference operator, i.e., carry out summation by parts on discrete jet spaces. Using the analogy with the continuous case [2, 7], a formula for the discrete homotopy operator will be presented without use of discrete higher-order Euler operators [2, 3].

One area of application [3, 4, 8] is the symbolic computation of conservation laws of multi-dimensional nonlinear PDEs and nonlinear differential-difference equations. Using the short pulse equation, the Boussinesq and Zakharov-Kuznetsov equations, and the Toda lattice as examples, it will be shown how homotopy operators are used to compute conserved fluxes.

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Invariants of ternary forms under the orthogonal group.
The even degree case.

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Classical invariant theory has essentially addressed the action of the general linear group on homogeneous polynomials. Yet the orthogonal group arises in applications as the relevant group of transformations, especially in 3 dimensional space. Having a *complete* set of invariants for its action on ternary quartics, i.e. degree 4 homogeneous polynomials in 3 variables, is, for instance, relevant in determining biomarkers for white matter from diffusion MRI.

We characterize a generating set of rational invariants of the orthogonal group acting on even degree forms by their restriction on a *slice*. These restrictions are invariant under the octahedral group and their explicit formulae are given compactly in terms of equivariant maps. The invariants of the orthogonal group can then be obtained in an explicit way, but their numerical evaluation can be achieved more robustly using their restrictions. The exhibited set of generators futhermore allows us to solve the inverse problem and the rewriting.

Central in obtaining the invariants for higher degree forms is the preliminary construction, with explicit formulae, for a basis of harmonic polynomials with octahedral symmetry, different, though related, to cubic harmonics.

This is joint work with Paul Görlach (now at MPI Leipzig), in a joint project with Téo Papadopoulo (Inria Méditerranée)

Görlach, P. Hubert, E. and Papadopoulo, T., Rational invariants of ternary forms under the orthogonal group. *To appear in* Foundation of Computational Mathematics.
Already available on `https://hal.inria.fr/hal-01570853` and Arxiv (2017).

Geometric structures for difference equations

Peter Hydon, University of Kent

Many useful differential equations have Lie symmetries, conservation laws, (multi-)symplectic structures and other geometric features. Recently, these ideas have been shown to apply also to difference equations. For instance, there are difference analogues of Noether's two theorems on variational symmetries, together with a new intermediate result. An important application is to determine which finite difference approximations retain conservation laws, Bianchi identities and other essential structures.

In this talk, I will describe aspects of the basic theory, including symbolic and numerical computation, and show how the resulting techniques are used in practice.

“Skew-symmetry and computation”

Arieh Iserles

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Abstract A most welcome feature of orthogonal bases employed in spectral methods is that their differentiation matrix is skew-symmetric, since this ensures conservation of energy in time-evolving problems: familiar examples are a Fourier basis in $[-1, 1]$ and a basis of Hermite functions on $(-\infty, \infty)$.

In this talk, which represents joint work with Marcus Webb (Leuven), we characterise *all* such bases on $(-\infty, \infty)$ and on the Paley–Wiener space of a symmetric subinterval of $(-\infty, \infty)$. Essentially, we prove that for every Borel measure $\mathrm{d}\mu$ which is anti-symmetric with respect to the origin there exists such a basis and it lives on $(-\infty, \infty)$ if this is the support of $\mathrm{d}\mu$, otherwise in a suitable Paley–Wiener space. We also present a constructive algorithm for generating such bases, as well as a number of examples, corresponding e.g. to Freud and Konoplev measures.

Invariants and covariants in Solid Mechanics

Boris Kolev*

Symmetry & Computation
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Abstract

In solids mechanics when the matter is slightly deformed, the local state of strain is modelled, at each material point, by a second-order symmetric tensor $\boldsymbol{\varepsilon}$ (the *strain*). The local stress resulting from the imposed strain is classically described by another second-order symmetric tensor, the Cauchy *stress* $\boldsymbol{\sigma}$. The way stress and strain are related is defined by a *constitutive law*. Among them, linear elasticity is one of the simplest model. It supposes a linear relationship between the strain and the stress tensor *at each material point*, $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, in which $\mathbf{C}$ is a fourth-order tensor, and an element of a 21-dimensional vector space $\mathbb{E}\mathrm{la}$. From a physical point of view, this relation, which is the 3D extension of Hooke's law for a linear spring, $F = k\Delta x$, encodes the elastic properties of a body in the small perturbation hypothesis.

