SUR LE CALCUL FORMEL ET NUMÉRIQUE DES SINGULARITÉS

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Journées Nationales de Calcul formel

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Ce mini-cours décrit les idées qui permettent d'aborder le calcul de racines de problèmes singuliers. Ce problème est énoncé dans le livre Condition. The Geometry of Numerical Algorithms. de Felipe Cuker et Peter Bürgisser [4] en ces termes

Extend the Shub–Smale theory from Chaps. 15–17 to systems with multiple zeros.

Je parlerai de travaux en cours dont certains ne sont pas publiés. Si l'accent est mis sur le coté numérique, il n'en reste pas moins que l'aspect symbolique est fondamental non seulement d'un point de vue théorique mais également du point de vue pratique. En général l'ordre des calculs se fait dans le sens symbolique puis numérique. On montrera sur les problèmes d'évaluation ou de détermination de rang comment utiliser des calculs numériques pour mener à bien des calculs exacts.
J'ai conçu ce cours en quatre parties :

0– Bref historique de la méthode de Newton.

1– Homotopie et problèmes singuliers : le cas univarié.

2– Approximation rapide des variétés singulières.

3– Structure algébrique des racines de systèmes polynomiaux. Quelques méthodes symboliques-numériques.

4– Approximations de racines isolées de racines de sytèmes analytiques.

La partie 0 décrit un bref aperçu historique de l'évolution des problèmes et leur traitement du point de vue de la méthode de Newton.

Un des problèmes est de trouver un point proche d'une racine multiple d'un système d'équation. Le cas régulier est le $17^{\text{ème}}$ problème de Smale résolu en 2015 par Pierre Lairez[16]. Dans le cas singulier la difficulté est la rencontre de la courbe d'homotopie avec la variété singulière quand celle ci est de dimension strictement positive. Ce phénomène n'arrive pas dans le cas univarié. J'expliquerai la dynamique des courbes d'homotopie en me basant sur l'ouvrage de Blum-Cucker-Shub-Smale [2] et l'article de 2005 de Giusti-Lecerf-Salvy-Yakoubsohn [12].

Un problème peu abordé est la méthode de Newton dans le cas non surjectif et non injectif. C'est le cas si la variété est de dimension positive et singulière. Je parlerai

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d'un travail non publié en collaboration avec Gregorio Malajovich `www.labma.ufrj.br/~gregorio/`.

Dans la troisième partie il sera question de la structure algébrique des racines. Les méthodes symboliques-numériques consistent à déterminer un système régulier partir d'un système singulier en une racine. Ce type de méthode est connu sous la dénomination de méthode de déflation. J'insisterai sur la dernière méthode de Hauenstein-Mourrain-Szanto [14]. Ces auteurs propose une méthode de construction d'un système régulier qui ajoute un nombre polynomial de variables contrairement à d'autre auteurs comme dans [18] dont le nombre de variables est exponentiel.

Enfin la dernière partie exposera un travail récent avec Marc Giusti `www.lix.polytechnique.fr/~giusti/`. C'est une extension d'un travail paru dans le volume 604 de Contemporary Mathematics [11]. On tente de faire l'analyse numérique d'une méthode de déflation sans ajout de nouvelles variables. À cette occassion on aborde les questions de la détermination numérique du rang d'une matrice et de l'évaluation d'une fonction. Notre but est d'obtenir des algorithmes libres de ϵ.

La bibliographie proposée est sélective.

Sur la question des méthodes d'approximation des racines de polynômes d'une variable on pourra se rapporter pour l'analyse de méthodes de subdivision à Becker-Sagraloff-Yap et [1], Batra-Sharma [28]. Pour des analyses fines de méthodes de points fixes voir Proinov [24], [25]. Concernant le champ de la dynamique de la méthode de Newton voir Hubbard-Schleicher-Sutherland [15] et le très beau et récent travail de Schleicher-Stoll [27].

Sur les travaux les plus récents concernant les méthodes de déflation et les structures algébriques des racines multiples on peut se rapporter à :

1– Leykin-Verschelde-Zhao [18],
2– Dayton-Li-Zeng [5] qui généralisent le travail précédent aux système de fonctions analytiques.
3– Les divers travaux de Bernard Mourrain sur la dualité par exemple [22] et celui de Emsalem sur la gométrie des points épais[9] (de très belles maths en français !).
4– Les travaux de Li-Zhi[19],[20].
5– Le papier fondateur de Lecerf sur l'itération de Newton pour des racines multiples [17].

Sur les études à tendances numériques on peut citer :

1– un travail fondateur sur un théorème de Rouché pour l'existence de zéros double-simple Dedieu-Shub [7].
2– un papier difficile à lire mais avec encore un potentiel inexploité qui généralise le précédent dans le cas de plongement 1 par Giusti-Lecerf-Salvy-Yakoubsohn [10].

Enfin il y a les livres :

1– Blum-Cucker-Shub-Smale [2]
2– Bürgisser-Cucker [4]
3– McNamee-Pan [21]
4– Elkadi-Mourrain [8]

Sans oublier les deux remarquables ouvrages en français

1– Jean-Pierre Dedieu, Points fixes zéros et méthode de Newton [6], source d'idées inépuisable!

2– Bostan-Chyzak-Giusti-Lebreton-Lecerf-Salvy-Schost, Algorithmes efficaces en calcul formel [3].

Je remercie le comité d'organisation des JNCF 2018 pour cette invitation à donner ce mini-cours.

Je dédie ma conférence à mon ami Jean-Pierre Dedieu, décédé en 2012, qui m'a précédé en 2010 avec le mini-cours :

Complexité des méthodes homotopiques pour la résolution des systèmes polynomiaux.

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Centro Nacional de Jubilación Científico

Sur le calcul formel et numérique des singularités

Jean-Claude Yakoubsohn

Journées Nationales de Calcul Formel

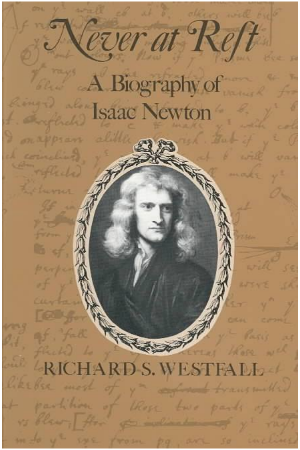
Luminy, 22-26 Janvier 2018

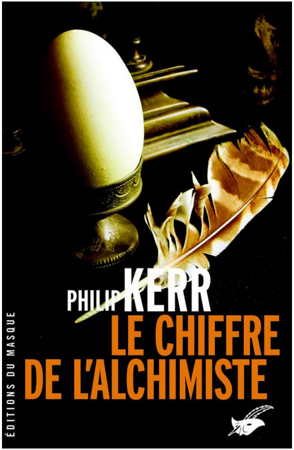
Contents

Dédié à Jean-Pierre Dedieu

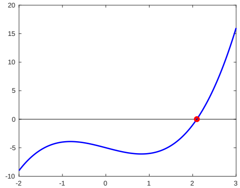
Partie 0. Bref historique de la méthode de Newton

Biographie et polar

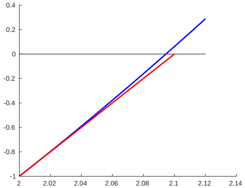


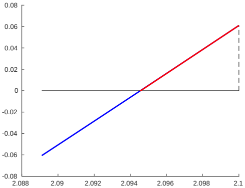


C'est magique!



$y^3 - 2y - 5$





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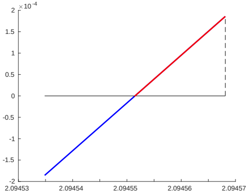
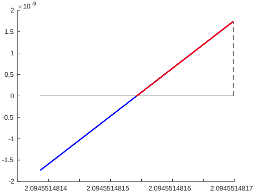




Newton $f'(x)(x-y)+f(x)=0$

Table I. Newton's solution of cubic equation $y^3-2y-5=0$ (from *De analysi per aequationes infinitas* (1669), reproduced from *The mathematical papers of Isaac Newton*, vol. 11, p. 218)

Numeralis æquationum affectarum resolutio.

EXEMPLA PER RESOLUTIONEM ÆQUATIONUM AFFECTARUM. Quia tota difficultas in Resolutione latet, modū quo ego utor in æquatione numerali primùm illustrabo.(45)

Sit $y^3-2y-5=0$ resolvenda: Et sit 2 numerus qui minùs quàm decimâ sui parte differt a radice quæsitâ. Tum pono $2+p=y$, & substituo hunc sibi valorem in Æquationem; & inde nova prodit

$$p^3+6p^2+10p-1=0,$$

cujus radix p exquirenda est ut quotienti addatur: Nempe (neglectis

$$p^3+6p^2$$

ob parvitatem) $10p-1=0$, sive

$$p=0,1$$

prope veritatē est;(46) itaq; scribo 0, 1 in quotiente,

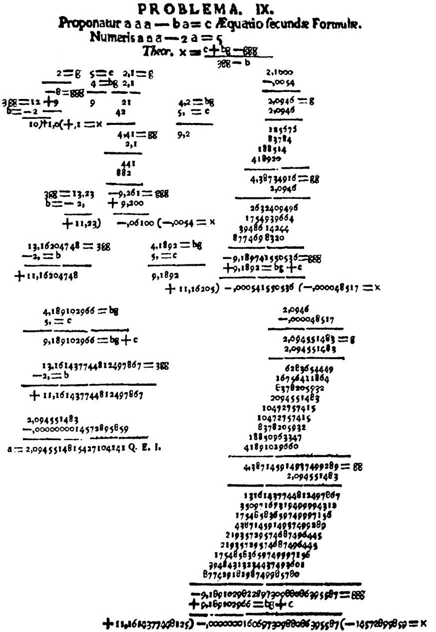
$$\begin{pmatrix} +2,10000000 \\ -0,00544853 \end{pmatrix}$$
$$2,09455147$$

$2+p=y)$	y^3	$+8+12p+6pp+p^3$
	$-2y$	$-4-2p$
	-5	-5
	Summa	$-1+10p+6p^2+p^3$
$0,1+q=p)$	$+p^3$	$+0,001+0,03q+0,3q^2+q^3$
	$+6p^2$	$+0,06\ \ +1,2\ \ +6,0$
	$+10p$	$+1,\ \ +10,$
	-1	$-1,$
	Summa	$+0,061+11,23q+6,3q^2+q^3$
$-0,0054+r=q)$	(47) $6,3q^2$	$+0,000183708-0,06804r+6,3r^2$
	$+11,23q$	$-0,060642\ \ +11,23$
	$+0,061$	$+0,061$
	Summa	$+0,000541708+11,16196r+6,3rr$

Raphson

$$y = x - \frac{f(x)}{f'(x)}$$

Aequationum . . . (1690), p. 13, courtesy of British Library, London). Note successive guesses ($2 = g$, $2{\cdot}1 = g$, $2{\cdot}0946 = g$, $2{\cdot}094551483 = g$), residuals ($0{\cdot}1 = x$, $-0{\cdot}0054 = x$, $-0{\cdot}000048517 = x$, $-0{\cdot}0000000014572895859 = x$) and final solution ($a = 2{\cdot}09455148154271 04141$)



PROBLEMA. IX.

Proponatur a a a — b a = c Æquatio secundæ Formulæ.

Numeris a a a — 2 a = 5

Theor. x = (c + bg − ggg) / (3gg − b)

2,1 = g

2,0946 = g

−,06100 (−,0054 = x

+ 11,16205) −,000541550536 (−,000048517 = x

2,094551483 = g

2,094551483

−,0000000014572895859

a = 2,09455148154271 04141 Q. E. I.

+ 11,161437744812 5) −,00000001606973098... (−14572895859 = x

Chronologie

1– Jean-Raymond Mouraille, 1768

2– Augustin-Louis Cauchy, 1829

3– Karl Weierstrass, 1870

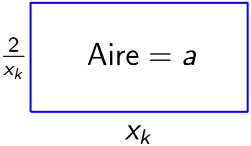
4– Carl Runge, 1898

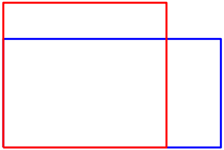
5– Leonid Kantorovich, 1939

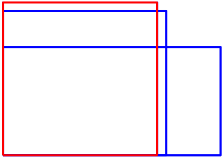
6– Steve Smale, 1980

Mais bien avant ... la méthode babbylonnienne

$$x_0 = a, \quad x_{k+1} = \frac{x_k}{2} + \frac{a}{2x_k} \quad = x_k - \frac{x_k^2 - a}{2x_k} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k \geq 0$$







Approximation de $\sqrt{2}$

$x_0 = 2$

erreur

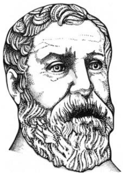
0.585786437626905

0.085786437626905

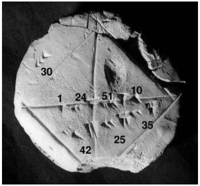
0.002453104293572

0.000002123901415

0.000000000001595



Héron d'Alexandrie, 10-75

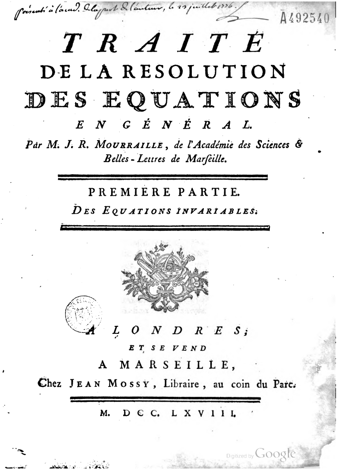


Tablette YBC 7289, 1900 av J.-C.

1– $\sqrt{2} \sim 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \frac{305470}{216000} \sim 1.41421296$

2– diagonale$=30 \times \left(1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}\right) = 42 + \frac{25}{60} + \frac{25}{60^2} \sim 42.4263889.$

Jean-Raymond Mourraille, 1720-1808



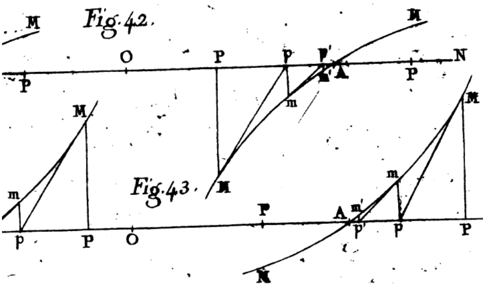
TRAITÉ
DE LA RESOLUTION
DES EQUATIONS
EN GÉNÉRAL.

Par M. J. R. MOURRAILLE, de l'Académie des Sciences & Belles-Lettres de Marſeille.

PREMIERE PARTIE.
DES EQUATIONS INVARIABLES.

A LONDRES;
ET SE VEND
A MARSEILLE,
Chez JEAN MOSSY, Libraire, au coin du Parc.

