Méthodes formelles pour les équations aux dérivées partielles

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1 Introduction

Given a system of differential equations, we would like to be able to solve the following tasks:

(a) determine all analytic solutions;

(b) obtain an overview of all consequences of the system; in particular, given another differential equation, decide whether it is a consequence of the system or not;

(c) among the consequences find the ones which involve only certain specified unknowns.

Throughout these notes we shall consider partial differential equations (PDEs) for unknown functions $u_1(z_1, \ldots, z_n)$, $\ldots$, $u_m(z_1, \ldots, z_n)$. Since we are going to employ formal methods, we restrict our attention to formal power series solutions in (a). Convergence of these power series on certain regions of $\mathbb{R}^n$ or $\mathbb{C}^n$ is to be investigated after the formal treatment. In fact, the formal treatment may reveal conditions on how the region in $\mathbb{R}^n$ or $\mathbb{C}^n$ should be chosen. Singular points will be excluded from consideration.

One of the first existence theorems for a large class of PDEs is the Cauchy-Kovalevskaya Theorem [Kov75], [RR04], [Eva10].

Theorem 1.1 (Cauchy-Kovalevskaya, 1875). *The Cauchy problem*

$$\left\{\begin{array}{rcl}
\dfrac{\partial u_1}{\partial z_1} & = & \displaystyle\sum_{j=2}^{n}\sum_{k=1}^{m} a_{1,j,k}(z_2, \ldots, z_n, u_1, \ldots, u_m)\,\frac{\partial u_k}{\partial z_j} + b_1(z_2, \ldots, z_n, u_1, \ldots, u_m),\\
\vdots & & \\
\dfrac{\partial u_m}{\partial z_1} & = & \displaystyle\sum_{j=2}^{n}\sum_{k=1}^{m} a_{m,j,k}(z_2, \ldots, z_n, u_1, \ldots, u_m)\,\frac{\partial u_k}{\partial z_j} + b_m(z_2, \ldots, z_n, u_1, \ldots, u_m),\\
u_1(0, z_2, \ldots, z_n) & = & 0 \qquad \textit{for all } z_2, \ldots, z_n,\\
\vdots & & \\
u_m(0, z_2, \ldots, z_n) & = & 0 \qquad \textit{for all } z_2, \ldots, z_n,
\end{array}\right.$$

where $a_{i,j,k}$ and b_i are real analytic functions around the origin of $\mathbb{R}^{m+n-1}$, has a unique real analytic solution $(u_1, \ldots, u_m)$ in a neighborhood of $(z_1, \ldots, z_n) = (0, \ldots, 0)$.

Note that any system of differential equations can be rewritten as a system of *first order* differential equations by introducing new unknown functions, if necessary. The differential equations in Theorem 1.1 are *quasilinear* in the sense that each equation is linear in the highest derivatives of the unknown functions. Analytic coordinate changes may be used to transform boundary data on an analytic hypersurface which is non-characteristic for the first order PDE system to the hypersurface $z_1 = 0$. Theorem 1.1 is also valid for complex analytic functions. However, the assumption of analyticity is necessary (cf. [Lew57]).

In work of C. Méray [Mér80] and C. Riquier [Riq10] in the second half of the 19th century a generalization of the Cauchy-Kovalevskaya Theorem was obtained. Riquier's Existence Theorem asserts the existence of analytic solutions to systems of PDEs of a certain class (cf. also [Tho28, Tho34], [Rit34, Chap. IX], [Rit50, Chap. VIII]). The equations are assumed to be solved for certain distinct partial derivatives and their right hand sides are analytic functions of $z_1, \ldots, z_n$ and of partial derivatives of $u_1, \ldots, u_m$ which are less than the ones on the respective left hand side with respect to a certain kind of total ordering. Moreover, the system is supposed to incorporate all integrability conditions in some sense.

It is a non-trivial task to include here all relevant references. Among the most important historical ones we select: C. Méray [Mér80], C. Riquier [Riq10], M. Janet (1888–1983) [Jan29], J. M. Thomas (1898–1979) [Tho37, Tho62], J. F. Ritt [Rit34], [Rit50], E. R. Kolchin [Kol73] and A. Seidenberg [Sei56]. Related references are [Olv93], [Pom78], [Pom94], [Sch08a] and many more.

Closely related to the method of Thomas decomposition discussed in these notes is the Rosenfeld-Gröbner algorithm and its implementation in the Maple package `diffalg` resp. `DifferentialAlgebra` (cf., e.g., [BLOP95], [BLOP09], [Hub97], [Hub00], [Bou]), but also the method of regular chains [LMMX05] and the rifsimp algorithm [RWB96]. Moreover, the notion of a characteristic set, introduced by Ritt and Wu, again belongs to the same circle of ideas, cf., e.g., [Wu00], [Wu89] [Wan98], [Wan01], [Wan04], [Dio92]. Janet bases to be introduced in Section 2 are related to Gröbner bases [Buc06, Buc87] as well as involutive bases [GB98a, GB98b, ZB96].

It is essential to note that the presented methods are also fundamental for further effective module-theoretic constructions for rings of linear functional operators and their implementations, on which applications, e.g., to systems theory are built (cf., e.g., [CQR05], [CQR07], [CQ08], [CQ09], [Rob15]). The algorithms discussed in these notes have been implemented in Maple packages (`Involutve`, `Janet`, `JanetOre`, `LDA`, `AlgebraicThomas`, `DifferentialThomas`).

This exposition is based, in particular, on [Rob07], [Rob14], [LHR], [GLR].

2 Systems of linear differential equations

In this section we assume that the given system of differential equations is *linear* (and homogeneous). In other words, for some l, m, $n \in \mathbb{N}$, some ring D of differential operators, some matrix of operators $R \in D^{l \times m}$ and some left D-module $\mathcal{F}$ we can write the system as

$$R\,u \;=\; 0\,, \qquad \text{where} \quad u \;=\; \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \tag{2.1}$$

for the unknown functions $u_i = u_i(z_1, \ldots, z_n) \in \mathcal{F}$, $i = 1, \ldots, m$. The *consequences* of (2.1) are the left D-linear combinations of the rows of R, i.e., the elements of $D^{1 \times l}\, R$. (The functions in $\mathcal{F}$ need to be infinitely often differentiable at least.)

Example 2.1. An example of a system of linear PDEs with constant coefficients is

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x^2} & = & 0\,, \\[2ex] \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} & = & 0\,, \end{array} \right. \tag{2.2}$$

where $u = u(x, y)$ depends on $x = z_1$ and $y = z_2$. We may choose $D = K[\partial_x, \partial_y]$, where $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots\}$ and where ∂_x and ∂_y are the partial differential operators with respect to x and y, respectively. The multiplication in D is composition of operators.

Example 2.2 ([Rob14], Ex. 3.2.38)**.** A system of linear PDEs with non-constant coefficients for $u = u(x, y)$ is given by

$$\left\{ \begin{array}{rcl} \dfrac{\partial^3 u}{\partial x\, \partial y^2} - \dfrac{\partial^3 u}{\partial y^3} - (2y+1)\, \dfrac{\partial^2 u}{\partial y^2} - 4\, \dfrac{\partial u}{\partial y} & = & 0\,, \\[2ex] \dfrac{\partial^3 u}{\partial x^2\, \partial y} - \dfrac{\partial^3 u}{\partial y^3} - 2\,(2y+1)\, \dfrac{\partial^2 u}{\partial x\, \partial y} + (4y^2 + 4y - 5)\, \dfrac{\partial u}{\partial y} & = & 0\,. \end{array} \right.$$

We may choose K to be $\mathbb{Q}(x, y)$ or the field or meromorphic functions on some open and connected subset Ω of $\mathbb{C}^2$. Moreover, we let $D = K\langle \partial_x, \partial_y \rangle$ be the ring of differential operators

$$\sum_{i,j \geq 0} a_{i,j}\, \partial_x^i\, \partial_y^j\,, \qquad a_{i,j} \in K\,,$$

which are (skew) polynomials in ∂_x and ∂_y, where composition is non-commutative in general.