Working with elastic materials implies the need to identify and distinguish them. A natural question is "*How to give different names to different homogeneous elastic materials ?*". Despite its apparent simplicity, this question formulated for 3D elastic media is a rather hard problem to solve. An elasticity tensor $\mathbf{C}$ represents a homogeneous material in a specific orientation and a rotation of the body results in another elasticity tensor $\overline{\mathbf{C}}$ representing the same material. From a mathematical point of view, classifying anisotropic homogeneous materials amounts to describing the orbits of the action of the rotation group $\mathrm{SO}(3,\mathbb{R})$ on $\mathbb{E}\mathrm{la}$. This can be achieved by determining a finite system of *invariants* which *separates* the orbits.

I will present some recent progresses on this problem. A minimal integrity basis of 297 invariants of this tensor has been produced recently (2017) and "coordinate-free" characterization of the *symmetry classes* of this tensor have been formulated (2018) using polynomial covariants. I will explain the main steps and the mathematical tools which lead to these results.

This is a joint work with Nicolas Auffray, Boris Desmorat, Rodrigue Desmorat, Marc Olive and Michel Petitot.

*Boris Kolev, LMT, ENS Cachan, CNRS, Université Paris Saclay, France.

Coordinate-independent criteria for Hopf bifurcations

N. Kruff, Sebastian Walcher

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We discuss the occurrence of Poincaré-Andronov-Hopf bifurcations in parameter dependent ordinary differential equations, with no a priori assumptions on special coordinates. The first problem is to determine critical parameter values from which such bifurcations may emanate, a solution for this problem was given by W.-M. Liu. We add a few observations from a different perspective. Then we turn to the second problem, viz., to compute the relevant coefficients which determine the nature of the Hopf bifurcation. As shown by J. Scheurle and co-authors, this can be reduced to the computation of Poincaré-Dulac normal forms (in arbitrary coordinates) and subsequent reduction, but feasibility problems quickly arise. We present a streamlined and less computationally involved approach to the computations. The efficiency and usefulness of the method is illustrated by examples.

see: https://arxiv.org/abs/1708.06545

The Hopf algebra of Lie group integrators and
planarly branched rough paths

Dominique Manchon

The Hopf algebra of Lie group integrators has been introduced by H.Munthe-Kaas and W.Wright as a tool to handle Runge-Kutta numerical methods on homogeneous spaces. It is spanned by planar rooted forests, possibly decorated. Planarly branched rough paths are characters of this Hopf algebra subject to Hölder-type estimates. We will show how these are used in resolving singular differential equations on homogeneous spaces.

Joint work with Charles Curry (NTNU Trondheim),
Kurusch Ebrahimi-Fard (NTNU) and Hans Z. munthe-Kaas (Univ. Bergen).

Noether's Theorem, then and now

E. L. Mansfield

It is now 100 years since Emmy Noether published her celebrated paper, proving that Lie group symmetries of an action functional give rise to conservation laws, such as the conservation of energy, linear and angular momentum. The proof of the result consists of a process to construct the laws from the Lie group action. Simpler versions of the result were rediscovered many times, but Noether did have the most general result for a very long time.

In this talk I will give an Introduction to the Theorem, and will indicate progress since then, both for the understanding of the smooth laws themselves and for various discrete versions. For smooth systems, we have general formulae which allow the laws to be written down for arbitrary order Lagrangians [1] by symbolic software, such as Maple's `DifferentialGeometry` package. Advances using moving frames have allowed us to to see the structure of the laws, in terms of invariants and an equivariant frame [2, 3]. The laws have been written down for various cases of the finite difference Lagrangians by many authors, the most general result differential-difference Lagrangians is proved in [5], and for Finite Element Lagrangians in [6]. Noether's second theorem, where the Lie group depends on arbitrary smooth functions, has been generalised and adapted to difference systems [4].

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Evolutions of polygons and Soliton equations

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Annalisa Calini, College of Charleston

March 1, 2018

Abstract

The relation between the discrete geometry of surfaces and completely integrable systems has been well stablished in the last few decades, through work of Bobenko, Suris and many others. The recent introduction of discrete moving frames by Mansfield, Mari-Beffa and Wang, and the study of the pentagram map by Richard Schwartz and many others, has produced a flurry of work connecting the discrete geometry of polygons to some completely integrable systems in any dimension, including connections to Combinatorics and the study of the role that the background geometry has in the generation of algebraic structures that often describe integrability. In this talk I will review definitions and background, and will describe recent advances in the proof of the integrability of discretizations of Adler-Gelfand-Dikii systems (generalized KdV), aided by the use of the geometry of polygons in $\mathbb{RP}^m$.

*speaker

ABOUT GORDAN'S ALGORITHM ON BINARY FORMS

MARC OLIVE

Classical invariant theory springs from the works of Boole and Gauss, and was then developed by Clebsch, Gordan, Sylvester, Cayler, etc in the 19th century.