M. DCC. LXVIII.



Augustin-Louis Cauchy, 1789-1857

Note sur la détermination approximative des racines d'une équation algébrique ou transcendante, 1829

576 NOTE SUR LA RÉSOLUTION

seront affectées de signes contraires. Donc, en vertu du théorème I, l'équation (1) aura une seule racine réelle comprise entre les limites a, $a+2i$.

THÉORÈME III. — *Concevons que, la quantité i étant déterminée par la formule (8), on pose*

$$b = a + i \tag{15}$$

et

$$j = -\frac{f(b)}{f'(b)}. \tag{16}$$

Soient d'ailleurs A *la plus petite valeur numérique que puisse acquérir la fonction $f'(x)$ entre les limites $x = a$, $x = a + 2i$, et* B *la plus grande valeur numérique que puisse acquérir entre les mêmes limites la fonction $f''(x)$. Si la valeur numérique du rapport*

$$\frac{2Bi}{A} \tag{17}$$

est inférieure à l'unité, celle de j ne surpassera pas le produit

$$\frac{B}{2A}i^2, \tag{18}$$

et l'équation (1) admettra une racine réelle comprise non seulement entre les limites a, $a+2i$, mais encore entre les limites b, $b+2j$.

Demonstration. — Si, comme on le suppose, la valeur numérique

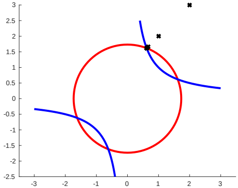
Karl Runge, 1815-1897

Méthode de Newton pour les systèmes d'équations

$$N_f(x) = x - Df(x)^{-1} f(x)$$

Intesection cercle-hyperbole

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2x_1 & 2x_2 \\ x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} x_1^2 + x^2 - 3 \\ x_1 x_2 - 1 \end{pmatrix}$$



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0.001433489714274
0.000000649225547
0.000000000000133
0.000000000000000

.

Ce qu'il faut retenir

Pour approcher une racine il faut en connaître une première approximation x_0.
Puis on calcule la suite

$$x^{k+1} = N_f(x^k), k \geq 0.$$

Si le point initial est suffisamment proche d'une racine simple, l'ordre de convergence de la suite vers la racine est quadratique.
Comment trouver le point initial x_0 ?
On utilise une méthode d'homotopie ou méthode de déformation.

$$h(x, t) = (1 - t)g(x) + tf(x)$$

Karl Weierstrass, 1815-1897

Calcul simultané de toutes les racines d'un polynôme f de degré d.

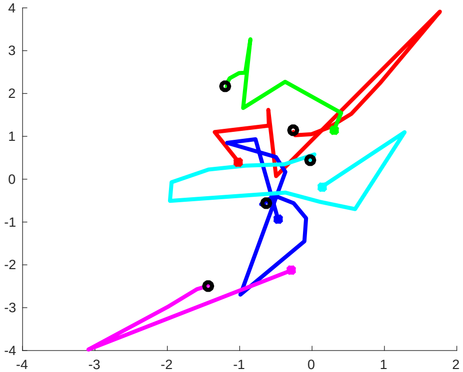
$$x = (x_1, \ldots, x_d) \in \mathbb{C}^d \to W(x) = (W_1(x), \ldots, W_d(x))$$

avec

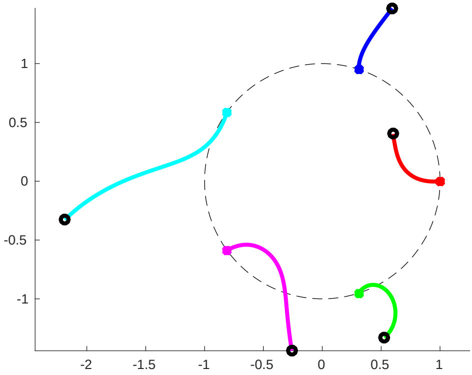
$$W_k(x) = x_k - \frac{f(x_k)}{a_d \prod_{\substack{j=1 \\ j \neq k}}^{d} (x_k - x_j)}$$

Conjecture

La suite $x^{k+1} = W(x^k)$, $k \geq 0$ converge pour presque tout $x^0 \in \mathbb{C}^d$ sauf sur un ensemble de mesure nul.



Calcul simultané et homotopie



Leonid Kantorovich, 1912-1986

Méthode de Newton dans des espaces de Banach

Soient
$K(f, x_0, r) = \sup_{||x - x_0|| \leq r} ||Df(x_0)^{-1} D^2 f(x)||$ et $\beta(f, x_0) = ||Df(x_0)^{-1} f(x_0)||$.

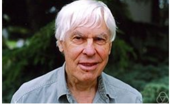
Si $2\beta(f, x_0)\, K(f, x_0, r) \leq 1$ alors il existe une unique racine de $f(x) = 0$ dans la boule $B(x_0, r)$. La suite de Newton

$$x^{k+1} = N_f(x^k), k \geq 0$$

converge vers ζ et vérifie

$$||x_k - \zeta|| \leq 1.6329 \left(\frac{1}{2}\right)^{2^k - 1} \beta(f, x_0), \quad k \geq 0.$$

Théorie α de Steve Smale, 1930-



Méthode de Newton pour des fonctions analytiques dans des espaces de Banach

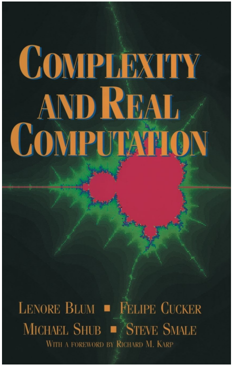
Soient $\gamma := \gamma(f, x_0) = \sup_{k\geq 2} \left(\frac{||Df(x_0)^{-1} D^k f(x_0)||}{k!} \right)^{\frac{1}{k-1}}$, $\beta := \beta(f, x_0) = ||Df(x_0)^{-1} f(x_0)||$ et $\alpha := \alpha(f, x_0) = \beta(f, x_0)\gamma(f, x_0)$. Soient

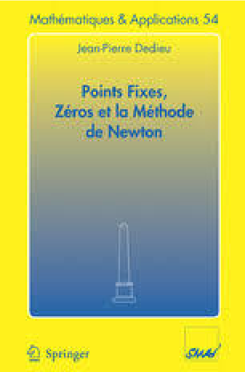
$$r_0 = \frac{1 - \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}, \qquad q = \frac{1 - \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}{1 - \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}$$

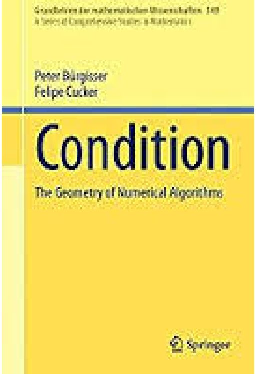
Si $\alpha \leq 3 - 2\sqrt{2}$ il existe une racine de f, $\zeta \in B(x_0, r_0)$. La suite de Newton $x^{k+1} = N_f(x^k), k \geq 0$ converge vers ζ en vérifiant pour tout $k \geq 0$:

$$||x_k - \zeta|| \leq \begin{cases} r_0 q^{2^k - 1}, \quad k \geq 0, & \text{si } \alpha < 3 - 2\sqrt{2} \\ r_0 \left(\frac{1}{2}\right)^k & \text{si } \alpha = 3 - 2\sqrt{2} \end{cases} .$$

Les sources


COMPLEXITY AND REAL COMPUTATION
LENORE BLUM ■ FELIPE CUCKER
MICHAEL SHUB ■ STEVE SMALE
WITH A FOREWORD BY RICHARD M. KARP


Mathématiques & Applications 54
Jean-Pierre Dedieu
Points Fixes, Zéros et la Méthode de Newton
Springer
SMAI

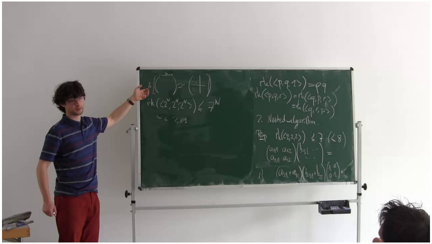

Peter Bürgisser
Felipe Cucker
Condition
The Geometry of Numerical Algorithms
Springer

Le 17$^{\text{ème}}$ de Smale, 1990

"Can a zero of *n* complex polynomial equations in *n* unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?"

Jean-Pierre Dedieu-Mike Shub

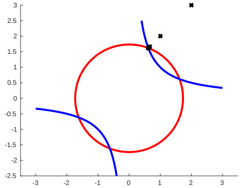
Carlos Beltrán-Luis Miguel Pardo



Résolution par Pierre Lairez, 2015.

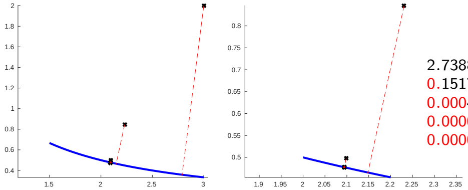
Les méthodes de Newton classiques

Cas régulier :
autant d'inconnues que d'équations
$N_f(x) = x - Df(x)^{-1} f(x)$



.

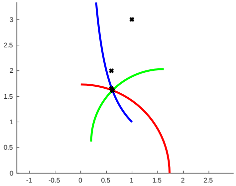
Cas surjectif :
plus d'inconnues que d'équations
$N_f(x) = x - (Df(x)^T Df(x))^{-1} Df(x)^T f(x)$



erreur
2.738874334670264
0.151710449476887
0.000454566780298
0.000000002243294
0.000000000000000

.

Cas injectif :
moins d'inconnues que d'équations
$N_f(x) = x - Df(x)^T (Df(x) Df(x)^T)^{-1} f(x)$



erreur
1.433781046743484
0.383751790557458
0.044534025609583
0.000780912646009
0.000000252646914
0.000000000000027

.

Les méthodes de Newton à découvrir

Une racine est singulière si $Df(x)$ n'est pas de rang plein.

1– Cas des racines singulières isolées.
Marc Giusti, Jean-Claude Yakoubsohn.
Numerical approximation of multiple isolated roots of analytic systems.
http://arxiv.org:443/find/all/1/au:+yakoubsohn/0/1/0/all/0/1

2– Approximation de sous-variétés de dimension positive.

Marc Giusti

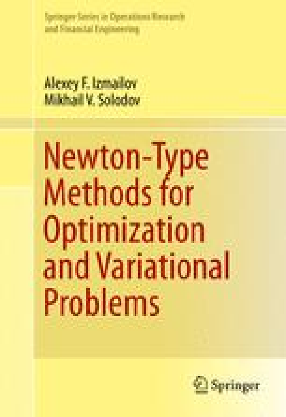
Gregorio Malajovich

Vers des algorithmes libres de ϵ

1– Détermination du rang numérique d'une matrice.

2– Décider quand $f(x_0)$ est proche ou non de zéro.

Ce dont je ne parlerai pas



Centro Nacional de Jubilación Científico

Part 1.Homotopy and singular problem : the univariate case.

Foundation

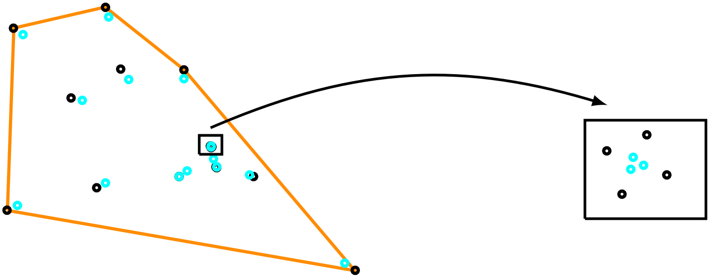
Polynomial and Fundamental theorem of Algebra

$$f(x) = a_d x^d + \ldots + a_1 x + a_0$$
$$= (x - \zeta_1)^{m_1} \ldots (x - \zeta_p)^{m_p}$$

with $m_1 + \ldots + m_p = d$.

Gauss-Lucas theorem

The roots of f' lie within the convex hull of the roots of f.



http://images.math.cnrs.fr/Si-nous-faisions-danser-les-racines

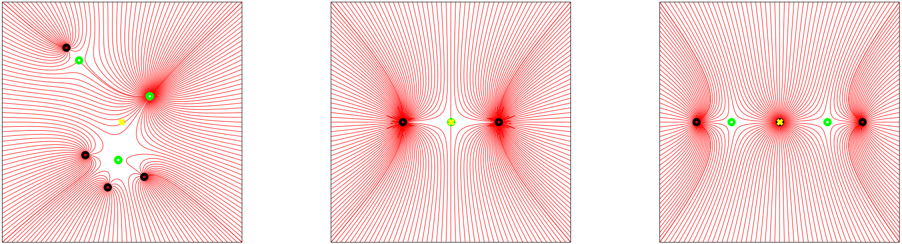
A physical interpretation

Potential. $\pm \log \overline{f(z)}$.

Field.

$$\pm \frac{\overline{f'(z)}}{\overline{f(z)}} = \pm \sum_{k=1}^{p} \frac{m_k}{\overline{z - z_k}} = \pm \sum_{k=1}^{p} \frac{m_k}{R_k} e^{i\theta_k} = \qquad \pm \left| \frac{f'(z)}{f(z)} \right|^2 \frac{f(z)}{f'(z)}.$$

with $z - z_k = R_k e^{i\theta_k}$, $k = 1 : p$.



It will not be the subject today

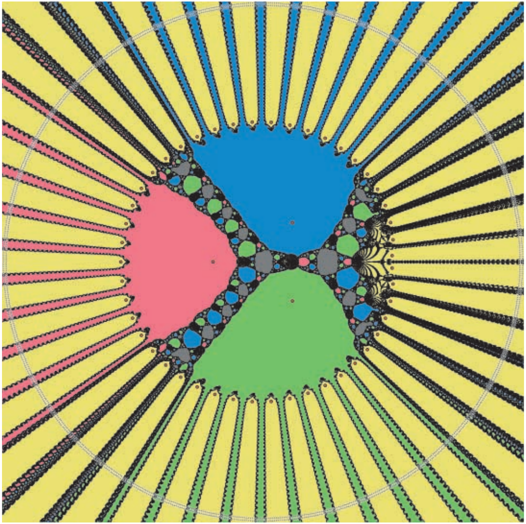


Fig. 2 A set of starting points as specified by our theorem for degree 50, indicated by small crosses distributed on two large circles. Also shown is the Julia set for the Newton map of the degree 50 polynomial $z^{50} + 8z^5 - \frac{80}{3}z^4 + 20z^3 - 2z + 1$ (*black*). There are 47 roots near the unit circle, and 3 roots well inside, all marked by *red disks*. As in Fig. 1, there is an attracting periodic orbit (*basin in gray*). A close-up is shown below.