Example 2.3 ([Rob14], Ex. 2.1.46)**.** Linearizing the system on nonlinear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial x} - u^2 & = & 0\,, \\[2ex] \dfrac{\partial^2 u}{\partial y^2} - u^3 & = & 0\,, \end{array} \right. \tag{2.3}$$

for one unknown function u of x and y, we obtain the system of linear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial U}{\partial x} - 2\,u\,U & = & 0\,, \\[2ex] \dfrac{\partial^2 U}{\partial y^2} - 3\,u^2\,U & = & 0\,, \end{array} \right. \tag{2.4}$$

for one unknown function U of x and y, where u is a solution of (2.3). In this case a preparatory treatment of the nonlinear system (2.3) is necessary to deal with the linearized system (2.4). The methods to be discussed in Section 3 allow to split system (2.3) into two systems

$$\boxed{\begin{array}{rcl} \underline{u_x} - u^2 & = & 0 \quad \{ \partial_x, \partial_y \} \\[1ex] 2\,\underline{u_y}^2 - u^4 & = & 0 \quad \{ \ast, \partial_y \} \\[1ex] u & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} & & \\[1ex] & & \\[1ex] u & = & 0 \quad \{ \partial_x, \partial_y \} \end{array}}$$

(where subscripts indicate differentiation and where the meaning of the sets on the right will become clear later). The set of analytic solutions of the original system (2.3) is the disjoint union of the set of analytic solutions of the above two systems. We define the polynomial ring $R = \mathbb{Q}(\sqrt{2})[u, u_x, u_y, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ and the ideal I of R which consists of all R-linear combinations of

$$\begin{array}{lllll} u_x - u^2\,, & \partial_x \left(u_x - u^2\right), & \partial_y \left(u_x - u^2\right), & \partial_x^2 \left(u_x - u^2\right), & \ldots \\[1ex] u_y - \frac{\sqrt{2}}{2}\,u^2\,, & \partial_x \left(u_y - \frac{\sqrt{2}}{2}\,u^2\right), & \partial_y \left(u_y - \frac{\sqrt{2}}{2}\,u^2\right), & \partial_x^2 \left(u_y - \frac{\sqrt{2}}{2}\,u^2\right), & \ldots \end{array}$$

Then R/I is an integral domain, and we may choose K as the field of fractions of R/I. Moreover, we define $D = K\langle \partial_x, \partial_y \rangle$. (Instead of $u_y - \frac{\sqrt{2}}{2}\,u^2$ one may also choose $u_y + \frac{\sqrt{2}}{2}\,u^2$.)

Remark 2.4. An essential remark for what follows is that the given *linear* PDEs translate into *linear* equations for the Taylor coefficients $c_{i,j}$ of power series solutions

$$u(x, y) \;=\; \sum_{i,j \geq 0} c_{i,j}\, \frac{(x - x_0)^i}{i!}\, \frac{(y - y_0)^j}{j!}$$

by substituting this ansatz into the PDEs and comparing coefficients (and similarly for a different number of independent variables and unknown functions). However, in order for the resulting system of linear equations in $c_{i,j}$ to characterize the power series solutions of the PDE system correctly (around a sufficiently generic point (x_0, y_0)), an overview of all consequences of the PDE system needs to be obtained first. Interesting new consequences are usually found by differentiating two known consequences so that in a suitable linear combination of these derivatives the highest derivatives of the unknown function cancel. Considering again Example 2.3, differentiation of the two PDEs in (2.4) yields

$$\begin{aligned} \frac{\partial^2}{\partial y^2} \left(\frac{\partial U}{\partial x} - 2\,u\,U \right) \;&=\; \frac{\partial^3 U}{\partial x\,\partial y^2} - 2 \left(\frac{\partial^2 u}{\partial y^2}\,U + 2\,\frac{\partial u}{\partial y}\,\frac{\partial U}{\partial y} + u\,\frac{\partial^2 U}{\partial y^2} \right) \\ &=\; \frac{\partial^3 U}{\partial x\,\partial y^2} - 2\,u^3\,U - 2\,\sqrt{2}\,u^2\,\frac{\partial U}{\partial y} - 6\,u^3\,U \end{aligned}$$

and

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 U}{\partial y^2} - 3\,u^2\,U \right) \;=\; \frac{\partial^3 U}{\partial x\,\partial y^2} - 3 \left(2\,u\,\frac{\partial u}{\partial x}\,U + u^2\,\frac{\partial U}{\partial x} \right) \;=\; \frac{\partial^3 U}{\partial x\,\partial y^2} - 6\,u^3\,U - 6\,u^3\,U\,.$$

Hence, we obtain

$$\frac{\partial^2}{\partial y^2} \left(\frac{\partial U}{\partial x} - 2\,u\,U \right) - \frac{\partial}{\partial x} \left(\frac{\partial^2 U}{\partial y^2} - 3\,u^2\,U \right) \;=\; 4\,u^3\,U - 2\,\sqrt{2}\,u^2\,\frac{\partial U}{\partial y}\,,$$

which yields the consequence

$$\frac{\partial U}{\partial y} - \sqrt{2}\,u\,U \;=\; 0\,.$$

2.1 Monomial ideals

Let $\partial_1, \ldots, \partial_n$ be the partial differential operators with respect to $z_1, \ldots, z_n$ and define $D = K[\partial_1, \ldots, \partial_n]$ for some field K. For the sake of simplicity, we assume in this section that the operators ∂_i act trivially on K, i.e., K consists of constants, and that we deal with a system of differential equations for one unknown function only.

The simplest operators in D are given by monomials

$$\partial^J \;:=\; \partial_1^{j_1} \ldots \partial_n^{j_n}\,, \qquad \text{where} \quad J = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n\,.$$

For $\mu \subseteq \{\,\partial_1, \ldots, \partial_n\,\}$ we consider the monoid

$$\mathrm{Mon}(\mu) \;:=\; \{\,\partial^J \mid J = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n,\ j_i = 0 \text{ for all } i \text{ such that } \partial_i \notin \mu\,\}$$

with the usual divisibility relation $|$, and we let $\mathrm{Mon}(D) := \mathrm{Mon}(\{\,\partial_1, \ldots, \partial_n\,\})$. An ideal of D which is generated by monomials is called a *monomial ideal*.

Example 2.5. The system of linear PDEs

$$\frac{\partial^2 u}{\partial x\,\partial y} \;=\; 0\,, \qquad \frac{\partial^4 u}{\partial x^3\,\partial z} \;=\; 0\,, \qquad \frac{\partial^4 u}{\partial x\,\partial y^2\,\partial z} \;=\; 0\,, \qquad \frac{\partial^5 u}{\partial x^2\,\partial y\,\partial z^2} \;=\; 0 \tag{2.5}$$

for the unknown function $u = u(x, y, z)$ defines the monomial ideal I of $K[\partial_x, \partial_y, \partial_z]$ which is generated by $\partial_x\partial_y$, $\partial_x^3\partial_z$, $\partial_x\partial_y^2\partial_z$, $\partial_x^2\partial_y\partial_z^2$. The ideal I encodes all consequences of (2.5).