Cayley first made a significant achievement in the field by introducing what is now known as the *Cayley Omega operator*. For about fifteen years (until Cayley's seventh memoir [4] in 1861), the English school of invariant theory, mainly led by Cayley and Sylvester, developed important tools to compute *invariant generators* of binary forms. Calculation was the backbone of this initial approach in invariant theory.

Meanwhile, a German school principally conducted by Clebsch, Aronhold and Gordan, developed their own approach, using the *symbolic method* (also present with slight differences in the English school). In 1868, Gordan, the so-called "king of invariant theory", presented a first proof that *covariant algebras* of binary forms are finitely generated [6]. This result was moreover endowed with a constructive proof: the English and the German schools were both preoccupied by explicit calculations of invariants and covariants.

From 1868 to 1875, Gordan's constructive approach led to several new explicit results. First, Gordan [7] computed covariant bases for the quintic and the sextic. Then, Von Gall, completing some partial results from Gordan, produced a complete covariant basis for the septimic [12] and for the octic [11].

In 1890, Hilbert made a critical advance in the field. Using a totally new approach [8], which is the cornerstone of today's algebraic geometry, he proved a finiteness theorem in the general case of *linear reductive groups* [5]. However, his first proof [8] was criticized at the time as not being constructive. Facing these critics, Hilbert produced later a second proof [8]. This second more effective approach is nowadays widely used to obtain a finite set of generators for invariant algebras [2, 3].

Methods to compute generating sets of invariants for binary forms are not limited to Hilbert. For a single binary form, Olver [10] suggested another constructive approach, which was later generalized to a single n-ary form and also specified with a "running bound" by Brini–Regonati–Creolis [1]. We could also cite Kung–Rota [9] but the combinatorial approach developed there becomes increasingly complex for non-trivial examples.

We will illustrate here the second version of Gordan's algorithm, developing background material in classical invariant theory like the Cayley operator, the polarization operator and transvectants. Then we define *molecules* and *molecular covariants*, which correspond to graphical representations of $\mathrm{SL}(2,\mathbb{C})$ equivariant homomorphisms build up from the Cayley and the polarization operators. Finally, Gordan's algorithm for joint covariants is explained, being deeply connected to some linear diophantine equations.

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Computation with moving frames

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The equivariant method of moving frames, for both finite-dimensional Lie group actions and infinite-dimensional Lie pseudo-groups, provides a practical tool for systematically computing invariant objects, including differential invariants, joint invariants, joint differential invariants, invariant differential operators, invariant differential forms, invariant variational problems, invariant numerical schemes, etc. The powerful recurrence formulas provide a symbolic calculus for understanding the underlying structure of these objects. I will cover the basics, and include some new developments and new applications.

Representation theory of the symmetric group, numerically.

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In a narrow sense, the representation theory of the symmetric group S_n concerns the study of sets of matrices viewed as operators acting on the vector space $\mathbb{C}^m$ who have the group structure of the symmetric group. A basic computational question we address is the following: given matrix representations of the generators of S_n, calculate the reduction of the group to a direct sum of irreducible representations. In linear algebra terms, we wish to find a change-of-basis that simultaneously block-diagonalises the generators of the group. We investigate this question by combing basic tools of numerical linear algebra with concepts from representation theory, such as Young–Jucys–Murphy elements (which span a maximal commutative subalgebra of $\mathbb{C}[S_n]$, namely, the Gelfand–Tsetlin algebra).

The motivation for this work arose from connections with the eigenvalues of random matrices, via known results from integrable probability (Borodin, Okounkov, and Olshanski) and free probability (Biane). The algorithm is also used to give a numerical answer to the open question of calculating Kronecker coefficients of the symmetric group.

Joint work with Oded Yacobi (U. Sydney).

Symmetries of Differential-Difference Equations and Noether's Conservation Laws

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In this talk, we mainly consider continuous symmetries and conservation laws of differential-difference equations. Spotting the incommutability of shift operators and differential operators acted by a group action, prolongations of continuous symmetries are investigated; their characteristic form immediately yields a differential-difference version of the Noether's theorem, connecting symmetries of variational problems and conservation laws of Euler-Lagrange equations. Illustrative examples of differential-difference equations are studied.

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SEPARATING INVARIANTS OF FINITE GROUPS

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ABSTRACT. Let G be a finite group and let X be an affine variety (over an algebraically closed field K) on which G acts through automorphisms. The invariant ring $K[X]^G$ is a finitely generated subalgebra of the ring $K[X]$ of polynomial functions on X. A subset or a subalgebra $S \subseteq K[X]^G$ is called separating if for all $x, y \in X$ with different orbits $Gx \neq Gy$ there exists an invariant $f \in S$ with $f(x) \neq f(y)$. Subsets that generate $K[X]^G$ as a K-algebra are always separating, but separating sets can be much smaller than generating sets.