Motivation and goal in the future

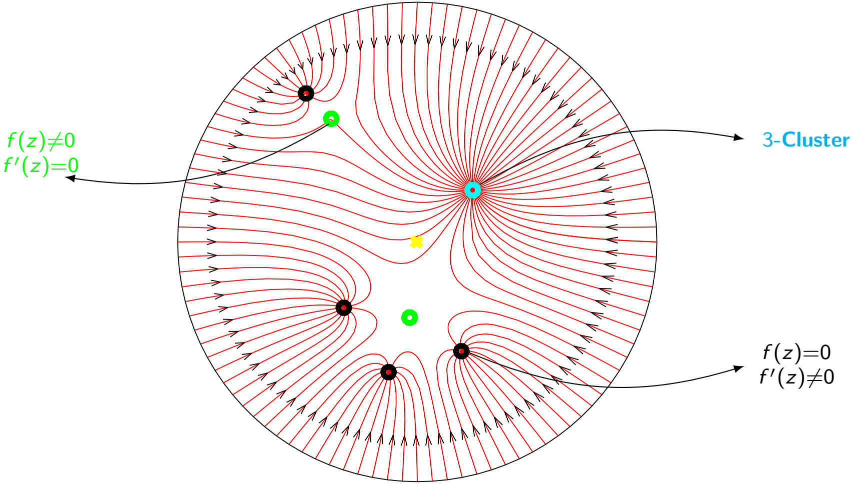
1– With Marc Giusti we have studied the problem of the approximation local of multiple roots with quadratic converence for the systems of analytic equations.

https://arxiv.org/find/all/1/all:+AND+giusti+yakoubsohn/0/1/0/all/0/1

2– The goal is to branche an method homotopy in order to find an initial point closed the multiple root. If this goal is possible then the 17 problem of Smale will be solved in a deterministic way in the general case.
The difficulty is that the homotopy curve can to encounter the singular variety, which is of dimension strictly positive,for a value of the parameter $t < 1$.

3– In the univariate case the dimension of the singular variety is equal to 0. For this reason the problen is less difficult.

Geometry of the dynamic global Newton



Notations

1– f a univariate complex polynomial of degree d.

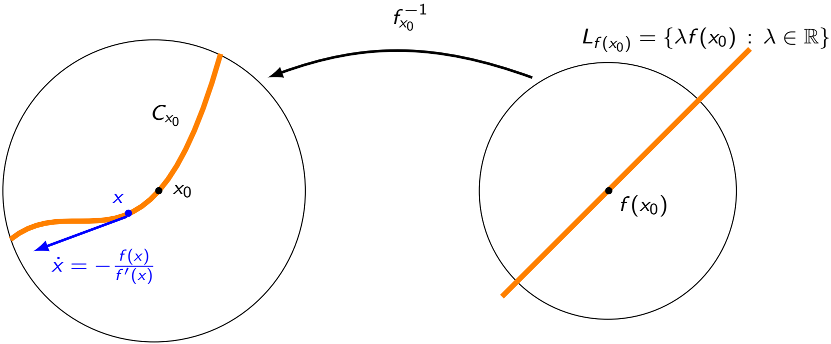
2– $N_f(x) = x - \dfrac{f(x)}{f'(x)}$.

3– $\beta(f,x) = \left|\dfrac{f(x)}{f'(x)}\right|$.

4– $\gamma(f,x)) = \max_{k=2:d} \left|\dfrac{f^{(k)}(x)}{k!f'(x)}\right|$.

5– $\alpha(f,x) = \beta(f,x)\gamma(f,x)$.

Quantitative implicit function theorem



Theorem. If $f'(x_0) \neq 0$ then f is a a diffeomorphism from the ball $B\left(x_0, \frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,x_0)}\right)$ to the ball $B\left(f(x_0), \frac{3-2\sqrt{2}}{|f'(x_0)^{-1}|\gamma(f,x_0)}\right)$.

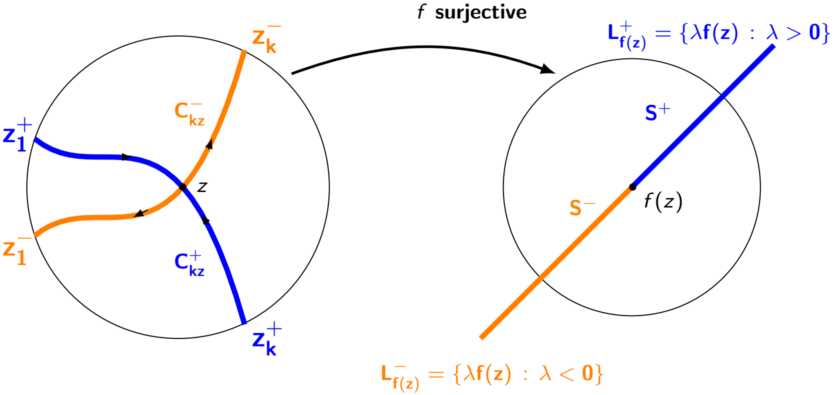
The implicit curve

Theorem. Let $x_0 \in \mathbb{C}$ s.t. $f'(x_0) \neq 0$. There exists a curve $C_{x_0} \subset B\left(x_0, \frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,x_0)}\right)$ solution of the Cauchy problem :

$$\dot{x} = -\frac{f(x)}{f'(x)}, \quad x(0) = x_0.$$

Geometry of the singular points : $f(x) = f(z) + (x-z)^m g(z)$

$f(z) \neq 0,\ f'(z) = 0$



Proposition. For $k = 1 : m$ we have :

1– $|f(z_k^+)| \geq |f(z)|$.

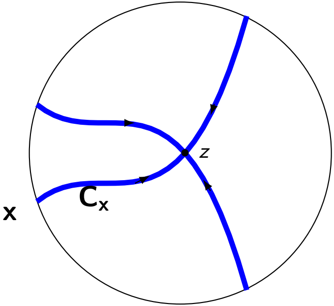
2– $|f(z_k^-)| \leq |f(z)|$.

Proposition. For $k = 1 : m$ we have :

1– $f(C_{kz}^+) = S^+$ and the curves C_{kz}^+ point in.

2– $f(C_{kz}^-) = S^-$ and the curves C_{kz}^- point out.

Geometry of a root (regular or singular) : $f(x) = (x - z)^m g(z)$



Proposition. The curves C_x point in.

Level curves

Proposition. The level curves of $x \to |f(x)|^2$ are orthogonal to the curves C_x.

Proof. In fact $\nabla|f(x)|^2 = 2f(x)\overline{f'(x)} = 2|f'(x)|^2 \dfrac{f(x)}{f'(x)}$.

Consequently the vector $-\nabla|f(x)|^2$ is tangent to C_x.

The conclusion follows. □

Behaviour of curves C_x near a root

Near a root ζ we have $f(x) = (x-\zeta)^m g_m(x-\zeta)$ for $m \geq 1$.

With $h_m(y) = mg_m(y) + (x-\zeta)g'_m(y)$ we obtain:

$$\begin{aligned}
\langle \nabla |f(x)|^2, x-\zeta \rangle &= \langle 2f(x)\overline{f'x)}, x-\zeta \rangle \\
&= Re\left((x-\zeta)^m g_m(x-\zeta) \left(\overline{(x-\zeta)^{m-1} h_m(x-\zeta)}\right) \overline{(x-\zeta)} \right) \\
&= Re\left(|x-\zeta|^{2m} g_m(x-\zeta)\overline{h_m(x-\zeta)} \right).
\end{aligned}$$

But $g_m(x-\zeta)\overline{h_m(x-\zeta)} = m|g_m(x-\zeta)|^2 + g_m(x-\zeta)\overline{(x-\zeta)g'_m(x-\zeta)}$.

We deduce for x closed to ζ that $\langle \nabla |f(x)|^2, x-\zeta \rangle > 0$.

Hence $\langle -\frac{f(x)}{f'(x)}, \zeta - x \rangle > 0$.

Consequently if the orientation of the curveC_x is given by $-\dfrac{f(x)}{f'(x)}$, the curve C_x point in ζ.

The monodromy for the dummies.
Behaviour of curves C_x near a singularity : $f(z) \neq 0,\ f'(z) \neq 0$.

There exists m such that

$$f(x) = f(z) + (x-z)^m g_m(x-z)$$

with $g_m(y) = \sum_{k \geq m} a_k y^{m-k}$, $a_k = \frac{f^{(k)}(z)}{k!}$ and $a_m \neq 0$.

Let $r < \dfrac{1-2^{-1/m}}{\gamma_m}$, $x_0^+ = z + re^{i\theta_0}$ and $x_0^- = z + re^{i\left(\theta_0 + \frac{\pi}{m}\right)}$ be such that

$$(x_0^+ - z)^m a_m = \lambda f(z) \quad \text{and} \quad (x_0^- - z)^m a_m = -\lambda f(z)$$

Then $\lambda = r^m |a_m / f(z)|$ and $\theta_0 = \frac{1}{m} Arg\left(\frac{f(z)}{a_m}\right)$.

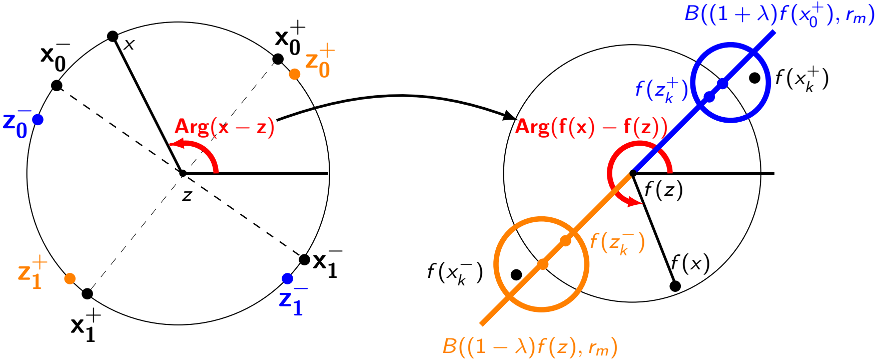
Let the m complex numbers $x_k^{\pm}$, $k = 0 : m$ defined by $x_k^{\pm} = z + (x_0^{\pm} - z)e^{\frac{2ik\pi}{m}}$, $k = 0 : m-1$. We can prove that

$$|f(x_k^{\pm}) - f(z) - (x_0^{\pm} - z)^m a_m| \leq r_m := \frac{|a_m| \gamma_m r^m}{1 - \gamma_m r}, \quad k = 0 : m-1. \tag{1}$$

This implies that the $f(x_k)'$s are in the ball

$$f(x_k^{\pm}) \in B\,(\,f(z)(1 \pm \lambda),\ r_m\,) \quad k = 0 : m-1.$$

Geometry of the singular points : $f(x) = \underset{\neq 0}{f(z)} + (x-z)^m g(z)$



The monodromy for the dummies.
Behaviour of curves C_x near a singularity : $f(z) \neq 0,\ f'(z) \neq 0$.

Lemma Let $r \leq \dfrac{1 - 2^{-1/m}}{\gamma_m}$.
The map $arg(x - z) \to arg(f(x) - f(z))$ increases for all $x \in B(z, r)$.

Proof. We have $arg(f(x) - f(z)) = m\, arg(x - z) + arg(g_m(x - z))$. Let $x - z = re^{i\theta}$ and $G(\theta) = m\theta + arg(g_m(re^{i\theta})$. Let us differentiate with respect to θ. Remember that $arg(z) = Im \log z)$. Then we can prove :

$$\begin{aligned} G'(\theta) &= m + Im \frac{g_m'(re^{i\theta})}{g_m(re^{i\theta})} rie^{i\theta} \\ &\geq m - \left| \frac{g_m'(re^{i\theta})}{g_m(re^{i\theta})} \right| r \\ &\geq m - \frac{r}{(1 - \gamma_m r)(1 - 2\gamma r)} \\ &> m - \frac{(2 + \sqrt{2})(1 - \sqrt{2}/2)}{\gamma_m} \\ &> m - \frac{1}{\gamma_m} > 0. \end{aligned}$$

Next we can Hence for r sufficiently small one has $G'(\theta) > 0$. We deduce that the function $G(\theta)$ increases.

The monodromy for the dummies.
Behaviour of curves C_x near a singularity : $f(z) \neq 0$, $f'(z) \neq 0$.

Then there exists m complex numbers $z_k^+ \in S(z,r)$, $k = 0 : m-1$ such that $f(z_k^+) \in L^+_{f(z)} = \{\mu f(z) : \mu > 1\}$.

Similarly there exists m complex numbers $z_k^- \in S(z,r)$, $k = 0 : m-1$ such that $f(z_k^-) \in L^-_{f(z)} = \{\mu f(z), : \mu < 1\}$.
By varying r, toward 0 we conclude that the set $f^{-1}(L^+_{f(z)})$ (respectively $f^{-1}(L^-_{f(z)})$) is constituted of m curves C^+_{kz} (respectively C^-_{kz}).

Behaviour of curves C_x near a singularity : $f(z) \neq 0$, $f'(z) \neq 0$.

Proposition.

1– $|f(z_k^+)| \geq |f(z)|$.

2– $|f(z_k^-)| \leq |f(z)|$.

Proof. By definition of z_k^+ we have

$$\begin{aligned} |f(z_k^+)| =& |f(z)(1+\lambda) + (z_k^+ - z)^{m+1} g_{m+1}(z_k^+ - z)| \\ & \geq |f(z)(1+\lambda)| - |(z_k^+ - z)^{m+1} g_{m+1}(z_k^+ - z)| \end{aligned}$$

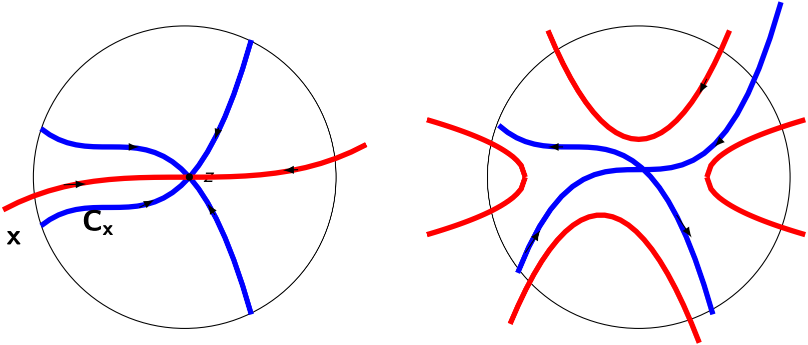
Hence for x closed to z we obtain : $|f(z_k^+)| \geq |f(z)$.

Similarly by definition of z_k^- we have :

$$\begin{aligned} |f(z_k^-)| &= |f(z)(1-\lambda) + (z_k^- - z)^{m+1} g_{m+1}(z_k^- - z)| \\ & \leq |f(z)(1-\lambda)| + |(z_k^- - z)^{m+1} g_{m+1}(z_k^- - z)| \end{aligned}$$

Hence $|f(z_k^-)| \leq |f(z)$.