Remark 2.6. Let the ideal I of D be generated by monomials $m_1, \ldots, m_r$. Then every *monomial* in I is a multiple of some m_i. The set of all monomials in I is a multiple-closed subset of $\mathrm{Mon}(D)$ in the sense of the following definition.

Definition 2.7. A set $S \subseteq \mathrm{Mon}(D)$ is said to be $\mathrm{Mon}(\mu)$-*multiple-closed*, $\mu \subseteq \{\,\partial_1, \ldots, \partial_n\,\}$, if

$$m\,s \in S \qquad \text{for all} \quad m \in \mathrm{Mon}(\mu)\,, \quad s \in S\,.$$

Every set $G \subseteq \mathrm{Mon}(D)$ satisfying

$$\mathrm{Mon}(\mu)\,G \;=\; \{\,m\,g \mid m \in \mathrm{Mon}(\mu),\, g \in G\,\} \;=\; S$$

is called a *generating set* for the $\mathrm{Mon}(\mu)$-multiple-closed set S.

Example 2.8. Let $D = K[\partial_1, \partial_2]$ and $G := \{\,\partial_1\partial_2^2,\ \partial_1^3\partial_2,\ \partial_1^4\,\}$. We consider the $\mathrm{Mon}(D)$-multiple-closed set S generated by G. If we visualize the monomial $\partial_1^i\partial_2^j$ as the point (i, j) in the positive quadrant of a two-dimensional coordinate system, then the set S of monomials can be viewed as the discrete set of points in the upper-right region in Figure 1.

The following combinatorial fact is also referred to as Dickson's Lemma.

Lemma 2.9. *Every* $\mathrm{Mon}(D)$*-multiple-closed subset of* $\mathrm{Mon}(D)$ *has a finite generating set. Equivalently, every ascending chain of* $\mathrm{Mon}(D)$*-multiple-closed subsets of* $\mathrm{Mon}(D)$ *terminates.*

In other words, every sequence of monomials in which no monomial has a divisor among the previous ones is finite.

Exercise. Prove Lemma 2.9 by induction on n.

Remark 2.10. Every multiple-closed set has a unique minimal generating set. It is obtained from any generating set G by removing all elements which have a proper divisor in G.

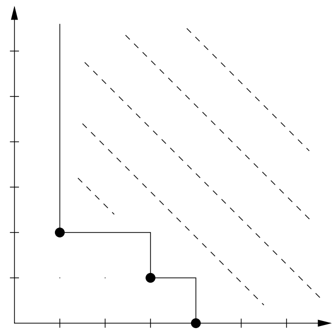

Figure 1: $\mathrm{Mon}(D)$-multiple-closed set S

Example 2.11. The multiple-closed set generated by $\partial_x\partial_y$, $\partial_x^3\partial_z$, $\partial_x\partial_y^2\partial_z$, $\partial_x^2\partial_y\partial_z^2$ in Example 2.5 has minimal generating set $\{\, \partial_x\partial_y,\ \partial_x^3\partial_z \,\}$.

We are going to partition multiple-closed sets (and, more importantly, their complements in $\mathrm{Mon}(D)$) into cones of monomials, one instrumental fact being that the latter are again $\mathrm{Mon}(\mu)$-multiple-closed sets for some $\mu \subseteq \{\, \partial_1, \ldots, \partial_n \,\}$. For a set S let $\mathcal{P}(S)$ be its power set.

Definition 2.12. (a) A pair $(C, \mu) \in \mathcal{P}(\mathrm{Mon}(D)) \times \mathcal{P}(\{\, \partial_1, \ldots, \partial_n \,\})$ is called a *cone* if there exists $v \in C$ such that

$$\mathrm{Mon}(\mu)\, v \;=\; \{\, m\, v \mid m \in \mathrm{Mon}(\mu) \,\} \;=\; C\,.$$

The elements of μ are called the *multiplicative variables*, those of $\overline{\mu} := \{\, \partial_1, \ldots, \partial_n \,\} \setminus \mu$ the *non-multiplicative variables* for (C, μ) (or simply for C, or for v). We often also refer to the cone C by the pair (v, μ), where v is the generator of C.

(b) Let $S \subseteq \mathrm{Mon}(D)$. A *cone decomposition of* S is a finite set $\{\, (m_1, \mu_1), \ldots, (m_r, \mu_r) \,\}$ of cones such that $C_i := \mathrm{Mon}(\mu_i)\, m_i$, $i = 1, \ldots, r$, satisfy

$$\bigcup_{i=1}^{r} C_i \;=\; S \quad \text{and} \quad C_i \cap C_j \;=\; \emptyset \quad \text{for all} \quad i \neq j\,.$$

Example 2.13. A cone decomposition of the multiple-closed set S defined in Example 2.8 is

$$\{\, (\partial_1^4, \{\partial_1, \partial_2\}), \quad (\partial_1^3\partial_2, \{\partial_2\}), \quad (\partial_1^2\partial_2^2, \{\partial_2\}), \quad (\partial_1\partial_2^2, \{\partial_2\}) \,\}\,,$$

which is visualized in Figure 2.

Remark 2.14. A cone decomposition of $S \subseteq \mathrm{Mon}(D)$ defines a restriction of the usual divisibility relation of monomials as follows. A monomial $m \in \mathrm{Mon}(D)$ is divisible by a generator m' of a cone (m', μ) if and only if there exists $m'' \in \mathrm{Mon}(\mu)$ such that $m = m''\, m'$. The disjointness of the cone decomposition entails that among cone generators the divisor is unique.

Given a finite set $\{\, m_1, \ldots, m_r \,\}$ of monomials, there are many possible ways of how to arrange sets of multiplicative variables $\mu_1, \ldots, \mu_r$ such that $\{\, (m_1, \mu_1), \ldots, (m_r, \mu_r) \,\}$ is a set of disjoint cones. These possibilities are addressed by the notion of *involutive division* which was introduced by V. P. Gerdt, Y. A. Blinkov, A. Y. Zharkov [GB98a, GB98b, ZB96], cf. also [Ape98]. Important for this exposition is only the Janet division:

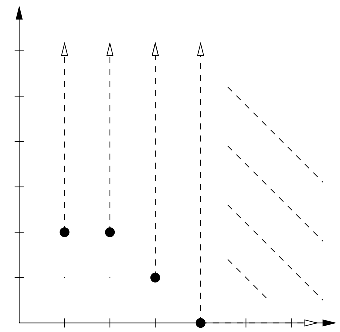

Figure 2: A cone decomposition of S

Definition 2.15. For a finite subset M of $\mathrm{Mon}(D)$ *Janet division* defines the set $\mu = \mu(m, M)$ of multiplicative variables for each $m \in M$ as follows. Let $m = \partial_1^{i_1} \ldots \partial_n^{i_n}$.

$$\partial_k \in \mu \quad :\Longleftrightarrow \quad i_k = \max \{ \, j_k \mid \partial_1^{j_1} \ldots \partial_n^{j_n} \in M \text{ with } j_1 = i_1, \, j_2 = i_2, \, \ldots, \, j_{k-1} = i_{k-1} \, \}.$$

This definition assumes the ordering $\partial_1, \partial_2, \ldots, \partial_n$ of the variables; a different ordering may be used as well. There are also other common involutive divisions. For instance, J. Thomas [Tho37] proposed another way of defining the multiplicative variables of cones; still another one is named after J.-F. Pommaret (cf., e.g., [Jan29, no. 58], [Pom94, p. 90], [Sei10]).