If γ_{sep} denotes the smallest size of a separating set, then we always have $n \leq \gamma_{\text{sep}} \leq 2n+1$ where n is the transcendence degree of $K[X]^G$, i.e., since G is finite, the dimension of X.

In the case of linear actions on vector spaces several results, proved by Serre, Dufresne, Kac-Watanabe and Gordeev, and Jeffries and Dufresne exist that relate properties of the invariant ring or a separating subalgebra to properties of the group action. In this talk we present generalizations of these results to the case of (possibly) non-linear actions on affine varieties.

Under mild assumptions on the variety and the group action we show that $\gamma_{\text{sep}} = n$ can only occur if G is generated by reflections, while $\gamma_{\text{sep}} = n+1$ (or more generally, the existence of a separating subalgebra which is a complete intersection) can only occur if G is generated by bireflections.

Computing the homology of symmetric semi-algebraic sets

Cordian Riener, UiT – The Arctic University of Norway

Let $\mathbf{R}$ be a real closed field, $S \subset \mathbf{R}^k$ be a semi-algebraic set and consider the rational (co-)homology groups of S. It is a fundamental problem in computational real algebraic geometry to compute the dimensions of these rational vector spaces. We consider the special case, when the semi-algebraic set is defined by symmetric polynomials of fixed degree. The action of the symmetric group $\mathfrak{S}_k$ on $\mathbf{R}^k$ gives these groups then the structure of a $\mathfrak{S}_k$-module. We study the associated isotypic decomposition and show bounds on the multiplicities of the irreducible representation appearing in this decomposition. In particular, we study the trivial representation, which is naturally isomorphic to the equivariant Homology groups, and given an algorithm with polynomially bounded (in k) complexity for computing these equivariant Betti numbers. We then discuss how this algorithm can be extended to an algorithm to compute the (ordinary) Betti numbers of S.
(joint work with Saugata Basu)

Discrete moving frames, evolution of curvature invariants and discrete integrability.

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Abstract

Discrete moving frames have been proven useful for the study of discrete integrable systems, which arise as analogues of curvature flows for polygon evolutions in homogeneous spaces [3]. In [4] a method that provides the evolution equation for the curvature invariants of a curve is presented. It is shown that it derives from a syzygy between sets of invariants. The study in [4] further makes a comparison between the symmetry condition of the curve evolutions and the curvature evolutions.

In this talk, I will introduce a brief discrete moving frame formalisation and I will compare the evolutions in the Lie algebra and the evolutions in the Lie Group in the continuous and in the discrete case. I will also derive the analogue of the compatibility condition in the discrete case to the one given in [4] in order to answer when a symmetry of the curvature evolution gives rise to a symmetry of the discrete curve evolution. I will present a few examples in order to illustrate the theorems and relate them with discrete integrable systems.

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Shape analysis on homogeneous spaces

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November 15, 2017

Shape analysis methods have in the past few years become very popular, both for theoretical exploration as well as from an application point of view. Originally developed for planar curves, these methods have been expanded to higher dimensional curves, surfaces, activities, character motions and many other objects. Here by shape we mean an unparametrized curve evolving on a vector space, a Lie group or on a manifold. Spaces of these curves, the so called Shape spaces are studied using infinite dimensional Riemannian geometry to compare and analyse shapes.

We are concerned with one particular approach to shape analysis, based on the Square Root Velocity Transform (SRVT) [SKJJ11]. Originally developed for vector spaces, the SRVT maps parametrised curves to appropriately scaled tangent vector fields along them. The transformed curves are compared computing geodesics in the L^2 metric. In our work, we have generalised the SRVT to Shape spaces with values in Lie groups and homogeneous manifolds. Key to our approach is the idea to use the transitive Lie group action available on these spaces. This additional geometric information provided by the group action enabled us (see [CES16, CES17]) to construct structure preserving numerical algorithms. The use of group actions is an alternative to what was earlier proposed in [SKKS14, Le 16] on Riemannian manifolds.

It turns out that for reductive homogeneous manifolds, there is an explicit way to construct these algorithms and we present some results concerning the realisation of our methods on reductive homogeneous spaces. In particular, we have applied these methods to several concrete problems from computer vision and motion capturing. For example in [CES16], our methods yield a curve closing algorithm allowing one to remove discontinuities from motion capturing data while preserving the general structure of the movement. (see Figure 1)

Our results on shape spaces with values in homogeneous manifolds (cf. [CES17, CEES17] are visualised by considering quotients of real (matrix) Lie groups (where the subgroups arise by canonically embedding the smaller Lie groups). To illustrate the performance of the proposed approach we compute geodesics between curves on the 2-sphere, see Figure 2 for an example.