Conclusion : behaviour of the curves near a root and/or a singularity



Case of a root

Case of a singularity

Local convergence : regular case

Theorem. Let $u \in [0, 1/2]$. For all $t \in]0, 1[$ such that

$$(1-t)\alpha(f, x_0) < u - \frac{u^2}{1-u} = \frac{u(1-2u)}{1-u}$$

and

$$(1-t)\alpha(f, x_0) < 3 - 2\sqrt{2} \sim 0.17,$$

the polynomial $h(t, x) = f(x) - tf(x_0)$ has only one root in the ball $B\left(x_0, \frac{u}{\gamma(f, x_0)}\right)$.
Moreover the Newton sequence

$$x_{k+1} = N_{h(.,t)}(x_k), \quad k \geq 0,$$

converges quadratically towards the root $\zeta \in B\left(x_0, \frac{u}{\gamma(f, x_0)}\right)$ of $h(x, t)$

Particularly, this assertion holds in the ball $B\left(x_0, \frac{1-\sqrt{2}/2}{\gamma(f, x_0)}\right)$.

Cluster of roots

Definition. A cluster of m roots of f is a ball, denoted by $cl_m(z,r)$, containing m roots.

α_m, β_m, γ_m.

1– $$\beta_m(f,x) = \max_{0\leq k\leq m-1}\left|\frac{m!f^{(k)}(x)}{k!f^{(m)}(x)}\right|^{\frac{1}{m-k}}.$$

2– $$\gamma_m(f,x)) = \max_{k\geq m+1}\left|\frac{f^{(k)}(x)}{k!f'(x)}\right|^{\frac{1}{k-m}}.$$

3– $$\alpha_m(f,x) = \beta_m(f,x)\gamma_m(f,x).$$

Found. Comput. Math. 257–311 (2005)

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FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

On Location and Approximation of Clusters of Zeros of Analytic Functions

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Abstract. At the beginning of the 1980s, M. Shub and S. Smale developed a quantitative analysis of Newton's method for multivariate analytic maps. In particular, their α-theory gives an effective criterion that ensures safe convergence to a simple isolated zero. This criterion requires only information concerning the map at the

Existence of a cluster of roots.

Theorem. Let $m \geq 1$ and $x_0 \in C$. If for k, $0 \leq k \leq m-1$, one has

$$\frac{(m-k)(m+1)}{m(m+1-k)}\alpha_m(f, x0) \leq \frac{1}{9}$$

then the polynomial $f^{(k)}(x)$ has $m-k$ roots each one counted with its multiplicty in the ball $B\left(x_0, \frac{u}{\gamma_m(f, x_0)}\right)$ where $u \in]u_1, u_2[$ and u_1, u_2 are the roots of

$$2u^2 - \left(3\frac{m-k}{m}\alpha_m(f, x_0) + \frac{m+1-k}{m+1}\right)u + 2\frac{(m+1-k)(m-k)}{m(m+1)}\alpha_m(f, x_0).$$

Moreover

$$\frac{2(m-k)}{m}\alpha_m(f, x_0) < u_1 < \frac{3(m-k)}{m}\alpha_m(f, x_0) < \frac{m+1-k}{3(m+1)} < u_2 < \frac{m+1-k}{2(m+1)}.$$

Separation number and diameter of Cluster of roots

Let a cluster $cl_m(\zeta, r)$ of diameter D.

1–

$$sep(f, \zeta) \leq \frac{1}{2\gamma_m(f, \zeta)} - \frac{3}{2}\beta(f, \zeta).$$

2– If $12\alpha_m(f, \zeta) < 1$ then

$$\frac{D}{3} \leq \beta_m(f, \zeta) \leq 24m^2 D.$$

Behaviour of one iteration of a Schröder iteration

$$x_1 = S_{f,m}(x_0) := x_0 - m\frac{f(x_0)}{f'(x_0)}$$

Lemma. Let $m \geq 1$ and $9\alpha_m(f, x_0) \leq 1$. Let $\zeta \in B(x_0, 3\beta_m(f, x_0))$

and $u = \gamma_m(f, \zeta)|x_0 - \zeta|$ such that $f^{(m)}(\zeta) \neq 0$ satisfying

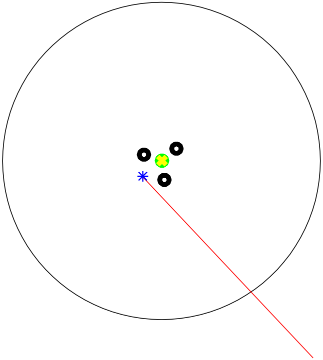
$$\alpha_m(f, \zeta) \leq u^2$$

Let $\psi(u) = 2(1-u)^2 - 1$. Then

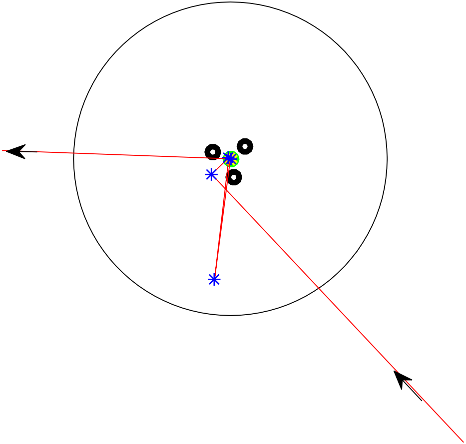
$$|x_0 - \zeta| \leq \frac{\gamma_m(f,\zeta)}{m\psi(u)}|x_0 - \zeta|^2$$

Question : How to choose ζ to get a practical algorithm ?

How to control the approximation of a cluster?



Behaviour of the Schröder iteration



How to control the approximation of a cluster?

Let :

1– $\psi_m(u) = 2(1-u)^{m+1} - 1$ et $\psi(u) := \psi_1(u)$.

2– $e(u) = \dfrac{1}{(1-u)\psi_m(u)}$.

3– $q(u) = \dfrac{e(u)}{m\psi(ue(u))}$.

It is based on the point estimate

$$\gamma_m(f,z) \leq \frac{\gamma_m(f,x)}{\psi(u)(1-u)}$$

with $u = \gamma_m(f,x)\,|x-z|$.

How to control the approximation of a cluster?

Theorem. (part 1)
Let $m \geq 1$ and $9\alpha_m(f, x_0) < 1$..
We let $r = \max\left(\left|\frac{mf(x_0)}{f'(x_0)}\right|, \quad \left|\frac{f^{(m-1]}(x_0)}{f^{(m)}(x_0)}\right|\right)$. The Newton sequence

$$z_{-p} = x_0, \qquad z_{k+1} = N_{f^{(m-1)}}(z_k), \qquad k \geq -p$$

converges towars the unique root of $f^{(m-1)}$ in the ball $B\left(x_0, \frac{3}{m}\beta_m(f, x_0)\right)$. One has

$$|z_k - z_{k-1}| \leq 2^{1-2^{k+p-1}}|z_0 - x_0|.$$

How to control the approximation of a cluster?

Theorem. (part 2)
The Schröder sequence defined by

$$x_{k+1} = S_{f,m}(x_k) := x_k - m\frac{f(x_k)}{f'(x_k)}$$

satisfies

$$|x_k - z_{k-1}| \leq 2^{1-2^k} r,$$

for all $k \geq 0$ such that

$$q(u_k, m, k) := \frac{2(1 + 2^{2^k - 2^{k+p-1}})(c_k + 1)\gamma_m(f, z_k)r}{m\psi(u_k)} \leq 1$$

with

$$u_k := \gamma(f, z_k)|x_k - z_k|, \qquad c_k := \frac{\alpha_m(f, z_k)}{u_k^2}, \qquad r = \max(|z_0 - x_0|, |x_1 - x_0|).$$

How to control the approximation of a cluster?

Theorem. (part 3)
Let us suppose that the diameter D of the cluster satisfies

$$D < \frac{r}{3m} < 1.$$

Let us also suppose there exists an integer ℓ such that

$$12\alpha_m(f, z_\ell) \leq 1.$$

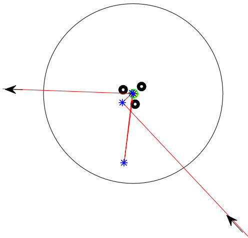
Let us consider the integer K defined by

$$K = \left\lceil \frac{1}{\log 2} \log\left(\frac{\log D^{-1}}{\log 2}\right) \right\rceil.$$

Then

$$q(u_K, m, K) > 1.$$

How to control the approximation of a cluster?



$D = 10^{-N}$, $m = 3$.

N \ it	1	2	3	4	5
4	$-.95$				
8	$-.95$	$-.96$			
16	$-.95$	$-.96$	$-.97$		
32	$-.95$	$-.96$	$-.97$	$-.97$	
64	$-.95$	$-.96$	$-.97$	$-.97$	$-.97$

Behaviour of $q(u_k, m, z_k)$

N \ it	1	2	3	4	5
4	0.03				
8	0.03	0.01			
16	0.03	0.06	0.06		
32	0.03	0.06	10^{-4}	10^{-4}	
64	0.03	0.06	0.0002	10^{-7}	10^{-4}

Behaviour of c_k

Global Newton homotopy algorithm

Let $h(x,t) = f(x) - tf(x_0)$.

1- Input : $it \geq 1$, $x_0 \in \mathbb{C}$ s.t. $f'(x_0) \neq 0$, $k = 0$.
Step k

2- Compute $E_k = \{p \geq 1 \,:\, \alpha_p(x_k) < \alpha_0\}$, where $\begin{matrix} \alpha_0 = 3 - 2\sqrt{2} & \\ \alpha_0 = 1/9 & \text{for } p > 1 \end{matrix}$

3- If $E_k = \emptyset$ then (hence $\alpha(f, x_k) > 3 - 2\sqrt{2}$)

3- $t_k = 1 - \dfrac{3 - 2\sqrt{2}}{\alpha(f, x_k)}$, $z_0 = x_k$

4- for $j = 0 : it$ $z_{j+1} = N_{h(.,t_k)}(z_j)$ end

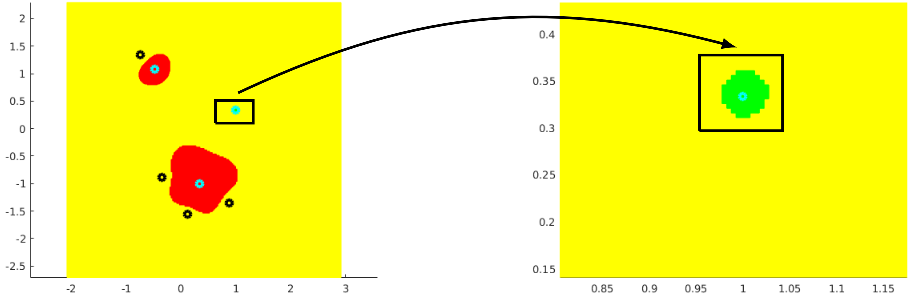
5- $x_{k+1} = z_{it}$, $k = k + 1$

6- else $m = \min E_k$, break

7- fi

8- Output : x_K an approximate point of a m cluster of f.

Values of $\alpha(f,x)$



red : $\alpha(f,x) \geq 1$ yellow : $\alpha(f,x) < 1$ green : $\alpha_m(f,x) < 1/9$

$$\alpha(x^d, x_0) = \left|\frac{x^d}{dx^{d-1}}\right| \max^{k=2:d} \left|\frac{d(d-1)\dots(d-k+1)x^{d-k}}{k!dx^{d-1}}\right|^{\frac{1}{k-1}}$$
$$= \frac{d-1}{2d} < \frac{1}{2}$$

Number of steps of the global Newton homotopy method

Theorem. Let a curve C_{x_0} solution of the Cauchy problem

$$\dot{x} = -\frac{f(x)}{f'(x)}, \quad x(0) = x_0.$$

The number of steps of the global Newton method to obtain an approximate zero z of a m-cluster $cl_m(\zeta)$ is bounded by

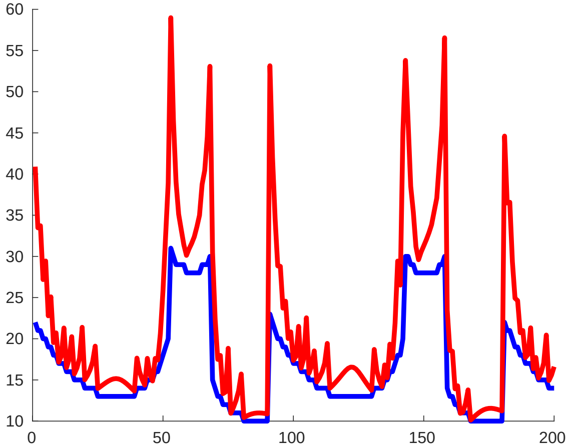
$$\left\lceil \frac{\log \dfrac{\alpha_0^m \; |f^{(m)}(z)|}{m! \; \gamma_m(z) \; |f(x_0)|}}{\log \left(1 - \dfrac{3 - 2\sqrt{2}}{\alpha}\right)} \right\rceil$$

where

$$\alpha = \sup_{x \in C_{x_0}} \{\alpha(f, x) \, : \, \alpha(f, x) \geq 3 - 2\sqrt{2}\}$$

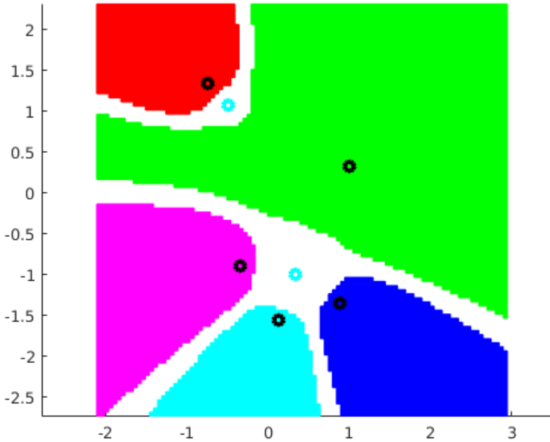
and $\alpha_0 = 3 - 2\sqrt{2}$ if $m = 1$ and $\alpha_0 = 1/9$ if $m > 1$.

Comparison between practical and theoretical number of steps



Attraction Bassins of global Newton homotopy

We represent by different colors the initial points x_0 which the curves C_{x_0} point in towards a cluster of root such that : $\alpha(f, x) < 1$ for all $x \in C_{x_0}$



an algorithm that can be applied routinely to find all roots without deflation and with the inherent numerical stability of Newton's method.

We specify an algorithm that provably terminates and finds all roots of any polynomial of arbitrary degree, provided all roots are distinct and exact computation is available. It is known that Newton's method is inherently stable, so computing errors do not accumulate; we provide an exact bound on how much numerical precision is sufficient.