Example 2.16. For $M = \{ \, \partial_1^2 \partial_2, \, \partial_1^2 \partial_3, \, \partial_2^2 \partial_3, \, \partial_2 \partial_3^2 \, \}$ Janet division associates the sets $\mu(m, M)$ of multiplicative variables to the elements $m \in M$ as indicated in the following table, where we replace non-multiplicative variables in the set $\{ \, \partial_1, \ldots, \partial_n \, \}$ with the symbol '$*$'.

$$\begin{array}{rl} \partial_1^2 \, \partial_2, & \{ \partial_1, \partial_2, \partial_3 \} \\ \partial_1^2 \, \partial_3, & \{ \partial_1, \, * \, , \partial_3 \} \\ \partial_2^2 \, \partial_3, & \{ \, * \, , \partial_2, \partial_3 \} \\ \partial_2 \, \partial_3^2, & \{ \, * \, , \, * \, , \partial_3 \} \end{array}$$

Definition 2.17. A finite subset M of $\mathrm{Mon}(D)$ is said to be *Janet complete* if

$$\bigcup_{m \in M} \mathrm{Mon}(\mu(m, M)) \, m \; = \; \bigcup_{m \in M} \mathrm{Mon}(D) \, m \, ,$$

i.e., if every monomial which is divisible by some monomial in M is obtained by multiplying a certain $m \in M$ by multiplicative variables for m only. (Recall that the left hand side of the above equation is a disjoint union.)

Example 2.18. The set M in Example 2.16 is not Janet complete because, e.g., the monomial $\partial_1 \, \partial_2^2 \, \partial_3$ is not obtained as a multiple of any $m \in M$ when multiplication is restricted to multiplicative variables for m. By adding this monomial and the monomial $\partial_1 \, \partial_2 \, \partial_3^2$ to M, we obtain the following Janet complete superset of M in $\mathrm{Mon}(D)$.

$$\begin{array}{rl} \partial_1^2 \, \partial_2, & \{ \partial_1, \partial_2, \partial_3 \} \\ \partial_1^2 \, \partial_3, & \{ \partial_1, \, * \, , \partial_3 \} \\ \partial_1 \, \partial_2^2 \, \partial_3, & \{ \, * \, , \partial_2, \partial_3 \} \\ \partial_1 \, \partial_2 \, \partial_3^2, & \{ \, * \, , \, * \, , \partial_3 \} \\ \partial_2^2 \, \partial_3, & \{ \, * \, , \partial_2, \partial_3 \} \\ \partial_2 \, \partial_3^2, & \{ \, * \, , \, * \, , \partial_3 \} \end{array}$$

Remark 2.19. Every finite subset M of $\mathrm{Mon}(D)$ can be augmented to a Janet complete finite set by adding certain monomials which are products of some $m \in M$ and a monomial which is divisible by at least one non-multiplicative variable for m.

Proposition 2.20. *Let I be a monomial ideal of D. Let an ordering of $\partial_1, \ldots, \partial_n$ be fixed. There exists a unique finite Janet complete generating set of monomials for I which is minimal with respect to set inclusion.*

Definition 2.21. Let $S \subseteq \mathrm{Mon}(D)$. We refer to the minimal Janet complete superset of S as the *Janet completion* of S. Its elements are the generators of cones in a cone decomposition of the multiple-closed set generated by S, which we call a *Janet decomposition*.

Exercise. Write an algorithm which computes a Janet decomposition of the complement of a multiple-closed set of monomials in $\mathrm{Mon}(D)$.

Cone decompositions of the *complement* of a multiple-closed set in $\mathrm{Mon}(D)$ which are defined by Janet division will be referred to as *Janet decompositions* as well.

Exercise. Write an algorithm which computes the Janet completion of a given finite set of monomials.

Definition 2.22. For any set $S \subseteq \mathrm{Mon}(D)$ of monomials, the *generalized Hilbert series* of S is the formal power series

$$H_S(\partial_1, \ldots, \partial_n) \; := \; \sum_{m \in S} m \in \mathbb{Z}[[\partial_1, \ldots, \partial_n]] \, .$$

Remark 2.23. The Hilbert series usually encountered in commutative algebra is obtained from the generalized Hilbert series as $H_S(\lambda, \ldots, \lambda)$ for an indeterminate λ.

The next remark shows that the computation of the generalized Hilbert series of a set S of monomials is trivial if a decomposition of S into disjoint cones is available.

Remark 2.24. Let $C = (m, \mu)$ be a cone. We use the geometric series

$$\frac{1}{1-x} \; = \; \sum_{i \geq 0} x^i$$

to write down the generalized Hilbert series $H_C(\partial_1, \ldots, \partial_n)$ as follows:

$$H_C(\partial_1, \ldots, \partial_n) \; = \; \frac{m}{\prod_{x \in \mu}(1-x)} \, .$$

More generally, every decomposition of a $\mathrm{Mon}(D)$-multiple-closed set S into disjoint cones allows to compute the generalized Hilbert series of S by adding the generalized Hilbert series of the cones. In an analogous way this applies to the complements of multiple-closed sets.

Example 2.25. The complement in $\mathrm{Mon}(D)$ of the multiple-closed set generated by $\partial_x \partial_y$, $\partial_x^3 \partial_z$, $\partial_x \partial_y^2 \partial_z$, $\partial_x^2 \partial_y \partial_z^2$ in Example 2.5 admits the following Janet decomposition:

$$\begin{aligned} 1, &\quad \{\, * \,, \partial_y, \partial_z \,\}, \\ \partial_x, &\quad \{\, * \,, \, * \,, \partial_z \,\}, \\ \partial_x^2, &\quad \{\, * \,, \, * \,, \partial_z \,\}, \\ \partial_x^3, &\quad \{\, \partial_x, \, * \,, \, * \,\}. \end{aligned}$$

The corresponding generalized Hilbert series is

$$\frac{1}{(1-\partial_y)(1-\partial_z)} + \frac{\partial_x}{1-\partial_z} + \frac{\partial_x^2}{1-\partial_z} + \frac{\partial_x^3}{1-\partial_x} \, .$$

2.2 Janet's algorithm

Given a system of linear PDEs, *Janet's algorithm* computes an equivalent system, called a *Janet basis*, for which it is a straightforward task to decide whether another linear PDE is a consequence of the system or not. The answer is obtained by trying to express the PDE as a linear combination of partial derivatives of the Janet basis elements. This process is based on a multivariate polynomial division for elements of $D = K[\partial_1, \ldots, \partial_n]$, which requires a choice of most significant term in each non-zero polynomial, called *leading term*.

Suppose that a total ordering $>$ on $\mathrm{Mon}(D)$ is chosen which is compatible with multiplication (i.e., composition of operators). By defining leading terms of PDEs with respect to $>$, the leading terms of consequences of *one* PDE are predictable: the leading term of a derivative of a PDE is the derivative of the leading term of the PDE.

A total ordering $>$ with the above property also allows to easily determine the monomials in $\partial_1, \ldots, \partial_n$ that do *not* occur in leading terms of consequences of a system of linear PDEs. Hence, a Janet basis then also allows to determine all analytic solutions (around a sufficiently generic point). By choosing the total ordering $>$ appropriately, further tasks, e.g., elimination of variables, can be solved as well.