*joint with E. Celledoni and S. Eidnes (NTNU Trondheim)

Applications of Cumulative Distance Histograms in Diagnosing Breast Cancer

In this talk we build on the Euclidean-invariant distance histogram function for curves originally introduced by Brinkman, D., and Olver.
Based on this cumulative histogram methodology we consider three specialized histograms centroid distance, curvature, and derivative of curvature and one specialized noncumulative histogram centroid distance. We will review a change in concavity point grading algorithm that has been shown to be effective in analyzing cumulative and noncumulative centroid distance histograms. We extend our methodology to calculate the area under cumulative curvature and derivative of curvature histograms and will discuss an application of this methodology to demonstrate that there is a statistically significant difference between the cumulative histogram of benign and malignant tumors in breast cancer.

Symmetry constraints for the modeling and numerical simulation of turbulent flows

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January 12, 2018

Abstract It is well known that the governing equations of fluid dynamics, the Navier–Stokes equations, are invariant under certain transformations, such as instantaneous rotations of the coordinate system and the Galilean transformation. These transformations, also referred to as symmetries of the equations, play an important physical role because they ensure that the description of fluids is the same in all inertial frames of reference. Furthermore, they relate to conservation and scaling laws. It has since long been realized that it is desirable that these symmetries are also satisfied in large-eddy simulations [1, 4].

Using large-eddy simulations one aims to predict the large-scale behavior of turbulent flows. This is done through numerical solution of the Navier–Stokes equations, on grids that are too coarse to resolve all the relevant physical details. An extra forcing term called a turbulence, or subgrid-scale, model is introduced to represent the effects of the (unresolved) small scales on the (resolved) large-scale motions.

We present a framework of constraints for the creation and assessment of subgrid-scale models for large-eddy simulation [2, 3], based on the idea that it is desirable that subgrid-scale models are consistent with the symmetries, as well as with other mathematical and physical properties of the Navier–Stokes equations. We further discuss issues of numerical implementation, including that of conservation of energy in simulations [5], and we show results of large-eddy simulations of turbulence, obtained using a new subgrid-scale model with built-in desirable properties.

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Walks, Groups, and Difference Equations

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Many questions in combinatorics, probability and statistical mechanics can be reduced to counting lattice paths (walks) in regions of the plane. A standard approach to counting problems is to consider properties of the associated generating function. These functions have long been well understood for walks in the full plane and in a half plane. Recently much attention has focused on walks in the first quadrant of the plane and has now resulted in a complete characterization of those walks whose generating functions are algebraic, holonomic (solutions of linear differential equations) or at least differentially algebraic (solutions of algebraic differential equations).

I will give an introduction to this topic, discuss previous work of Bousquet-Melou, Kauers, Mishna, and others and then present recent work by Dreyfus, Hardouin, Roques and myself applying the theory of QRT maps and Galois theory of difference equations to determine which generating functions satisfy differential equations and which do not.

HYBRID FINITE ELEMENT METHODS PRESERVING LOCAL SYMMETRIES AND CONSERVATION LAWS

ARI STERN

Abstract. Many PDEs arising in physical systems have symmetries and conservation laws that are *local* in space. However, classical finite element methods are described in terms of spaces of *global* functions, so it is difficult even to make sense of such local properties. In this talk, I will describe how hybrid finite element methods, based on non-overlapping domain decomposition, provide a way around this local-vs.-global obstacle. Specifically, I will discuss joint work with Robert McLachlan on multisymplectic hybridizable discontinuous Galerkin methods for Hamiltonian PDEs, as well as joint work with Yakov Berchenko-Kogan on symmetry-preserving hybrid finite element methods for gauge theory.

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Multivariate Symmetric Interpolation, Subresultants and Jacobi Polynomials

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Abstract

The theory of symmetric multivariate Lagrange interpolation is a beautiful but rather unknown tool that has many applications. In this talk I first describe how to derive from it an Exchange Lemma that allows to explain in a simple and natural way the full description of the double sum expressions introduced by Sylvester in 1853 in terms of subresultants and their Bézout coefficients. I will also report on generalizations to symmetric multivariate Hermite interpolation, and applications to get closed formulae for the subresultants in the case of root multiplicities. Finally I will describe the extremal case when the polynomials have only one root, and show the connection to Jacobi polynomials. This talk is based on my collaboration with Alin Bostan, Carlos D'Andrea, Teresa Krick and Marcelo Valdettaro.