A paper of Dirck Schleicher and Robin Stoll

1. Introduction

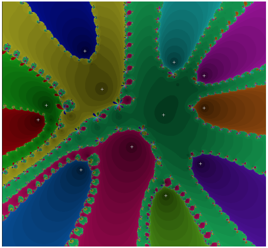


FIGURE 1. The dynamics of Newton's method for a polynomial of degree 12. Different colors indicate starting points that converge to different roots, and different shades of color indicate the speed of convergence to that root.

Finding roots of equations, especially polynomial equations, is one of the oldest tasks in mathematics; solving any equation $f(x) = g(x)$ means finding roots of $(f - g)(x)$. This task is of fundamental importance in modern computer algebra systems, as well as for geometric modelling. Newton's method, as the name indicates, is one of the oldest methods for approximating roots of smooth maps, and in many cases

1

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Part 2. Fast approximation of singular varieties

Gregorio Malajovich, LABMA

Ball of quadratic convergence of the Newton method in the regular case

$x \in \mathbb{C}^n$, $f(x) = (f_1(x), \ldots, f_n(x)) \in \mathbb{C}[x]^n$,
$degf_i = d_i$, $D = \max d_i \geq 2$.

$$N_f(x) = x - Df(x)^{-1} f(x).$$

Theorem. (From $\gamma-$Theorem of Schub-Smale)
Let ζ a regular zero of f : $f(\zeta) = 0$ and $Df(\zeta)^{-1}$ exists.
Then, for all x_0 verifying

$$\|x_0 - \zeta\| < \frac{5 - \sqrt{17}}{4 \, \frac{D^2}{2} \, \max(1, ||f||) \quad (1 + \|\zeta\|^2)^{\frac{D-2}{2}} \quad \|Df(\zeta)^{-1}\|}$$

the Newton sequence

$$x_{k+1} = N_f(x_k), k \geq 0,$$

converges quadratically towards ζ.

More on the ball of quadratic convergence of the Newton method in the regular case

$x \in \mathbb{C}^n,\ f(x) = (f_1(x), \dots, f_n(x)) \in \mathbb{C}[x]^n,$

$deg f_i = d_i,\ D = \max d_i \geq 2.$

$$\Delta(x) = \begin{bmatrix} \frac{1}{||x||^{d_1-1}} & & \\ & \ddots & \\ & & \frac{1}{||x||^{d_n-1}} \end{bmatrix}.$$

$$N_f(x) = x - (\Delta(x) Df(x))^{-1} \Delta(x) f(x).$$

Theorem. (From $\gamma-$Theorem of Schub-Smale)
Let ζ a regular zero of f : $f(\zeta) = 0$ and $Df(\zeta)^{-1}$ exists.
Then, for all x_0 vérifying

$$\|x_0 - \zeta\| \leq \frac{\left(\sqrt{2} + 2 - \sqrt{2\sqrt{2} + 4}\right)(\sqrt{2} - 1)}{D^{3/2}\, ||f||\, \|\zeta\|\, \|\, (\Delta(\zeta) Df(\zeta))^{-1}\, \|}$$

the Newton sequence

$$x_{k+1} = N_f(x_k), k \geq 0,$$

converges quadratically towards ζ.

The well known cases

$x \in \mathbb{C}^n$, $f(x) = (f_1(x), \ldots, f_m(x)) \in \mathbb{C}[x]^m$,

1– The regular case : $Df(x)$ invertible in a ball.

2– The surjective case : fewer equations than unknowns and $Df(x)$ has full rank in a ball.

3– The injective case : more equations than unknowns and $Df(x)$ has full rank in a ball.

4– The case where $Df(x)$ has constant rank $r \le n$: then $f^{-1}(0)$ is an analytic sub-variety of dimension $n - r$.

In these four cases the following operator

$$N_f(x) = x - Df(x)^{\dagger} f(x)$$

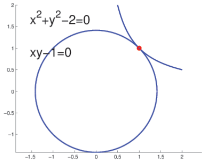
is well defined .

There are α-theorems (existence) and γ-theorems (behaviour of Newton iteration).

Case where $Df(x)$ has not a constant rank

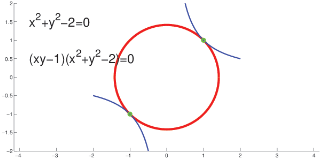
For instance :

1– If $f^{-1}(0)$ has an isolated multiple root .



In this case the multiplicity is the dimension of the local algebra.

2– If $f^{-1}(0)$ is a variety of positive dimension with drop in rank.



Low rank approximation (lra) Newton method

We consider $f \in \mathcal{H}_{n+1;(d_1,\dots,d_m)}$ with $D = \max_i d_i \geq 2$.

If u is a vector, $\|u\|$ is the Euclidean norm.

If A is a matrix, $\|A\|$ is the operator 2-norm.

If $f \in \mathcal{H}_{n+1;(d_1,\dots,d_m)}$, $\|f\|$ is Weyl's invariant norm.

We define

$$d_{\mathbb{P}}(x, y) = \min_{\lambda \in \mathbb{C}} \left(\frac{\|x - \lambda y\|}{\|x\|} \right)$$

the projective distance, and

$$d_T(x, y) = \min_{\substack{\lambda \in \mathbb{C}, \\ \lambda y \perp x}} \left(\frac{\|x - \lambda y\|}{\|x\|} \right) = \tan\left(\arcsin(d_{\mathbb{P}}(x, y))\right)$$

which does not satisfy the triangular inequality.

Low rank approximation (lra) Newton method

Definition

Let B be a complex $m \times n$ matrix. Let $\epsilon > 0$ be a fixed real number. The low rank approximation of B is defined by

$$\mathrm{lra}(B) = \mathrm{lra}_\epsilon(B) = U diag(\sigma_1, \ldots, \sigma_r, 0_{\min(n,m)-r})\, V^H$$

where

$$B = U\, diag(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{\min(n,m)-r})\, V^H$$

is the singular value decomposition of B with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \epsilon\sigma_1 \geq \sigma_{r+1} \geq \cdots \geq \sigma_{\min(m,n)}.$$

Low rank approximation (lra) Newton method

Let $f \in \mathcal{H}_{n+1;(d_1,\ldots,d_m)}$. Define

$$\Delta(x) = \begin{bmatrix} \frac{1}{\sqrt{d_1}\,\|x\|^{d_1-1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{d_m}\,\|x\|^{d_m-1}} \end{bmatrix}$$

Definition The low rank approximation Newton operator in $\mathbb{C}^{n+1} \setminus \{0\}$ associated to f is defined by

$$N_{\mathrm{lra}}(f,x) = x - \left(\mathrm{lra}\left(\Delta(x)\, Df(x)\right)\right)^{\dagger} \Delta(x)\, f(x).$$

Lemma
The operator $N_{\mathrm{lra}} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$ induces a well-defined mapping $N_{\mathrm{lra}} : \mathbb{P}^n \to \mathbb{P}^n$, i.e.,

$$\forall \lambda \neq 0 \in \mathbb{C}, \quad N_{\mathrm{lra}}(f, \lambda x) = \lambda N_{\mathrm{lra}}(f,x).$$

$\gamma-$theorem for lra Newton method

We note

$$\mu(f,\zeta) = \|f\| \left\| \left(\Delta(\zeta) Df(\zeta)\right)^{\dagger} \right\|$$

When $Df(\zeta)$ is surjective, this is the Shub and Smale condition number.

Theorem

There is a constant $u_0 \le \frac{1}{28\sqrt{2}} \left(9\sqrt{2} + 140 - \sqrt{19762 + 952\sqrt{2}}\right) \sim 0.18791...$ with the following property :

Let $Z(f)$ the set of zeros of $f \in \mathcal{H}_{n+1;(d_1,\ldots,d_m)}$ such that $\mathcal{Z} \subseteq Z(f)$ where $\mathcal{Z}$ is a projective variety of codimension r with the following properties:

1– $\mathrm{rank} Df(\zeta) = r$ for all $\zeta \in \mathcal{Z}$

2–

$$\mu = \mu(f, \mathcal{Z}) = \sup_{\zeta \in \mathcal{Z}} \mu(f,\zeta) < \infty.$$

Let us consider $x_0 \in \mathbb{C}^n$ such that

$$\mathrm{rank}(\mathrm{lra}(A)) = r \qquad \text{with} \qquad A = \Delta(x_0) Df(x_0).$$

$\gamma-$theorem for lra Newton method

Theorem (conclusion)
Assume that the parameter ϵ of the lra satisfies

$$\sqrt{D}\mu\epsilon < \frac{1}{2} \qquad \text{and} \qquad \tilde{u}_0 \leq \min\left(u_0,\ \frac{\epsilon\mu\|A\|}{(\frac{\sqrt{2}}{4}\mu\|A\| + e_1)\epsilon + e_1}\right),$$

with $e_1 = 1 + \frac{\sqrt{2}}{42} \sim 1.0336...$.
If $x_0 \in \mathbb{C}^{n+1}$ satisfies

$$D^{3/2}\mu\, d_T(x_0, \mathcal{Z}) \leq \tilde{u}_0,$$

then the sequence $x_{i+1} = N_{\text{lra}}(f, x_i)$ converges and satisfies

$$d_T(x_i, \mathcal{Z}) \leq 2^{-2^i+1} d_T(x_0, \mathcal{Z}).$$

Idea of the proof

1– Notations:
$A = lra(Delta(\zeta)Df(\zeta)), \quad B = \Delta(x)Df(x), \quad C = \mathrm{lra}(B), \qquad s = D\frac{\|x-\zeta\|}{\|\zeta\|}.$

2– We know that :ζ minimizes $\|x-\zeta\|/\|x\|$. Hence $x - \zeta \perp T_\zeta \mathcal{Z} = \ker Df(\zeta)$.

3– We write :

$$\begin{aligned} y - \zeta &= x - \zeta - \mathrm{lra}\left(\Delta(x)\, Df(x)\right)^{\dagger} \Delta(x)\, f(x) \\ &= C^{\dagger} \Delta(x)\, \Delta(\zeta)^{-1} \left(\Delta(\zeta)\, Df(x)\, (x-\zeta) - \Delta(\zeta)\, f(x)\right) + P_{\ker(C)}\, P_{\Im(A^H)}(x-\zeta) \end{aligned}$$

4– This gives the following point estimate using the useful lemma above

$$\|y-\zeta\| \leq \frac{1}{1-\delta\|A\|\mu} \left(1 + \frac{\|x-\zeta\|}{\|\zeta\|}\frac{s}{1+s}\right) \|A^{\dagger}\| D^{3/2} \frac{\|x-\zeta\|}{\|\zeta\|} \|x-\zeta\| + \frac{s\|A\|\|A^{\dagger}\|}{1-s\|A\|\|A^{\dagger}\|} \|x -$$

5– We bound $\|A\| \leq \sqrt{D}, \quad \|A^{\dagger}\| \leq \mu,$ and $s = \leq \frac{\tilde{u}}{\sqrt{D}}$.

6– Therefore with $\tilde{u} = \sqrt{D}\mu\delta$, and assuming $D \geq 2$, we finally get

$$\|y-\zeta\| \leq \frac{\tilde{u}+e(\tilde{u})}{1-e(\tilde{u})}\|x-\zeta\| \qquad \text{with } e(u) = \frac{ue1}{1-\frac{u}{2\sqrt{2}}}.$$

An useful lemma

Lemma

Let A be an $m \times n$ complex matrix of rank r. Let B be a general $m \times n$ matrix with $\|B - A\| \leq \delta\|A\|$ and let $C = \text{lra}(B)$. If

$$0 \leq \delta < \min\left(\frac{\epsilon}{1+\epsilon}, \frac{1 - \epsilon\|A\|\|A^{\dagger}\|}{(1+\epsilon)\|A\|\|A^{\dagger}\|}\right),$$

Then,

1– $\mathrm{rank}(C) = r$,

2– $\|B - C\| \leq \delta\|A\|$,

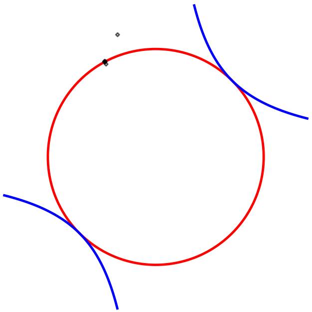
3– $\|C^{\dagger}\| \leq \frac{\|A^{\dagger}\|}{1-\delta\|A\|}$,

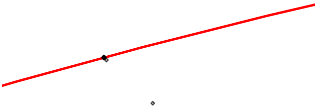
4– $\|P_{\Im(A)} \circ P_{\ker(C^H)}\| = \|P_{\Im(C)} \circ P_{\ker(A^H)}\| \leq \frac{\delta\|A\|\|A^{\dagger}\|}{1-\delta\|A\|\|A^{\dagger}\|}$.

5– $\|P_{\Im(A^H)} \circ P_{\ker(C)}\| = \|P_{\Im(C^H)} \circ P_{\ker(A)}\| \leq \frac{\delta\|A\|\|A^{\dagger}\|}{1-\delta\|A\|\|A^{\dagger}\|}$.

lra **Newton method : example**

$f = [x^2 + y^2 - 2, (x^2 + y^2 - 2)^2(xy - 1)]$, $\epsilon = 1e - 2$.





Applications

1– Fast computation of the eigenpair problem in the case of multiple eigenvalues.

2– idem for the SVD.

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Part 3. Algebraic structure of the roots of the polynomial systems.

Some symbolical-numerical methods

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5– Two papers of M. Giusti,G.Lecerf,B. Salvy, J.-C. Yak., in FOCM, 2005, 2007.

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7– J.-P. Dedieu, M. Shub, On Simple Double Zeros and Badly Conditioned Zeros of Analytic Functions of n Variables. Mathematics of Computation, 70 (2001) 319-327.

Book

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Number of roots of a polynomial system

Theorem Let I be the ideal $< f_1, \ldots, f_m >$ and $V(I) \subset \mathbb{C}^n$ the associated variety.

1– The dimension of $C[x]/I$ is finite iff the dimension of $V(I)$ is zero.

2– In the finite dimension case one has

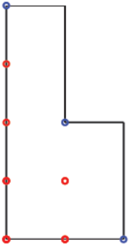
$$dim C[x]/I = \#V(I)$$

where $\#V(I)$ is the number of points of $V(I)$ counted with multiplicities.