The methods to be discussed in this section can be applied in a similar way to other types of linear equations, e.g., difference equations, multidimensional discrete equations, time-delay equations and other functional equations. The coefficients of these equations may be constant or not, corresponding to commutative or non-commutative rings of operators, e.g., Ore algebras (cf., e.g., [CS98], [CQR05]). For example, singular points of differential equations may be studied in terms of D-modules [Kas03, Cou95], i.e., modules over Weyl algebras and related rings of differential operators.

Last but not least, Janet's algorithm applies in the same way to systems of polynomial equations, i.e., equations defining algebraic varieties. Hence, it is an alternative to Buchberger's algorithm computing Gröbner bases. In fact, every Janet basis is a Gröbner basis. Generalizations of Gröbner bases to non-commutative polynomial algebras have been studied since a couple of decades, cf., e.g., [KRW90], [Kre93], [Mor94], [Lev05], [GL11]; for rings of differential operators, cf., e.g., [CJ84], [Gal85], [IP98], [SST00]. Buchberger's algorithm was adapted to Ore algebras by F. Chyzak (cf. [Chy98], [CS98], where it is also applied to the study of special functions and combinatorial sequences). Involutive divisions were studied for the Weyl algebra in [HSS02] and were extended to non-commutative rings in [EW07].

In this section we confine ourselves to linear PDEs with constant coefficients, but these may involve q unknown functions. Note that we ignore efficiency issues in favor of a concise formulation of Janet's algorithm.

Let $D = K[\partial_1, \ldots, \partial_n]$, where K is a field of constants, $q \in \mathbb{N}$ and $e_1, \ldots, e_q$ the standard basis vectors of the free left D-module $D^{1\times q}$. We define the set of monomials of $D^{1\times q}$ to be

$$\mathrm{Mon}(D^{1\times q}) \ := \ \bigcup_{i=1}^{q} \mathrm{Mon}(D)\, e_i \, .$$

Every $p \in D^{1\times q}$ has a unique representation

$$p \ = \ \sum_{k=1}^{q} \sum_{m \in \mathrm{Mon}(D)} c_{k,m}\, m\, e_k \tag{2.6}$$

as linear combination of monomials in $\mathrm{Mon}(D^{1\times q})$ with coefficients $c_{k,m} \in K$, where only finitely many $c_{k,m}$ are non-zero.

Definition 2.26. A *term ordering* $>$ on $\mathrm{Mon}(D^{1\times q})$ (or on $D^{1\times q}$) is a total ordering on $\mathrm{Mon}(D^{1\times q})$ which satisfies the following two conditions.

(a) For all $1 \le i \le n$ and $1 \le k \le q$ we have $\partial_j\, e_k > e_k$.

(b) For all $m_1\, e_k,\ m_2\, e_l \in \mathrm{Mon}(D^{1\times q})$ the following implication holds:

$$m_1\, e_k > m_2\, e_l \quad \Longrightarrow \quad \partial_j\, m_1\, e_k > \partial_j\, m_2\, e_l \qquad \text{for all } j = 1, \ldots, n\,.$$

Let a term ordering $>$ be fixed. For every $p \in D^{1\times q} \setminus \{0\}$ the greatest monomial, with respect to $>$, occurring (with non-zero coefficient) in the representation (2.6) of p is uniquely determined and is called the *leading monomial* of p, denoted by $\mathrm{lm}(p)$. The coefficient of $\mathrm{lm}(p)$ is called the *leading coefficient* of p, denoted by $\mathrm{lc}(p)$. For any subset $S \subseteq D^{1\times q}$ we define

$$\mathrm{lm}(S) \ := \ \{\, \mathrm{lm}(p) \mid 0 \neq p \in S \,\}\,.$$

Remark 2.27. Every term ordering on $D^{1\times q}$ is a well-ordering, i.e., every descending sequence of elements of $\mathrm{Mon}(D^{1\times q})$ terminates.

Example 2.28. The *lexicographical ordering (lex)* on $\mathrm{Mon}(D)$ (which extends the ordering $\partial_1 > \partial_2 > \ldots > \partial_n$) is defined for monomials $m_1 = \partial_1^{a_1} \ldots \partial_n^{a_n}$, $m_2 = \partial_1^{b_1} \ldots \partial_n^{b_n} \in \mathrm{Mon}(D)$ by

$$m_1 > m_2 \quad :\Longleftrightarrow \quad m_1 \neq m_2 \quad \text{and} \quad a_j > b_j \quad \text{for} \quad j = \min\{\, 1 \le i \le n \mid a_i \neq b_i \,\}\,.$$

Example 2.29. The *degree-reverse lexicographical ordering (degrevlex)* on $\mathrm{Mon}(D)$ (extending the ordering $\partial_1 > \ldots > \partial_n$) is defined for $m_1 = \partial_1^{a_1} \ldots \partial_n^{a_n}$, $m_2 = \partial_1^{b_1} \ldots \partial_n^{b_n} \in \mathrm{Mon}(D)$ by

$$m_1 > m_2 \quad :\Longleftrightarrow \quad \left\{ \begin{array}{l} \deg(m_1) > \deg(m_2) \quad \text{or} \\[1ex] \big(\ \deg(m_1) = \deg(m_2) \quad \text{and} \quad m_1 \neq m_2 \quad \text{and} \quad a_j < b_j \\[1ex] \quad \text{for} \quad j = \max\{\, 1 \le i \le n \mid a_i \neq b_i \,\}\ \big), \end{array} \right.$$

where deg refers to the total degree.

Example 2.30. Two ways of extending a given term ordering $>_1$ on $\mathrm{Mon}(D)$ to $\mathrm{Mon}(D^{1\times q})$ for $q > 1$ are often used. The *term-over-position ordering* (extending $>_1$ and the total ordering $e_1 > \ldots > e_q$ of the standard basis vectors) is defined for $m_1,\ m_2 \in \mathrm{Mon}(D)$ by

$$m_1\, e_i > m_2\, e_j \quad :\Longleftrightarrow \quad m_1 >_1 m_2 \quad \text{or} \quad (m_1 = m_2 \quad \text{and} \quad i < j)\,.$$

Accordingly, the *position-over-term ordering* (extending $>_1$ and $e_1 > \ldots > e_q$) is defined by

$$m_1\, e_i > m_2\, e_j \quad :\Longleftrightarrow \quad i < j \quad \text{or} \quad (i = j \quad \text{and} \quad m_1 >_1 m_2)\,.$$

In what follows, we assume that a term ordering $>$ on $D^{1\times q}$ is fixed.

Let M be a submodule of $D^{1\times q}$. Note that $\mathrm{lm}(M)$ is a $\mathrm{Mon}(D)$-multiple-closed set. More precisely, for each $k \in \{1, \ldots, q\}$,

$$\{\, m \in \mathrm{Mon}(D) \mid m\, e_k \in \mathrm{lm}(M) \,\}$$

is a $\mathrm{Mon}(D)$-multiple-closed set as discussed in Section 2.1.

Starting with a finite generating set L of M, Janet's algorithm possibly removes elements from L and inserts new elements of M into L repeatedly in order to finally achieve that the $\mathrm{Mon}(D)$-multiple-closed set generated by $\mathrm{lm}(L)$ equals $\mathrm{lm}(M)$. An element $p \in L$ is removed if it is reduced to zero by subtraction of suitable multiples of other elements of L.