Symmetries in Numerical Analysis

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Symmetries have always played a fundamental, but somewhat underestimated, role in numerical analysis. The goal of this talk is to review some of the manifestations of symmetries such as

- equivariance, natural transformations
- weak natural transformations (a generalisation of equivariance)
- prolongation of actions

to numerical analysis. We will focus mostly on the affine group, but also on the rotation and unitary groups. The numerical algorithms that we will look at include

- Runge–Kutta methods and B-Series
- Symplectic integrators on projective spaces
- Polynomial interpolation

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Dimension reduction for chemical reaction equations

We consider parameter-dependent polynomial (or rational) systems of ordinary differential equations, with an emphasis on equations derived from chemical reaction networks with mass-action kinetics. Such systems may be high-dimensional but frequently model assumptions or intuition suggest reduction to small dimension, for which there exist various methods and heuristics. The work outlined in the talk, done jointly with Alexandra Goeke, Eva Zerz and other co-authors, provides a systematic and mathematically sound approach to reduction methods, based on Tikhonov's and Fenichel's classical theorems on singular perturbation theory. The work (necessarily) relies on results and methods from algorithmic algebra.

Reduction with no a priori separation of slow and fast variables: Tikhonov's and Fenichel's theorems assume a separation of variables in two sets ("slow" and "fast"). While reaction equations frequently exhibit slow-fast phenomena, no slow and fast variables are generally known a priori. We obtain coordinate-independent criteria and a coordinate-independent reduction: If the system is cast in the form

$$\dot{x} = h(x, \varepsilon) = h^{(0)}(x) + \varepsilon h^{(1)}(x) + \cdots$$

with a small parameter $\varepsilon > 0$ then a Tikhonov-Fenichel reduction exists if and only if $h^{(0)}$ satisfies a number of requirements, the most important of which is the existence of non-isolated stationary points. The reduced equation itself is defined on an algebraic variety.

Finding "small parameters" in parameter-dependent systems: The reduction above requires a priori knowledge of a "small parameter" ε in a given parameter dependent system, or rather knowledge of so-called Tikhonov-Fenichel parameter values from which singular perturbations emanate. Thus, starting with an ODE

$$\dot{x} = H(x, p)$$

which depends on parameters $p \in \mathbb{R}^m$, the first task is to identify parameter values p^* so that a small perturbation (along a curve in parameter space) will lead to the setting of Tikhonov's and Fenichel's theorems. (Loosely speaking, p^* corresponds to $\varepsilon = 0$.) This problem is amenable to algorithmic algebra; in particular the existence of non-isolated stationary points at $p = p^*$ naturally brings elimination ideals into play. From a theoretical perspective, a complete characterization of Tikhonov-Fenichel parameter values can be given. As for applications, this allows to determine systematically all Tikhonov-Fenichel parameter values for standard systems from biochemstry.

SYMMETRY AND CUBATURE RULES

YUAN XU

Abstract

How many nodes do we need for a cubature rule of degree $2n-1$ on a domain Ω in $\mathbb{R}^d$? The answer depends on the symmetry of the integral. It is known that the Gaussian cubature rules of degree $2n-1$, which has $\binom{n+d-1}{d}$ nodes, exist for two family of integrals, both of which on domains that are not centrally symmetric. More nodes are needed for integrals on centrally symmetric domains. We discuss what is known on the number of nodes and the structure of cubature rules that have small number of nodes.

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MOVING FRAMES AND CONSERVATION LAWS: A LINEAR ACTION OF SU(2)

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In the study of variational problems, it is very useful to analyse the symmetries of a Lagrangian. We take a closer look to the case where a Lagrangian is symmetric with respect to a linear action of SU(2) on pairs of complex curves.

A milestone in the study of variational systems is Noether's theorem, which allows us to derive the conservation laws. Our work takes place in the setting of the invariant calculus of variations and we derive both invariantised Euler-Lagrange equations and conservation laws, in a similar way as in [1] and [2]. The conservation laws involve the adjoint representation of the moving frame, a vector of invariants ($\mathbf{v}$) and a vector of constants ($\mathbf{c}$). In this case we obtain:

$$Ad(g)^{-1}_{|frame}\mathbf{v} = \mathbf{c}$$

where $g \in SU(2)$. If the constants and the vector of invariants are known, we show a method that takes advantage of the geometrical setting to derive the parameters appearing in the adjoint representation of the moving frame. Once the frame has been found, we use it to find all the solutions to the variational problem.

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Posters

General self similar solutions and contour enhancement via nonlinear degenerate parabolic equation

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Abstract: We propose in this work to study a contour enhancement in image processing via the nonlinear degenerate parabolic equation

$$\frac{\partial \phi}{\partial t} = \phi_x^{-2(1+\alpha)} \phi_{xx}$$

Where $\phi(x,t)$ image intensity flux which takes values between 0 (black) and 1 (white), and $\alpha \geq 0$ is a positive constant which plays the role of an enhancement parameter.

We seek a general self similar solutions by a formulation of free-boundary problem describing the image intensity evolution in the boundary layer. In particular we study the contour enhancement for a large parameter enhancement.