Example

$f_1(x,y) = x^2 + x^3, \quad f_2(x,y) = x^3 + y^2$ and $V(I) = \{(0,0),(-1,1)\}$.
A Groebner basis of I is :
$g_1(x,y) = x^0y^4 - y^2, \quad g_2(x,y) = x^1y^2 + y^2, \quad g_3(x,y) = x^2y^0 - y^2,$
We deduce

$$dimC[x]/I = 6$$



Multiplicities and local rings

Let w an isolated root of $f = (f_1, \ldots, f_m)$ and $I =< f_1, \ldots f_m >$.
Let $\mathbb{C}\{x - w\}$ the local ring of convergent power series in n variables which the maximal ideal is generated by $x_1 - w_1, \ldots x_n - w_n$.
Let $I\mathbb{C}\{x - w\}$ the ideal generated by I in $\mathbb{C}\{x - w\}$
We note $A_w = \mathbb{C}\{x - w\}/I\mathbb{C}\{x - w\}$.

Theorem.

Let $V(I) = \{w_1, w_2, \ldots w_N\}$. Then

1– $\mathbb{C}[x]/I \sim A_{w_1} \times \ldots \times A_{w_N}$.

2– $dim\,\mathbb{C}[x]/I = \sum_{i=1}^{N} dim\,A_{w_i}$.

We define $dim\,A_{w_i}$ as the algebraic multiplicity of the root w_i.

Multiplicities and Rouché's theorem

Let $B(w, r)$, $f(w = 0$ and m the dimension of the local ring $A_w(f)$ associated to f at w.
All polynomial system g such that

$$||f(z) - g(z)|| < ||f(z)||, \qquad \forall z \in \partial B(w, r)$$

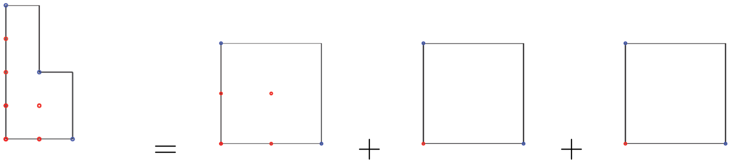
has p roots $w_1, \ldots, w_p$ in $B(w, r)$ such that

$$m = \sum_{k=1}^{p} dim\, A_{w_k}(g).$$

A proof is given p.86-99 in
Arnold, Gusein-Zade, Varchenko, Singularities of differentiable varieties

Example

$f_1(x,y) = x^2 + x^3, \quad f_2(x,y) = x^3 + y^2$ and
$V(I) = \{(0,0), (-1,1), (-1,1)\}$.
1– A standard basis of $I\mathbb{C}\{x,y\}$ is : $g_1 = x^2, \quad g_2 = y^2$.
Hence $dimA_{(0,0)} = 4$.
2– A standard basis of $I\mathbb{C}\{x+1, y-1\}$ (resp. $I\mathbb{C}\{x+1, y+1\}$) is :
$g_1 = 3x, \quad g_2 = 2y$.
Hence $dimA_{(-1,1)} = dimA_{(-1,-1)} = 1$.



Duality and Multiplicities

Let w an isolated root of $f = (f_1, \dots f_n)$. Let $x = (x_1, \dots, x_n)$, $\alpha = (\alpha_1, \dots \alpha_n)$ and

$$\partial_\alpha[w]f_i = \left.\frac{\partial^{|\alpha|} f_i(x)}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}\right|_{x=w} \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

We define

$$\mathcal{D}_w^j(f) = \{L = \sum_{|\alpha| \leq j} L_\alpha \partial_\alpha[w] \quad : \quad L(f) = 0\}$$

Theorem

1– The root w is isolated iff there exists k s.t. $\mathcal{D}_w^{k-1} = \mathcal{D}_w^k$.

2– In this case the dimension of $\mathcal{D}_w^k$ is equal to the multiplicity of w.

A way to compute the multiplicity of an isolated root: Macaulay matrices.

We consider the Macaulay matrices

$$S_k = (\partial_\alpha[w]((x-w)^\beta f_i(x))) \quad \text{for } |\beta| \leq k-1 \text{ and } i = 1 : m.$$

Theorem The multiplicity of of the isolated root w of f is

$$\mu = corank(S_{k-1}) = corank(S_k)$$

where k is the smallest index which satisfies this equality.

We name thickness this smallest index.

Ensalem, Géométrie des points épais, 1978 ,

Example

$f_1 = x^2 + y^2 - 2,\ f_2 = xy - 1.\ w = (1,1).$

		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}
S_0	f_1	0	2	2	2	0	2
	f_2	0	1	1	0	1	0
	− − −	−					
S_1							
	− − −	−	−	−			
S_2	$(x-1)f_1$	0	0	0	4	2	0
	$(x-1)f_2$	0	0	0	2	1	0
	$(y-1)f_1$	0	0	0	0	2	4
	$(y-1)f_2$	0	0	0	0	1	2

A basis of the kernel of S_2 is

$\{(1,0,0,0,0,0),\ (0,1,-1,0,0,0)\}$

Hence

$\{1, \quad \partial_1 - \partial_2\}$

is a basis for $\mathcal{D}_w^2$

$rank(S_0) = 0,\ corank(S_0) = 1$
$rank(S_1) = 1,\ corank(S_1) = 2$
$rank(S_2) = 4,\ corank(S_2) = 2.$

Hence $\mu = corank(S_1) = 2.$

Computation of a regular system

Step 1- Compute a basis for the Kernel of the Macaulay matrix S_μ. :
$\Lambda = (\Lambda_1, \ldots, \Lambda_\mu)$
Each Λ_k is such that : $\Lambda_k = \sum_\alpha A_\alpha \partial_\alpha[w]$.
Step 2- Compute the system $f^\Lambda := (\Lambda(f_1), \ldots, \Lambda(f_m))$.

Theorem The system f^Λ is regular at w.

Mantzaflaris and Mourrain, Deflation and certified isolation of singular zeros of polynomial systems, ISSAC 2011

Example.
$f_1(x, y) = x^2 + y^2 - 2,\ f_2 = xy - 1.\ w = (1, 1).$
$\Lambda = (1, \partial_1 - \partial_2)$.
$f^\Lambda = (f_1, f_2, x - y)$.

Counting multiplicities numerically

Let us consider a polynomial system

$$f(x) = (f_1(x), \ldots, f_n(x)), \quad x = (x_1, \ldots, x_n) \in \mathbb{C}^n.$$

Let $d = (d_1, \ldots, d_n)$ the vector of the degrees of $(f_1, \ldots, f_n)$.

Let the Bézout number of f : $D = d_1 d_2 \ldots d_n$.

Theorem. The number of isolated zeros of $f(x)$ is less than D.

Counting multiplicities numerically

From now we consider a polynomial system with D isolated zeros. Let us consider the homotopy function

$$h(t,x) = (1-t)g(x) + tf(x)$$

where

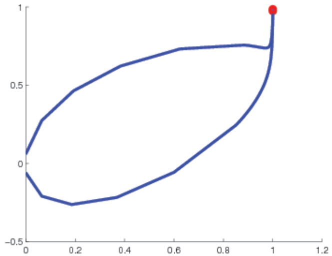
$$g(x) = (a_1^{d_1}x_1^{d_1} - b_1^{d_1}, \dots, a_n^{d_n}x_n^{d_n} - b_n^{d_n}).$$

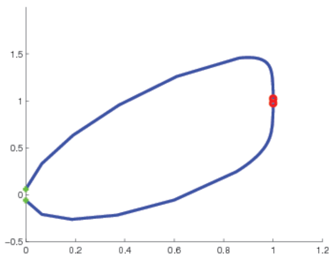
The polynomial system $g(x)$ has D regular isolated zeros

$$w_{k_1 \dots k_n} = \left(\frac{b_1}{a_1} e^{\frac{2i\pi k_1}{d_1}}, \dots \frac{b_n}{a_n} e^{\frac{2i\pi k_n}{d_n}} \right)$$

with $1 \leq k_i \leq d_i$, $i = 1 : n$.

Counting multiplicities numerically





Counting multiplicities numerically

Theorem.
For almost $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $b = (b_1, \ldots b_n) \in \mathbb{C}^n$, we have :

1– The set
$$\{t \in [0,1] \to x(t) \,:\, h(t, x(t)) = 0\}$$
is constituted of D curves $x_{k_1 \ldots k_n}$ such that $D_x h(t, x_{k_1 \ldots k_n}(t))$ have full rank for all $t \in [0, 1[$.

2– Let w a zero of f of multiplicity μ. The number of curves such that $x_{k_1 \ldots k_n}(1) = w$ is equal to μ

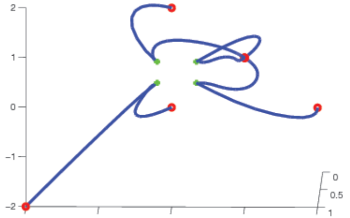
Open problem 1. Complexity for that?
Well known (Carlos Beltràn, Luis Miguel Pardo) in the regular case.
Open problem 2. Effective rouché theorem. Well known in the regular case and for the double root rank one case (Dedieu-Shub).

Counting multiplicities numerically

Let $$f_i = x_i^2 + \sum_{k=1}^{n} -2x_i - n + 1, \quad i = 1 : n$$

$(1, ..., 1)$ *multiplicity* 2^{n-1} *if n odd*

n	2	3	4	5	6	7	8	9	10	11
μ	3	4	11	16	42	64	?	256	?	1024
cpu	0.27	0.76	1.1	1.32	2.86	4.71	?	20	?	600



Case $n = 3$: projection on the two first real coordinates of the homotopy curves.

Recovering the quadratic convergence

The idea is to determine a sequence of systems which the last is regular at the multiple root of the original system.

The method of LVZ : Leykin-Vershelde-Zhao

Let r the rank of the jacobian matrix J of f at w.
The LVZ method consist to add at each step of deflation $r+1$ equations and unknowns at the initial system.

$$\begin{aligned} & f(x) = 0 \\ & J(x)B\lambda = 0 \qquad B \in \mathbb{C}^{n\times(r+1)} \textit{ random matrix} \\ & \lambda^T h - 1 = 0 \qquad h \in \mathbb{C}^{r+1} \textit{ random vector} \end{aligned}$$

The unknowns of this new system are $(x,\lambda) \in \mathbb{C}^{n+r+1}$.
The corank of the system $(J(x)B\lambda = 0, h^T\lambda - 1 = 0)$ is generalically equal to 1 and the multiplicty of the new system is less than the initial system.
We have added $m+r+1$ equations and $n+r+1$ unknowns.
The number of step of deflations to restore quadratic convergence for the Gauss-Newton method is bounded by the multiplicity of the root.

The first method of DLZ: Dayton-Li-Zeng

$$
\begin{aligned}
f(x) &= 0 \\
J(x)\lambda &= 0 \\
\lambda^T h - 1 &= 0
\end{aligned}
$$

At the end of this of this process we can to add $2^{\mu-1} \times m$ equations and $2^{\mu-1} \times n$ unknowns.

The second method of DLZ : case of breadth one.

Breadth one : if at each deflation step $\mathit{corank}(\mathit{system}) = 1$.
In this case the second method of DLZ constructs a regular system with at most $\mu \times m$ equations and $\mu \times n$ unknowns.

The last method of HMS : Hauenstein-Mourrain-Szanto.

$z = (z_1, \ldots, z_n)$. They define a parametric normal form

$$N_{z,\mu}(p) = \sum_{\gamma \in \mathbb{N}^n} \frac{1}{\gamma!} \partial_\gamma[w](p) M^\gamma(\mu)[1]$$

associated to a basis

$$B = \{(z - w)^{\alpha_0} = 1, (z - w)^{\alpha_1} \ldots, (z - w)^{\alpha_{\delta-1}}$$

where

$$M^\gamma = M_1^{\gamma_1}(\mu) \ldots M_n^{\gamma_n}$$

where each $M_i(\mu)$ is named paramatric multiplication matrix.

The last method of HMS : Hauenstein-Mourrain-Szanto.

Each $M_i(\mu)$ are defined by

$$M_i(\mu)^t = \begin{bmatrix} 0 & \mu_{\alpha_1,e_i} & \mu_{\alpha_2,e_i} & \cdots & \mu_{\alpha_{\delta-1},e_i} \\ 0 & 0 & \mu_{\alpha_2,\alpha_1+e_i} & \cdots & \mu_{\alpha_{\delta-1},\alpha_1+e_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{\alpha_{\delta-1},\alpha_{\delta-2}+e_i} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with

$$\mu_{\alpha_j,\alpha_k+e_i} = \begin{cases} 1 & \text{if } \alpha_j = \alpha_k + e_i \\ 0 & \text{if } \alpha_k + e_i \in E \text{ and } \alpha_j \neq \alpha_k + e_i \\ \mu_{\alpha_j,\alpha_k+e_i} & \text{if } \alpha_k + e_i \notin E \end{cases}$$

and

$$E = (\alpha_0 := 1, \alpha_1, \ldots, \alpha_{delta-1})$$

The last method of HMS : Hauenstein-Mourrain-Szanto.

Theorem

Let $f \in \mathbb{C}[x]^N$ and $w \in \mathbb{C}^n$ be an isolated solution of f. Let Q be the primary ideal at w and assume that B is a basis for $\mathbb{C}[x]/Q$

$$B = \{(z-w)^{\alpha_0} = 1, (z-w)^{\alpha_1} \dots, (z-w)^{\alpha_{\delta-1}}$$

and

$$E = (\alpha_0 := 1, \alpha_1, \dots, \alpha_{delta-1})$$

with the propery that B is connected to 1.
Then there exists (w, ν) an regular isolated root of the polynomial system in $\mathbb{C}[z,]$:

$$\begin{aligned} N_{z,\mu}(f_k) &= 0, \qquad k = 1..N \\ M_i(\mu)Mj(\mu) - M_j(\mu)Mi(\mu) &= 0, \qquad i,j = 1..n. \end{aligned}$$

The last method of HMS : Hauenstein-Mourrain-Szanto.

Number of equations : $N + \dfrac{n(n-1)}{2}$

Number of variables at most: $n + \frac{n\delta(\delta-1)}{2}$

In case of thickness one $(n-1)(\delta-1)$.

J. D. Hauenstein, B. Mourrain, A. Szanto, On deflation and multiplicity structure, Journal of Symbolic Computation, 83, 228–253, 2017.

The last method of HMS : Hauenstein-Mourrain-Szanto.

$f = (x - y + x^2, \quad x - y + y^2)$, $B = \{1, x_2\}$, $E = \{(0,0),(0,1)\}$.
$0,0)$ has multiplicity 2.

Alors $M_1(\mu)^t = \begin{pmatrix} 0 & \mu \\ 0,0 & \end{pmatrix} \qquad M_2(\mu)^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

On obtient

$$N_{x,y,\mu} = \begin{pmatrix} 2y + \mu \\ x^2 + x \\ x^2 + y^2 \\ 2x\mu + 2y \end{pmatrix}$$

The last method of HMS : Hauenstein-Mourrain-Szanto.