We denote by ${}_D\langle\, L\,\rangle$ the submodule of $D^{1\times q}$ generated by $L \subseteq D^{1\times q}$.

For $G \subseteq \mathrm{Mon}(D^{1\times q})$ we denote by $[\,G\,]$ the $\mathrm{Mon}(D)$-multiple-closed set generated by G. If $G = \{\, m_1, \ldots, m_r \,\}$, then we also write $[\, m_1, \ldots, m_r \,]$ for $[\,G\,]$.

Definition 2.31. Let $T = \{\, (b_1, \mu_1), \ldots, (b_t, \mu_t) \,\}$ with $b_i \in D^{1\times q} \setminus \{0\}$ and $\mu_i \subseteq \{\, \partial_1, \ldots, \partial_n \,\}$.

(a) The set T is *Janet complete* if $\{\, \mathrm{lm}(b_1), \ldots, \mathrm{lm}(b_t) \,\}$ equals its Janet completion and, for each $i \in \{1, \ldots, t\}$, μ_i is the set of multiplicative variables of the cone with generator $\mathrm{lm}(b_i)$ in the Janet decomposition $\{\, (\mathrm{lm}(b_1), \mu_1), \ldots, (\mathrm{lm}(b_t), \mu_t) \,\}$ of $[\, \mathrm{lm}(b_1), \ldots, \mathrm{lm}(b_t) \,]$.

(b) An element $p \in D^{1\times q}$ is *Janet reducible modulo* T if there exist $(b, \mu) \in T$ and a monomial $m \in \mathrm{Mon}(D^{1\times q})$ which occurs in p such that $m \in \mathrm{Mon}(\mu)\,\mathrm{lm}(b)$. In this case, (b, μ) is called a *Janet divisor* of p. If p is not Janet reducible modulo T, then p is also said to be *Janet reduced modulo* T.

The following algorithm subtracts suitable multiples of Janet divisors from a given element $p \in D^{1\times q}$ as long as a term in p is Janet reducible modulo T.

Algorithm 2.32 (Janet-reduce)**.**

Input: $p \in D^{1\times q}$, $T = \{\, (b_1, \mu_1), \ldots, (b_t, \mu_t) \,\}$, and a term ordering $>$ on $D^{1\times q}$, where T is Janet complete (with respect to $>$, cf. Def. 2.31)

Output: $r \in D^{1\times q}$ such that $r + {}_D\langle\, b_1, \ldots, b_t \,\rangle = p + {}_D\langle\, b_1, \ldots, b_t \,\rangle$ and r is Janet reduced modulo T

Algorithm:

$p' \leftarrow p$; $\quad r \leftarrow 0$
while $p' \neq 0$ **do**
  if $\exists\, (b, \mu) \in T$ such that $\mathrm{lm}(p') \in \mathrm{Mon}(\mu)\,\mathrm{lm}(b)$ **then** // (b, μ) *is a Janet divisor of* p'
    $p' \leftarrow p' - \frac{\mathrm{lc}(p')}{\mathrm{lc}(b)} \frac{\mathrm{lm}(p')}{\mathrm{lm}(b)}\, b$
  else
    subtract the term of p' with monomial $\mathrm{lm}(p')$ from p' and add it to r
  end if
end while
return r

Remark 2.33. Algorithm 2.32 terminates because, as long as p' is non-zero, the leading monomial of p' decreases with respect to the term ordering $>$, which is a well-ordering. Its correctness is clear. The result r is uniquely determined for the given input because every monomial has at most one Janet divisor in T, and also the course of Algorithm 2.32 is uniquely determined as opposed to reduction procedures which apply multivariate polynomial division without distinguishing between multiplicative and non-multiplicative variables.

Remark 2.34. Let p_1, $p_2 \in D^{1\times q}$ and T be as in the input of Algorithm 2.32. In general, the equality $p_1 + {}_D\langle\, b_1, \ldots, b_t \,\rangle = p_2 + {}_D\langle\, b_1, \ldots, b_t \,\rangle$ does not imply that the results of applying Janet-reduce to p_1 and p_2, respectively, are equal. However, in Thm. 2.38 (d) it is shown that, if T is a Janet basis, then the result of Janet-reduce constitutes a unique representative for every coset in $D^{1\times q}/{}_D\langle\, b_1, \ldots, b_t \,\rangle$. It is called the *Janet normal form of* p_1 *(or* p_2*) modulo* T.

Definition 2.35. Let $T = \{\, (b_1, \mu_1), \ldots, (b_t, \mu_t) \,\}$ be Janet complete (as in Definition 2.31 (a)). We write $\mathrm{NF}(p, T, >)$ for the result of Algorithm 2.32 (Janet-reduce) applied to p, T, $>$. The set T is said to be *passive* if

$$\mathrm{NF}(v \cdot b_i, T, >) = 0 \qquad \text{for all} \quad v \in \overline{\mu_i}, \quad i = 1, \ldots, t. \tag{2.7}$$

In this case T is also called a *Janet basis for* ${}_D\langle\, b_1, \ldots, b_t \,\rangle$, and $\{\, b_1, \ldots, b_t \,\}$ is often referred to as a Janet basis for ${}_D\langle\, b_1, \ldots, b_t \,\rangle$ as well.

Remark 2.36. Let M be a submodule of $D^{1\times q}$. By applying Janet's algorithm to a finite generating set L of M, an ascending chain of multiple-closed subsets of $\mathrm{lm}(M)$ is constructed. This chain terminates by Lemma 2.9. In each round, a Janet decomposition is computed for the current multiple-closed set generated by the leading monomials of a generating set for M. In order to obtain the minimal Janet complete set of monomials, the generating set for M is first turned into an auto-reduced one, i.e., no leading monomial of a generator divides (in the conventional sense) the leading monomial of another generator.

Let $M = {}_D\langle\, b_1, \ldots, b_t \,\rangle$. Then every element of M is a D-linear combination of b_1, …, b_t. Suppose that T is passive. Each summand $k_i\, m_i\, b_i$ in such a linear combination, where $k_i \in K$ and $m_i \in \mathrm{Mon}(D)$ involves some variable that is non-multiplicative for b_i, can be replaced with a K-linear combination of elements in $\mathrm{Mon}(\mu_1)\, b_1$, …, $\mathrm{Mon}(\mu_t)\, b_t$. Due to the passivity condition (2.7), this can be achieved by applying successively Algorithm 2.32 to terms involving only one non-multiplicative variable. This substitution process should deal with the largest term with respect to $>$ first. Elimination of all non-multiplicative variables demonstrates that the leading monomial of every $p \in M \setminus \{0\}$ has a Janet divisor in T. We conclude that passivity of the Janet complete set T is equivalent to $[\, \mathrm{lm}(b_1), \ldots, \mathrm{lm}(b_t) \,] = \mathrm{lm}(M)$.

Recall that for any set S we denote by $\mathcal{P}(S)$ the power set of S.