Keywords: Nonlinear diffusion equations - General self similar solutions - Free boundary problem

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A Newton-like Validation Method for Chebyshev Approximate Solutions of Linear Ordinary Differential Equations

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A wide range of efficient numerical routines exist for solving function space problems (ODEs, PDEs, optimization, etc.) when no closed form is known for the solution. While most applications prioritize efficiency, some safety-critical tasks, as well as computer assisted mathematics, need rigorous guarantees on the computed result. For that, rigorous numerics aims at providing numerical approximations together with rigorous mathematical statements about them, without sacrificing (too much) efficiency and automation.

In the spirit of Newton-like validation methods (see for example [3]), we propose a fully automated algorithm which computes both a numerical approximate solution in Chebyshev basis and a rigorous uniform error bound for a restricted class of differential equations, namely Linear ODEs (LODEs). Functions are rigorously represented using *Chebyshev models* [2], which are a generalization of Taylor models [4] with better convergence properties. Broadly speaking, the algorithm works in two steps: *(i)* After applying an integral transform on the LODE, an infinite-dimensional linear almost-banded system is obtained. Its truncation at a given order N is solved with the fast algorithm of [5]. *(ii)* This solution is validated using a specific Newton-like fixed-point operator. This is obtained by approximating the integral operator with a finite-dimensional truncation, whose inverse Jacobian is in turn approximated by an almost-banded matrix, obtained with a modified version of the algorithm of [5].

As an example, we propose to validate a satellite trajectory arising in a space rendezvous problem (a more in-depth study is led in [1]). A C library implementation is freely available online[1].

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Combinatorics of involutive divisions

M.Ceria

Given a semigroup ideal J and its minimal set of generators, Janet introduced in 1920 both the notion of multiplicative variables and the connected decomposition of J into disjoint cones, together with a procedure to produce such a decomposition. Moreover, in order to describe Riquier's formulation of the description for the general solutions of a PDE problem, Janet gave a similar decomposition also for the related escalier.

Later, from 1924, he gave a completely different decomposition (and the related algorithm for computing it) which labelled as involutive and which is behind both Gerdt-Blinkov procedure for computing Groebner bases and Seiler's theory of involutiveness.

Assuming to have a homogeneous ideal I, within a generic frame of coordinates, he reformulates Riquier's completion proposing essentially a Macaulay-like construction, iteratively computing the vector-spaces $I_d := \{f \in I : \deg(f) = d\}$ until Cartan test grants that Castelnuovo-Mumford regularity D has been reached.

Janet explicitly demotes the role of the ideal in this construction considering the whole set T_D of terms in degree D and decomposing it in terms of disjoint cones generated by multiplicative variables.

The aim of this paper is to discuss involutiveness following the approach proposed by Janet; in particular we postpone the discussion of ideal membership and related test only after having performed a deep reconsideration of the combinatorial properties of involutive divisions over T_D.

To do so, we apply the theory of involutive divisions, set up by Gerdt–Binklov but we have to adapt it, talking about relative involutive divisions, and requiring that the union of all the cones produces the ideal $T_{\geq D}$ and that the cones are disjoint.

Then we deal with the problem of membership. In particular, we define a directed graph with vertices in T_D such that

- if a vertex h is included in the ideal and we walk against the flow, we reach all the terms in T_D that must belong to the ideal; and
- if a vertex n is included in the escalier and we follow the flow, we reach all the terms in T_D which necessarily belong to the escalier as well.

First we show that such a graph can be easily obtained, adapting Ufnarovsky graph, for Pommaret division. However a general solution requires to build and prune a graph constructed using the lcm's of all the pairs of terms in T_D.

Volume of alcoved polyhedra and Mahler conjecture

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Abstract

The facet equations of a 3–dimensional alcoved polyhedron $\mathcal{P}$ are only of two types ($x_i = cnst$ and $x_i - x_j = cnst$) and the f–vector of $\mathcal{P}$ is bounded above by $(20, 30, 12)$. We represent an alcoved polyhedron by a real square matrix A of order 4 and we compute the exact volume of $\mathcal{P}$: it is a polynomial expression in the a_{ij}, homogeneous of degree 3 with rational coefficients. Then we compute the volume of the polar $\mathcal{P}^\circ$, when $\mathcal{P}$ is centrally symmetric. Last, we show that Mahler conjecture holds in this case: the product of the volumes of $\mathcal{P}$ and $\mathcal{P}^\circ$ is no less that $4^3/3!$, with equality only for boxes. Our proof reduces to computing a certificate of non–negativeness of a certain polynomial (in 3 variables, of degree 6, non homogeneous) on a certain simplex.