The construction depends of B. $f = (x - y + x^2, \quad x - y + y^2)$, $B = \{1, x_2\}$, $E = \{(0,0),(1,0)\}$.
Alors $M_1(\mu)^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad M_2(\mu)^t = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$
On obtient un système qui n'admet pas $(0,0,\mu)$ comme solution:

$$N_{x,y,\mu} = \begin{pmatrix} x^2 + x \\ 2y\mu + 1 \\ x^2 + y^2 \\ 2y\mu + 2x \end{pmatrix}$$

The last method of HMS : Hauenstein-Mourrain-Szanto.

Let the système
$f = (1/3\,x^3 + y^2x + x^2 + 2\,yx + y^2, x^2y - y^2x + x^2 + 2\,yx + y^2)$,
where $(0,0)$ has multiplicity 6.
Then $B = \{1, x, y, x^2, xy, x^3\}$ and

$$M_1(\mu)^t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_2(\mu)^t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu_2 & \mu_3 & \mu_4 \\ 0 & 0 & 0 & 0 & \mu_5 & \mu_6 \\ 0 & 0 & 0 & 0 & 0 & \mu_7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last method of HMS : Hauenstein-Mourrain-Szanto.

We obtain

$$N_{x,y,\mu}(f) = \begin{bmatrix} (x+1)\,\mu_2 + x + 1 \\ (x+1)\,\mu_3 + 2\,y + 2 \\ (x+1)\,\mu_4 + \mu_1\mu_3 + \mu_2 + 1/3, \\ 1/3\,x^3 + y^2x + x^2 + 2\,yx + y^2 \\ 2\,yx + 2\,x + 2\,y \\ x^2 + y^2 + 2\,x + 2\,y \\ (-x+1)\,\mu_2 + y + 1 \\ (-x+1)\,\mu_3 + 2\,x - 2\,y + 2 \\ (-x+1)\,\mu_4 - \mu_1\mu_3 - \mu_2 + \mu_1 \\ 2\,yx - y^2 + 2\,x + 2\,y \\ x^2 - 2\,yx + 2\,x + 2\,y \\ x^2y - y^2x + x^2 + 2\,yx + y^2 \\ \mu_1\mu_5 \\ -\mu_5 \\ \mu_1 - \mu_6 \\ \mu_1\mu_3 + \mu_2 - \mu_7 \end{bmatrix}$$

Solution

$$(x = 0, y = 0, \mu_1 = -1/3, \mu_2 = -1, \mu_3 = -2, \mu_4 = 0, \mu_5 = 0, \mu_6 = -1/3, \mu_7 = -1/3)$$

Centro Nacional de Jubilación Científico

Part 4. Numerical approximation of multiple isolated roots of analytic systems

Marc Giusti* and Jean-Claude Yakoubsohn**

What is the dream?

The quadratic convergence of the Newton method is lost in a neighborhood of a singular root.

To recover it : determine a regular system admitting the same root as the initial one .

Example :

$$\begin{bmatrix} x - y^2 \\ 2cy^3 - 2xy \end{bmatrix} = 0$$

Griewank and Osborne (1983)

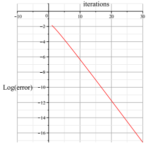


Fig. 1. Linear convergence of Newton sequence from $(0.1, -0.2)$ with $c = 5/32$.

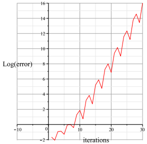
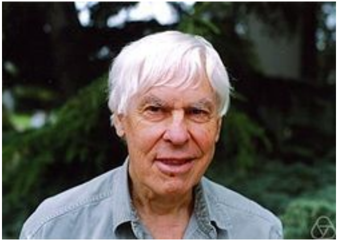


Fig. 2. Divergence of Newton sequence from $(0.1, -0.2)$ with $c = 29/32$.

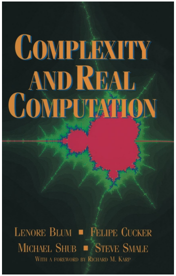
But the system $\begin{bmatrix} x - y^2 \\ y \\ 3cy^2 - x \end{bmatrix} = 0$ is regular at $(0, 0)$.

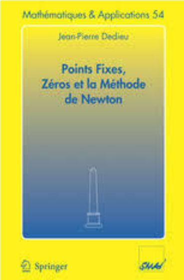
Our inspirers



Steve Smale

Mike Shub and Jean-Pierre Dedieu





Our inspirers

Carlos Beltràn and Luis Miguel Pardo

- random matrices
- foundations of numerical PDEs
- approximation theory
- computational number theory
- numerical linear algebra
- real-number complexity
- computational dynamics
- stochastic computation.

In addition to the workshops, eighteen plenary lectures, covering a broad spectrum of topics connected to computational mathematics, were delivered by some of the world's foremost researchers. This volume is a collection of articles, based on the plenary talks presented at FoCM 2005. The topics covered in the lectures and in this volume reflect the breadth of research within computational mathematics as well as the richness and fertility of interactions between seemingly unrelated branches of pure and applied mathematics.

We hope that this volume will be of interest to researchers in the field of computational mathematics and also to non-experts who wish to gain some insight into the state of the art in this active and significant field.

Like previous FoCM conferences, the Santander gathering proved itself as a unique meeting point of researchers in computational mathematics and of theoreticians in mathematics and computer science. While presenting plenary talks by foremost world authorities and maintaining the highest technical level in the workshops, the conference, like previous meetings, laid emphasis on multidisciplinary interaction across subjects and disciplines in

1

On the Complexity of Non Universal Polynomial Equation Solving: Old and New Results

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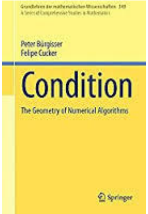
Abstract

These pages summarize some results on the efficiency of polynomial equation solving. We focus on semantic algorithms, i.e., algorithms whose running time depends on some intrinsic/semantic invariant associated with the input data. Both computer algebra and numerical analysis algorithms are discussed. We show a probabilistic and positive answer to Smale's 17th problem. Estimates of the probability distribution of the condition number

Les conditionneurs

Peter Burgisser

Felipe Cucker

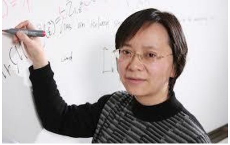


THE CONTRIBUTORS OF NUMERICAL APPROXIMATION OF SINGULARITIES

Bernard Mourrain

Jan Verschelde

Barry Dayton



Lihong Zhi

Symbolic studies with use of float numbers ... but not Numerical Analysis

Our Framework

1– $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $s \geq n$.

2– an analytic system $f = (f_1, \dots, f_s)$ defined in $B(\omega, R_\omega)$ such that

$$||f|||^2 = \int_{B(\omega, R_\omega)} f^2(z)dz < +\infty.$$

We said that $f \in \mathbf{A}^2(\omega, R_\omega)$.

3– $\zeta \in B(\omega, R_\omega)$ an isolated root of f.

4– ζ is a multiple isolated root if $Df(\zeta)$ has not a full rank.

5– Two systems are equivalent in ζ if ζ is an isolated root of each of them.

6– We said system for analytic system.

Propositions to use a well tested terminology

1– Embedding dimension not breadth

2– Thickness not depth

Emsalem uses épaisseur in "Géométrie des points épais" (1978)

3– Standard Basis not Gröbner Basis

see

M. Giusti, J.-C. Yakoubsohn, Multiplicity hunting and approximating multiple roots of polynomial systems, Recent Advances in Real Complexity and Computation, 604, p.105–128, 2014

Definition

The Schur complement of a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of rank $r > 0$ associated to an invertible A of size $r \times r$ is

$$\mathrm{Schur}(M) = D - CA^{-1}B.$$

If $r = 0$ we define $\mathrm{Schur}(M) := M$.

Definition

Let $\varepsilon \geq 0$. Let M a matrix with $U\Sigma V^T$ as SVD such that its singular values verify

$$\sigma_1 \geq \ldots \geq \sigma_{r_\varepsilon} > \varepsilon \geq \sigma_{r_\varepsilon+1} \geq \ldots \geq \sigma_n.$$

The real number r_ε is the ε-numerical rank of M.
Let Σ_ε the matrix obtained from Σ putting $\sigma_{r_\varepsilon+1} = \ldots = \sigma_n = 0$.
We note $M_\varepsilon = U\Sigma_\varepsilon V^*$.

Kerneling

Definition

Let $\varepsilon \geq 0$, $0 \leq r < n$ and $f = (f_1, \ldots, f_s) \in \mathbb{C}\{x - x_0\}^s$. Let us suppose $D_{1:r} f_{1:r}(x_0)$ has an ε-rank equal to r.
We define the kerneling operator

$$K : f \mapsto (f_1, \ldots, f_r, vec(\,\mathrm{Schur}(Df(x)))) \in \mathbb{C}\{x - x_0\}^{r+(n-r)\,(s-r)}.$$

We say that $K(f)$ is an ε-kerneling of f if we have

$$\|K(f)\| \leq \varepsilon.$$

We say that the kerneling is exact when $\varepsilon = 0$.

Deflation sequence

Definition
Let $\varepsilon \geq 0$, $x_0 \in \mathbb{C}^n$ and $f = (f_1, \ldots, f_s) \in \mathbb{C}\{x - x_0\}^s$. The sequence

$$\begin{aligned} F_0 &= f \\ F_{k+1} &= K(F_k), \quad k \geq 0. \end{aligned}$$

is named the deflation sequence.
The thickness is the index ℓ where the ε-rank of $DF_\ell(x_0)$ is equal to n, and not before.
We name *deflation system* $\mathrm{dfl}(f)$ of f a system of rank n extracted from F_ℓ.

The multiplicity drops through kerneling

Definition
The multiplicity of an isolated singular root is the dimension of the associated local algebra.

Theorem
Let us suppose that the rank of $Df(\zeta)$ is equal to r and that

$$Df(x) := \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$$

where $A(\zeta) \in \mathbb{C}^{r\times r}$ is invertible. Then the multiplicity of ζ as root of $K(f)$ is strictly lower than the multiplicity of ζ as root of f.

Example : exact computations

$$f_1(x,y) = x^3/3 + y^2x + x^2 + 2yx + y^2, \quad f_2(x,y) = x^2y - y^2x + x^2 + 2yx + y^2$$

The root $(0,0)$ has multiplicity 6. We have

$$Df(x,y) = \begin{pmatrix} x^2 + y^2 + 2x + 2y & 2xy + 2x + 2y \\ 2xy - y^2 + 2x + 2y & x^2 - 2xy + 2x + 2y \end{pmatrix}.$$

The rank of the Jacobian matrix is 0 at $(0,0)$. Hence kerneling consists just to replace the input system by the gradients of f_1 and f_2:

$$F_1 = K(f) = (x^2+y^2+2x+2y, \quad 2xy+2x+2y, \quad 2xy-y^2+2x+2y, \quad x^2-2xy+2x+2y)$$

Then

$$DF_1(x,y) = \begin{pmatrix} 2x+2 & 2y+2 \\ 2y+2 & 2x+2 \\ 2y+2 & 2x-2y+2 \\ 2x-2y+2 & -2x+2 \end{pmatrix} \qquad DF_1(0,0) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$$

The rank of $DF_1(0,0)$ is one.

Example : exact computations

The Schur complement of $DF_1(x, y)$ associated to $2x + 2$ is

$$\mathrm{Schur}(DF_1(x,y)) = \frac{2}{x+1}\left(\begin{array}{c} 2x - 2y + x^2 - y^2 \\ 2x - 3y + x^2 - xy - y^2 \\ -x - x^2 - xy + y^2. \end{array}\right)$$

Then we can easily check that the system

$$F_2 = (f_1, vec(\,\mathrm{Schur}(DF_1(x,y))$$

is a regular system equivalent at $(0, 0)$ to f. Let us remark also the truncated system of F_2 up to the order 1 namely

$$(x + y, \quad x - y, \quad 2x - 3y, \quad x)$$

is a regular system equivalent at $(0, 0)$ to f.

Theoretical algorithm for deflation sequence

```
1-  Inputs : x_0 ∈ ℂ^n, f ∈ A^2(x_0, R_{x_0})
2-  dfl(f) = {∅}
3-  F := f.
4-  η := 2α_0 / ((n+1)(n+2)(R_{x_0} + ‖F‖) R_{x_0}^{n-2})        Why this η ?
5-  if ‖F(x_0‖ ≤ η then     need to be justified by a result
6-     r := numerical rank (DF(x_0))        How to?
7-     if r < n then
8-        F := K(F)
9-        go to 2
10-    else
11-       Let dfl(f) a deflated system of numerical rank n from F.
12-    end if
13- end if
14- Output : dfl(f).
```

where

$$\eta := \frac{2\alpha_0}{(n+1)(n+2)(R_{x_0} + \|F\|)R_{x_0}^{n-2}}$$

Singular Newton operator

SINGULAR NEWTON

1- Inputs : $x_0 \in B(\omega, R_\omega)$, $f \in \mathbf{A}^2(\omega, R_\omega)$

2- $\mathrm{dfl}(f) =$ deflated system(f).

3- Output : If $\mathrm{dfl}(f) \neq \emptyset$ then $N_{\mathrm{dfl}(f)}(x_0)$ else x_0.

Evaluation map

$$\text{eval} \; : \; (f,x) \mapsto eval_x(f) = f(x) \qquad \text{from } (\mathbf{A}^2(\omega, R_\omega))^s \times B(\omega, R_\omega) \text{ to } \mathbb{C}^s.$$

Let $c_0 := \sum_{k\geq 0}(1/2)^{2^k-1}$ ($\sim 1.63...$), and α_0 ($\sim 0.13...$) be the first positive root of the trinomial $(1-4u+2u^2)^2-2u$.

We study the question: when the value $f(x)$ can be considered as small? We give a precise meaning of being small without the use of any ε.