Algorithm 2.37 (JanetBasis)**.**

Input: A finite set $L \subseteq D^{1\times q}$, a term ordering $>$ on $D^{1\times q}$, and an ordering of ∂_1, …, ∂_n for Janet division

Output: A finite subset J of $D^{1\times q} \times \mathcal{P}(\{\, \partial_1, \ldots, \partial_n \,\})$ such that ${}_D\langle\, p \mid (p, \mu) \in J \,\rangle = {}_D\langle\, L \,\rangle$ (and $J = \emptyset$ if and only if ${}_D\langle\, L \,\rangle = \{0\}$)

Algorithm:

$G \leftarrow L$
repeat
  $G \leftarrow \text{Auto-reduce}(G, >)$ // *cf. Rem. 2.36*
  $J \leftarrow \text{Janet-decompose}(G)$
  $P \leftarrow \{\, \mathrm{NF}(v \cdot p, J, >) \mid (p, \mu) \in J,\ v \in \overline{\mu} \,\}$ // *cf. Alg. 2.32*
  $G \leftarrow \{\, p \mid (p, \mu) \in J \,\} \cup P$
until $P \subseteq \{0\}$
return J

Theorem 2.38 ([Rob14], Thm. 2.1.43)**.**

(a) Algorithm 2.37 terminates and is correct.

(b) A K-basis of ${}_D\langle\, L\,\rangle$ is given by $\biguplus_{(p,\mu)\in J} \mathrm{Mon}(\mu)p$, where J is the result of Algorithm 2.37. In particular, every $r \in {}_D\langle\, L\,\rangle$ has a unique representation

$$r \;=\; \sum_{(p,\mu)\in J} c_{(p,\mu)}\, p,$$

where each $c_{(p,\mu)} \in D$ is a K-linear combination of elements in $\mathrm{Mon}(\mu)$.

(c) The cosets in $D^{1\times q}/{}_D\langle\, L\,\rangle$ with representatives in

$$\mathrm{Mon}(D^{1\times q}) \setminus [\, \mathrm{lm}(p) \mid (p,\mu)\in J\,]$$

form a K-basis of $D^{1\times q}/{}_D\langle\, L\,\rangle$.

(d) For every r_1, $r_2 \in D^{1\times q}$ the following equivalence holds.

$$r_1 + {}_D\langle\, L\,\rangle = r_2 + {}_D\langle\, L\,\rangle \iff \mathrm{NF}(r_1, J, >) = \mathrm{NF}(r_2, J, >).$$

We present a small example illustrating the idea of Janet's algorithm.

Example 2.39. Let $D = K[x, y]$ be the commutative polynomial algebra over a field K of arbitrary characteristic or over $K = \mathbb{Z}$. We choose the degree-reverse lexicographical ordering on $\mathrm{Mon}(D)$ satisfying $x > y$ (cf. Ex. 2.29). Let the ideal I of D be generated by

$$g_1 \;:=\; \underline{x^2} - y, \quad g_2 \;:=\; \underline{xy} - y\,.$$

Using the ordering x, y of the variables, the Janet decomposition of the multiple-closed set which is generated by the underlined leading monomials of g_1 and g_2 is

$$\{\, (x^2,\, \{x, y\}),\; (xy,\, \{y\})\,\}\,.$$

This result indicates that we need to check whether $f := x \cdot g_2$ can be written as

$$f = c_1 \cdot (x^2 - y) + c_2 \cdot (xy - y), \qquad c_1 \in K[x, y], \quad c_2 \in K[y]. \tag{2.8}$$

The monomials appearing in $f = x^2y - xy \in I$ lie in the cones $(x^2, \{x, y\})$ and $(xy, \{y\})$, respectively. Reduction yields $g_3 := y^2 - y \in I$, which does not have a representation as in (2.8). So, we include g_3 in our list of generators, and for this example, we already arrive at the (minimal) Janet basis $\{\, (g_1, \{x, y\}), (g_2, \{y\}), (g_3, \{y\})\,\}$ for I.

No division by any coefficient was necessary to arrive at a Janet basis for I. The statements above therefore hold for a field K of any characteristic and for $K = \mathbb{Z}$.

Remark 2.40. Janet's algorithm can also be performed over the integers $\mathbb{Z}$ to obtain Janet bases for submodules of $\mathbb{Z}[x_1, \ldots, x_n]^{1\times q}$. To this end, the monomial ordering $<$ is extended to terms $a \cdot m$, where $a \in \mathbb{Z}$ and m is a monomial, such that the absolute values of the coefficients of two terms with equal monomials are compared to break ties. Essentially the only other necessary modification of Janet's algorithm is to replace Algorithm 2.32 by a corresponding method which uses Euclidean division instead of exact division for the coefficients $\mathrm{lc}(p')$, $\mathrm{lc}(b)$.

Remark 2.41. The K-vector space $\mathcal{F} := \hom_K(D, K)$ is a (left) D-module with action

$$D \times \mathcal{F} \longrightarrow \mathcal{F} \colon (d, f) \longmapsto (a \mapsto f(a \cdot d))\,,$$

and the following K-bilinear form is non-degenerate in both arguments:

$$(\ ,\ ) \colon D \times \mathcal{F} \longrightarrow K \colon (d, f) \longmapsto f(d)\,. \tag{2.9}$$

Hence, D and $\mathcal{F}$ are dual to each other. The linear map $D \to D$ defined by right multiplication by d and the linear map $\mathcal{F} \to \mathcal{F}$ given by left multiplication by d are adjoint to each other:

$$(a \cdot d, f) = f(a \cdot d) = (d \cdot f)(a) = (a, d \cdot f), \qquad a \in D, \quad f \in \mathcal{F}\,. \tag{2.10}$$

Since every homomorphism $f \in \mathcal{F}$ is uniquely determined by its values for the elements of the K-basis $\mathrm{Mon}(D)$ of D, we can write f in a unique way as a (not necessarily finite) formal sum

$$\sum_{m \in \mathrm{Mon}(D)} (m, f)\, m\,. \tag{2.11}$$

Due to (2.10), for every $d \in D$ the representation of $d \cdot f$ can be obtained from

$$\sum_{m \in \mathrm{Mon}(D)} (m, d \cdot f)\, m \;=\; \sum_{m \in \mathrm{Mon}(D)} (m \cdot d, f)\, m\,. \tag{2.12}$$

By writing the monomials in the sum (2.11) in indeterminates $z_1, \ldots, z_n$, we identify $\mathcal{F}$ with the K-algebra $K[[z_1, \ldots, z_n]]$ of formal power series. It follows from (2.12) that the (left) action on $\mathcal{F}$ of any monomial in D effects a shift of the coefficients of the power series according to the exponent vector of the monomial, which is the same action as the one defined by partial differentiation. Therefore, the K-vector space bases $(z^\alpha/\alpha! \mid \alpha \in (\mathbb{Z}_{\geq 0})^n)$ and $(\partial^\beta \mid \beta \in (\mathbb{Z}_{\geq 0})^n)$ are dual to each other with respect to the pairing (2.9), i.e.,

$$\left(\partial^\beta, \textstyle\sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} c_\alpha \frac{z^\alpha}{\alpha!}\right) = c_\beta, \qquad \beta \in (\mathbb{Z}_{\geq 0})^n, \qquad \alpha! := \alpha_1! \cdot \ldots \cdot \alpha_n!\,.$$

Suppose that a system of (homogeneous) linear PDEs with constant coefficients for one unknown function of n arguments is given. We compute a Janet basis J for the ideal of D which is generated by the left hand sides p of these equations with respect to the term ordering $>$. The differential equations are considered as linear equations for (∂^β, f), $\beta \in (\mathbb{Z}_{\geq 0})^n$, where $f \in \mathcal{F}$ is a formal power series solution, and using the term ordering $>$, we may solve each of these equations for $(\mathrm{lm}(p), f)$. Then Janet's algorithm partitions $\mathrm{Mon}(D)$ into a set of monomials m for which $(m, f) \in K$ can be chosen arbitrarily and a set S of monomials for which $(\mathrm{lm}(p), f) \in K$ is uniquely determined by these choices. The latter set is the multiple-closed subset $S := [\,\mathrm{lm}(p) \mid (p, \mu) \in J\,]$ of $\mathrm{Mon}(D)$. In particular, the K-dimension of the space of formal power series solutions, if finite, can be computed as the number of monomials in the complement C of S in $\mathrm{Mon}(D)$. In fact, the generalized Hilbert series $H_C(\partial_1, \ldots, \partial_n)$ of C enumerates a basis for the Taylor coefficients (∂^β, f) of f whose values can be assigned freely.