Keywords and phrases: volume, alcoved polyhedron, tropical semiring, normal matrix, idempotent matrix, symmetric matrix, perturbation, Mahler Conjecture, polynomial, certificate of non–negativeness.

Joint work with P.J. Clavería

Computing Symmetric cubatures: A moment matrix approach.

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A quadrature is an approximation of the definite integral of a function by a weighted sum of function values at specified points, or nodes, within the domain of integration. Gaussian quadratures are constructed to yield exact results for any polynomials of degree $2r-1$ or less by a suitable choice of r nodes and weights. Cubature is a generalization of quadrature in higher dimension. Constructing a cubature amounts to find a linear form
$\Lambda : \mathbb{R}[x] \to \mathbb{R}$, $p \mapsto \sum_{j=1}^{r} a_j\, p(\xi_j)$ from the knowledge of its restriction to $\mathbb{R}[x]_{\leq d}$. The unknowns to be determined are the weights a_j and the nodes ξ_j.

An approach based on moment matrices was proposed in [?, ?, ?]. We give a basis-free version in terms of the Hankel operator $\mathcal{H}$ associated to Λ. The existence of a cubature of degree d with r nodes boils down to conditions of ranks and positive semidefiniteness on $\mathcal{H}$. We then recognize the nodes as the solutions of a generalized eigenvalue problem.

Standard domains of integration are symmetric under the action of a finite group. It is natural to look for cubatures that respect this symmetry [?, ?]. They are exact for all anti-symmetric functions beyond the degree of the cubature. Introducing adapted bases obtained from representation theory, the symmetry constraint allows to block diagonalize the Hankel operator $\mathcal{H}$. The size of the blocks is explicitly related to the orbit types of the nodes. From the computational point of view, we then deal with smaller-sized matrices both for securing the existence of the cubature and computing the nodes.

Joint work with Mathieu Collowald, Université Côte d'Azur & Inria Méditerranée [?].

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A procedure for resolving ambiguous planar regions and its applications to offsets and other morphological operations

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We consider regions on the complex plane which are defined by a closed and oriented planer curve. When the curve is simple the region is obtained without any ambiguity, however, when the curve is not simple there might be some ambiguity in determining the region. This is a real problem since, for example, the offset of a simple curve is not necessarily a simple curve. And yet, the offset of a region should again be a region. This creates the problem of resolving ambiguities in regions which are defined as having the boundary of a not necessarily simple curve. In this work we use the notion of a link [1] as a model for a continuous closed and oriented planar curve. We call it a complex-link and identify the euclidean plane with the complex numbers. The notion of a complex-link, as a mathematical structure (consisting of an indexing set, an endomap of indexes and a realization map into the complex numbers, see [1] for more details) has certain desirable properties. For example, it is an efficient way of encoding a planar curve in a clear and concise way. It is suitable for practical computational calculations as well as to produce mathematical consistency. On the top of the mathematical structure of a complex-link, we derive a general procedure that assigns to every such structure a classifying map. This map creates a partitioning of the complex plane into a family of regions indexed by the integers. The region labeled by 0 is unlimited, a region labeled by n+1 is contained into a region labeled by n. In particular, the partition can be transferred to the indexing sets and the ambiguity is resolved by choosing the appropriate indexing family. For example, in the case of offsets, the resulting region is the one whose boundary is the indexing family indexed by 0, if the region is limited, or -1 is the complement of the region is limited. Applications to other morphological operations are also derivable from this general procedure. For instance, the procedure outlined in [2] is covered by this approach.

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Symmetries and conservation laws as constraints for the modeling and numerical simulation of turbulent flows

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Abstract It is well known that the governing equations of fluid dynamics, the Navier–Stokes equations, are invariant under transformations like instantaneous rotations of the coordinate system and the Galilean transformation. These transformations, or symmetries of the equations, play an important physical role because they ensure that the description of fluids is the same in all inertial frames of reference. They further relate to conservation and scaling laws [2]. It has since long been realized that it is desirable that these symmetries are also satisfied in large-eddy simulations [1, 5].

Using large-eddy simulations one aims to predict the large-scale behavior of turbulent flows. This is done by numerically solving the Navier–Stokes equations, on grids that are too coarse to resolve all the relevant physical details. An extra forcing term, called a turbulence or subgrid-scale model, is introduced to model the effects of the (unresolved) small-scale motions on the (resolved) large scales.

We present a framework of constraints for the creation and assessment of subgrid-scale models for large-eddy simulation [3, 4], based on the idea that it is desirable that subgrid-scale models are consistent with the symmetries, as well as with other mathematical and physical properties of the Navier–Stokes equations. We also discuss issues of numerical implementation, including that of conservation of energy in simulations [6]. Finally, we wonder how the symmetries of the Navier–Stokes equations can be satisfied on the discrete level.

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