Since the evaluation map is surjective a Newton map associated to this evaluation map make sense :

$$\begin{aligned} N_{\text{eval}}(f,x) &= D\,\text{eval}(f,x)^\dagger\,\text{eval}(f,x) \\ &= D\,\text{eval}(f,x)^*\left(D\,\text{eval}(f,x) D\,\text{eval}(f,x)^*\right)^{-1} \end{aligned}$$

Evaluation map

Theorem

Let $f = (f_1, \ldots, f_s) \in \mathbf{A}^2(\omega, R_\omega)^s$. Let $x \in B(\omega, R_\omega)$ and $\|x - \omega\| = \rho_x$. If

$$\frac{c_0}{R_\omega}\left(R_\omega^2 - \rho_x^2\right)^{\frac{n+1}{2}} \|f(x)\| + \rho_x < R_\omega$$

and

$$\frac{(n+1)(n+2)}{2}\left(R_\omega^2 - \rho_x^2\right)^{(n-3)/2}\left(\|f\|\, R_\omega + (R_\omega^2 - \rho_x^2)\right)\, \|f(x)\| \leq \alpha_0$$

then $f(x)$ is small at the following sense : the Newton sequence defined by

$$(f^0, x_0) = (f, x), \quad (f^{k+1}, x_{k+1}) = ((f^k, x_k) - D\,\mathrm{eval}(f^k, x_k)^\dagger\, \mathrm{eval}(f^k, x_k)), \quad k \geq 0,$$

converges quadratically towards a certain $(g, y) \in (\mathbf{A}^2(\omega, R_\omega))^s \times B(\omega, R_\omega)$ satisfying $g(y) = 0$. More precisely we have

$$(\|f - g\| + \|x - y\|^2)^{1/2} \leq \frac{c_0}{R_\omega}\left(R_\omega^2 - \rho_x^2\right)^{\frac{n+1}{2}} \|f(x)\|.$$

Evaluation map

Corollary

Let $f = (f_1, \ldots, f_s) \in \mathbf{A}^2(x_0, R_{x_0})^s$. If

$$\|f(x_0)\| \leq \frac{2\alpha_0}{(n+1)(n+2)\, R_{x_0}^{n-2}\,(\|f\| + R_{x_0})}$$

then $f(x)$ is small at the following sense : the Newton sequence defined by

$$(f^0, x_0) = (f, x), \quad (f^{k+1}, x_{k+1}) = ((f^k, x_k) - D\,\mathrm{eval}(f^k, x_k)^\dagger\,\mathrm{eval}(f^k, x_k)), \quad k \geq 0,$$

converges quadratically towards a certain $(g, y) \in (\mathbf{A}^2(x_0, R_{x_0}))^s \times B(x_0, R_{x_0})$ satisfying $g(y) = 0$.

More precisely there exists $(g, y) \in (\mathbf{A}^2(x_0, R_{x0}))^s \times B(x_0, R_{x_0})$ such that $g(y) = 0$ and

$$(\|f - g\| + \|x_0 - y\|^2)^{1/2} \leq c_0\, R_{x_0}^n \|f(x_0)\|.$$

Tracking the rank of a matrix

Let M a matrix with singular values $\sigma_1 \geq \ldots \geq \sigma_n$.
We consider the elementary symmetric sums of the σ_i's, i.e.:

$$s_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \sigma_{i_1} \ldots \sigma_{i_k}, \quad k = 1 : n$$

which are the roots of

$$s(\lambda) = \prod_{i=1}^{n} (\lambda - \sigma_i) = \lambda^n + \sum_{1 \leq i \leq n} (-1)^{(n-i)} s_{n-i} \lambda^i.$$

By convention $s_0 = 1$. We introduce the quantities :

1– $b_k(M) := \max_{0 \leq i \leq k-1} \left(\frac{s_{n-i}}{s_{n-k}} \right)^{\frac{1}{k-i}}$.

2– $g_k(M) := \max_{k+1 \leq i \leq n} \left(\frac{s_{n-i}}{s_{n-k}} \right)^{\frac{1}{i-k}}$.

3– $a_k(M) := b_k(M)\, g_k(M)$.

with the convention $g_n(M) = 1$.
For simplicity let us denote by a_k, b_k, g_k the corresponding values $a_k(M)$, $b_k(M)$, $g_k(M)$.

Tracking the rank of a matrix

Theorem

Let us consider the polynomial $s(\lambda)$ previously defined.

1– If there exists m an integer be such that $1 \leq m \leq n$ satisfying $a_m < 1/9$ then the polynomial $s(\lambda)$ has m roots in the ball $B(0,\varepsilon)$ with

$$\varepsilon = \frac{3a_m + 1 - \sqrt{(3a_m+1)^2 - 16a_m}}{4g_m}.$$

2– If $a_1 > 1/9$ then $\sigma_n > \dfrac{1}{10g_1}$ where m is the integer satisfying $s_n \neq 0,\ s_{n-k} = 0,\ k = 1 : m-1$ and $s_{n-m} \neq 0$.

Definition

Let a matrix M be such that $\mathrm{rank}(M) = r$.
Let $E = \{k \geq 1 \,:\, a_k < 1/9\}$ and

$$m = \max(0, \quad \min_k E).$$

Let

$$\epsilon = \begin{cases} \frac{3a_m+1-\sqrt{(3a_m+1)^2-16a_m}}{4g_m} & \text{if } E \neq \emptyset \\ \frac{1}{10g_1} & \text{otherwise} \end{cases}$$

Numerical rank determination free of ε

1– Input : a matrix $M \in \mathbb{C}^{s \times n}$, $s \geq n$

2– Compute the singular values of M : $\sigma_1 \geq \ldots \geq \sigma_n$.

3– From these σ_i's, compute the elementary symmetric functions s_k's associated

4– Then compute the quantities b_k, g_k, a_k.

5– if there exists m s.t. $a_m :=< 1/9$ then the ε-rank of the matrix A is $n - m$.
where $\varepsilon := \dfrac{3a_m + 1 - \sqrt{(3a_m + 1)^2 - 16a_m}}{4g_m}$

6– else

7– we have $\varepsilon := \dfrac{1}{10g_m} < \sigma_n$ and the ε-rank of the matrix M is n
where m is the integer satisfying $s_n \neq 0$, $s_{n-k} = 0$, $k = 1 : m - 1$ and $s_{n-m} \neq 0$.

8– end if

9– Output : the ε-rank of the matrix M.

For $x \in B(\omega, R_\omega)$ we note $\rho_x := \|x - \omega\|$.

Proposition

Let $H(z,x) = \dfrac{R_\omega^2}{(R_\omega^2 - < z-\omega, x-\omega >)^{n+1}}$.

1– $H(x,x) = \|H(\bullet, x)\|^2 = \dfrac{R_\omega^2}{(R_\omega^2 - \rho_x^2)^{n+1}}$.

2– For all $f \in \mathbf{A}^2(\omega, R_\omega)$ one has

$$f(x) = \int_{B(\omega,R_\omega)} f(z)H(z,x)d\nu(z) \leq \frac{\|f\|\, R_\omega}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}}}.$$

For $x \in B(\omega, R_\omega)$ we note $\rho_x := \|x - \omega\|$.

Proposition

Let $k \geq 0$, $\omega \in \mathbb{C}^n$, $x \in B(\omega, R_\omega)$ and $u_i \in \mathbb{C}^n$, $i = 1 : k$. Let us introduce

$$H_k(z,x,u_1,\ldots,u_k) = \frac{(n+1)\ldots(n+k) < z-\omega, u_1 > \ldots < z-\omega, u_k >}{(R_\omega^2 - < z-\omega, x-\omega >)^k} H(z,x).$$

We have

1–
$$D^k f(x)(u_1, \ldots, u_k) = \int_{B(\omega, R_\omega)} f(z)\, H_k(z, x, u_1, \ldots, u_k)\, d\nu(z)$$

2– $$\|D^k f(x)\| \leq \frac{(n+1)\ldots(n+k)\, R_\omega^{1+k}}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}+k}} ||f||$$

$\beta,\ \kappa,\ \gamma.$

We consider as previously $\omega \in \mathbb{C}^n$ and the set $\mathbf{A}^2(\omega, R_\omega)$.
For $x \in B(\omega, R_\omega)$ we introduce the quantities

1– $\beta(f,x) = \|Df(x)^{-1}f(x)\|$

2– $\kappa_x = \max\left(1,\ \dfrac{R_\omega(n+1)}{R_\omega^2 - \rho_x^2}\right)$

3– $\gamma(f,x) = \max\left(1,\ \dfrac{\|f\|\,|Df(x)^{-1}\|\,R_\omega\,\kappa_x}{(R_\omega^2 - \rho_x^2)^{\frac{n+1}{2}}}\right)$

Theorem

Let ζ a regular root of an analytic system $f = (f_1, \ldots, f_n) \in \mathbf{A}^2(\omega, R_\omega)^n$.

Let us note γ for $\gamma(f, \zeta)$ and κ for κ_ζ . Then, for all x be such that

$$\kappa \|x - \zeta\| < \frac{2\gamma + 1 - \sqrt{4\gamma^2 + 3\gamma}}{\gamma + 1}$$

the Newton sequence

$$x_0 = x, \quad x_{k+1} = N_f(x_k), \quad k \geq 0,$$

converges quadratically towards ζ. More precisely

$$\|x_k - \zeta\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x - \zeta\|, \quad k \geq 0.$$

Let $f \in \mathbf{A}^2(\omega, R_\omega)^s$ and $\zeta \in B(\omega, R_\omega)$ such that $f(\zeta) = 0$.
Let us suppose there exists a index ℓ be such that

1– For all $0 \leq k < \ell$ each element $F_k = K(F_{k-1})$ satisfies $F_k(\zeta) = 0$ and $\mathrm{rank}(DF_k(\zeta)) < n$.

2– The assumptions of regular γ-theorem hold for the system F_ℓ at ζ.

Then, for all x be such that

$$\kappa \|x - \zeta\| < \frac{2\gamma + 1 - \sqrt{4\gamma^2 + 3\gamma}}{\gamma + 1}$$

the Newton sequence, computed thanks to Singular Newton algorithm,

$$x_0 = x, \quad x_{k+1} = N_{\mathrm{dfl}(f)}(x_k), \quad k \geq 0,$$

converges quadratically towards ζ.

Example : numerical computations

We give the behaviour of the deflation sequence.

1– The initial point $(x_0, y_0) = (-0.01, 0.02)$.

2– The system :

$$f = \begin{pmatrix} 1/3\,x^3 + y^2x + x^2 + 2\,xy + y^2 \\ x^2y - y^2x + x^2 + 2\,xy + y^2 \end{pmatrix}$$

3– The ball $B(x_0, R_\omega) := B(x_0, 1/4)$

4– Truncated expansion series of the system $F_0 = f(x + x_0, y + y_0)$ up to the order 3.

$$F_0 = \begin{pmatrix} 0.0000957 + 0.0205\,x + 0.0196\,y + 0.990\,x^2 + 2.04\,xy + 0.99\,y^2 + 0.333\,x^3 + y^2x \\ 0.000102 + 0.0201\,y + 0.0196\,x + 1.98\,xy + 1.02\,x^2 + y^2 + x^2y \end{pmatrix}$$

5– Evaluation of F_0 at $(0,0)$: $(0.000095, 0.000102)$.

6– We successively have $\|F_0\| = 8 \times 10^{-4}$,

$$\eta = \frac{2\alpha_0}{12(R_\omega + \|F_0\|)R_\omega^{n-2}} = 0.086 \ > \ \|F_0(0,0)\| \ = \ 0.000106.$$

7– Jacobian of F_0 at $(0,0)$: $DF_0(0,0) = \begin{pmatrix} 0.0205 & 0.0196 \\ 0.0196 & 0.0205 \end{pmatrix}$. The singular values of this jacobian are 0.039 and 0.0011. This jacobian has a $\epsilon = 0.086$ numerical rank equal to 0.

Example : numerical computations

8– Kerneling of F_0 at $(0,0)$:

$$F_1 = K(F_0) = \begin{pmatrix} 0.0205 + 1.98\,x + 2.04\,y + 1.0\,x^2 + y^2 \\ 0.0196 + 2.04\,x + 1.98\,y + 2.0\,xy \\ 0.0192 + 1.94\,y + 2.04\,x + 2xy - y^2 \\ 0.0205 + 1.94\,x + 2.02\,y + x^2 - 2xy \end{pmatrix}$$

9– Evaluation of F_1 at $(0,0)$: $F_1(0,0) = (0.0205, 0.0196, 0.01920.0201)$. We have $\|F_1\| = 0.1$ and
$\eta = \frac{2\alpha_0}{12(R_\omega + \|F_1\|)R_\omega^{n-2}} = 0.062 > \|F_1(0,0)\| = 0.034.$

10– Jacobian matrix of F_1 and its evaluation at $(0,0)$:

$$DF_1(x,y) = \begin{pmatrix} 1.98 + 2.0\,x & 2.04 + 2\,y \\ 2.04 + 2.0\,y & 1.98 + 2.0\,x \\ 2.04 + 2\,y & 1.94 + 2x - 2y \\ 1.94 + 2x - 2y\,x & 2.02 - 2x \end{pmatrix} \qquad DF_1(0,0) = \begin{pmatrix} 1.98 & 2.04 \\ 2.04 & 1.98 \\ 2.04 & 1.94 \\ 1.94 & 2.02 \end{pmatrix}$$

The singular values of $DF_1(0,0)$ are 5.6 and 0.06 and its $\epsilon = 0.21$ numerical rank is one.

11– Kerneling of F_1. We compute the truncated series at the order one in $(0,0)$ of each element of the Schur complement of $DF_1(x,y)$ associated to $1.98 + 2.0\,x$. We obtain

$$F_2 := K(F_1) = \begin{pmatrix} 0.0205 + 1.98\,x + 2.04\,y \\ -0.121 + 4.123\,x - 4.121\,y \\ -0.16 + 4.12x - 6. \\ -0.21 - 2.04\,x - 0.1y \end{pmatrix}$$

EXAMPLE : NUMERICAL COMPUTATIONS

12– Regular system from F_2 at $(0,0)$. The singular values of $DF_2(0,0)$ are 9.46 and 3.32. Hence $DF_2(0,0)$ has $\epsilon = 3.32$ rank equal to 2.

13– If we consider

$$dfl(f) = \begin{pmatrix} 0.0205 + 1.98\,x + 2.04\,y \\ -0.121 + 4.123\,x - 4.121\,y \end{pmatrix}$$

we find that the iterate of

$$(x_0, y_0) = (-0.01, 0.02)$$

by the singular Newton operator is

$$(-0.0001017, 0.00034)$$

This illustrates the manifestation of a quadratic convergence.

We show below quadratic convergence obtained thanks to the algorithm singular Newton.

$$[-0.01, 0.02]$$

$$[-0.00010175, 0.000343]$$

$$[-1.7 \times 10^{-8}, 8.1 \times 10^{-8}]$$

$$[-7.15 \times 10^{-16}, 4.2 \times 10^{-15}]$$

$$[-1.5 \times 10^{-30}, 1.06 \times 10^{-29}]$$

$$[-7.9 \times 10^{-60}, 6.55 \times 10^{-59}]$$

Open problems

1– Untreated problem : estimation of the quantity $\gamma(dfl(f), x_0)$ with $\gamma(f, x_0)$.

2– Truncation problem : to retrieve the quadratic convergence it is sufficient to deal with a truncated deflation sequence defined by

$$\begin{aligned} T_0 &= Tr_{x_0,\ell}(f) \\ T_{k+1} &= Tr_{x_0,\ell-k-1}\left(K\left(T_k\right)\right), \quad 0 \leq k \leq \ell. \end{aligned}$$

where ℓ is the thickness of the deflation sequence. How to determine a priori the thickness ℓ?

3– Connection with homotopy.