M. Janet calls the monomials ∂^β in $\mathrm{Mon}(D) \setminus S$ *parametric derivatives* because the corresponding Taylor coefficients (∂^β, f) of a formal power series solution f can be chosen arbitrarily. The monomials in S are called *principal derivatives* [Jan29, e.g., no. 22, no. 38]. The Taylor coefficients (∂^β, f) which correspond to principal derivatives ∂^β are uniquely determined by K-linear equations in terms of the Taylor coefficients of parametric derivatives. Of course, the extension of this method of determining the formal power series solutions of a system of linear partial differential equations is extended to the case of more than one unknown function in a straightforward way by using submodules of $D^{1 \times q}$ instead of ideals of D.

Note that convergence of series solutions is to be investigated separately.

For a similar treatment of partial difference equations, we refer to [OP01].

Remark 2.42. The previous remark also applies to linear systems of partial differential equations whose coefficients are rational functions in the independent variables $x_1, \ldots, x_n$, i.e. $D = K[\partial_1, \ldots, \partial_n]$ is replaced by $B_n(K) := K(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$. Of course, in this case a formal power series solution is only well-defined if the left submodule M of $B_n(K)^{1\times q}$ which represents the left hand sides of the equations is also a left submodule of $A[\partial_1, \ldots, \partial_n]^{1\times q}$, where A is a K-subalgebra of $B_n(K)$ whose elements do not have a pole in $0 \in K^n$ and the Janet basis for M is computed within $A[\partial_1, \ldots, \partial_n]^{1\times q}$. In other words, a formal power series solution is only well-defined if $0 \in K^n$ is not a zero of any denominator occurring in the course of Janet's algorithm.

Example 2.43. [Jan29, no. 23] Let us consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.13}$$

for an unknown real analytic function u of t and x. The corresponding operator is $p := \partial_t - \partial_x^2 \in D := K[\partial_t, \partial_x]$, where $K = \mathbb{Q}$ or $\mathbb{R}$. Choosing a degree-reverse lexicographical term ordering on D, the leading monomial of p is ∂_x^2. The polynomial p forms a Janet basis for the ideal of D it generates, and the parametric derivatives are given by ∂_t^i, $\partial_t^j \partial_x$, i, $j \in \mathbb{Z}_{\geq 0}$. Hence, any choice of formal power series in t for $u(t, 0)$ and $\frac{\partial u}{\partial x}(t, 0)$ uniquely determines a formal power series solution u to (2.13). In this case, every choice of convergent power series yields a convergent series solution u. On the other hand, using the lexicographical term ordering extending $t > x$, the parametric derivatives are given by ∂_x^i, $i \in \mathbb{Z}_{\geq 0}$. Now, the choice $u(0, x) = \sum_{i\geq 0} x^i$ determines a divergent series solution u.

Example 2.44. The (minimal) Janet basis for the system of linear PDEs in Example 2.5 is

$$\begin{aligned}
\frac{\partial^2 u}{\partial x\, \partial y} &= 0, \quad \{\, * , \partial_y, \partial_z \,\},\\
\frac{\partial^3 u}{\partial x^2\, \partial y} &= 0, \quad \{\, * , \partial_y, \partial_z \,\},\\
\frac{\partial^4 u}{\partial x^3\, \partial z} &= 0, \quad \{\, \partial_x, * , \partial_z \,\},\\
\frac{\partial^4 u}{\partial x^3\, \partial y} &= 0, \quad \{\, \partial_x, \partial_y, \partial_z \,\}.
\end{aligned}$$

A Janet decomposition of the set of parametric derivatives is (cf. also Example 2.25)

$$\begin{aligned}
1, &\quad \{\, * , \partial_y, \partial_z \,\},\\
\partial_x, &\quad \{\, * , * , \partial_z \,\},\\
\partial_x^2, &\quad \{\, * , * , \partial_z \,\},\\
\partial_x^3, &\quad \{\, \partial_x, * , * \,\}.
\end{aligned}$$

The corresponding generalized Hilbert series is

$$\frac{1}{(1 - \partial_y)(1 - \partial_z)} + \frac{\partial_x}{1 - \partial_z} + \frac{\partial_x^2}{1 - \partial_z} + \frac{\partial_x^3}{1 - \partial_x}.$$

Accordingly, a formal power series solution u of (2.5) is uniquely determined as

$$u(x, y, z) = f_0(y, z) + x\, f_1(z) + x^2\, f_2(z) + x^3\, f_3(x)$$

by any choice of formal power series f_0, f_1, f_2, f_3 of the indicated variables.

3 Systems of nonlinear differential equations

The methods to be developed in this section allow to solve tasks (a), (b), (c) as stated in the Introduction for systems of partial differential equations (PDEs) that are given by polynomials in the unknown functions and their derivatives.

A *system of partial differential equations and inequations* (or simply a *differential system*) S is given by

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0, \tag{3.1}$$

where $p_1, \ldots, p_s$ and $q_1, \ldots, q_t$ are polynomials in unknown functions $u_1, \ldots, u_m$ of independent variables $z_1, \ldots, z_n$ and their partial derivatives (of arbitrary order), $s, t \in \mathbb{Z}_{\geq 0}$.

Let Ω be an open and connected subset of $\mathbb{C}^n$ with coordinates $z_1, z_2, \ldots, z_n$. Then the *solution set* of S on Ω is

$$\begin{aligned}\mathrm{Sol}_\Omega(S) \;:=\; \{\, f = (f_1, \ldots, f_m) \;\mid\; & f_k \colon \Omega \to \mathbb{C} \text{ analytic}, \, k = 1, \ldots, m, \\ & p_i(f) = 0, \, q_j(f) \neq 0, \, i = 1, \ldots, s, \, j = 1, \ldots, t \,\}\,.\end{aligned}$$

Example 3.1. The following differential system for one unknown function u of independent variables t and x is a combination of the Korteweg-de Vries equation (KdV, [BC80]) and a (generalized) Wronskian determinant:

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} - 6\,u\,\dfrac{\partial u}{\partial x} + \dfrac{\partial^3 u}{\partial x^3} & = & 0\,, \\[2ex] u\,\dfrac{\partial^2 u}{\partial t\,\partial x} - \dfrac{\partial u}{\partial t}\,\dfrac{\partial u}{\partial x} & = & 0\,. \end{array} \right.$$

If we denote partial derivatives by repeated indices, we may write it also as

$$\left\{ \begin{array}{rcl} u_t - 6\,u\,u_x + u_{x,x,x} & = & 0\,, \\ u\,u_{t,x} - u_t\,u_x & = & 0\,. \end{array} \right.$$

Example 3.2. The Navier-Stokes equations describe the flow of an incompressible fluid, where x, y, z are the spatial coordinates, t the time coordinate, (u_x, u_y, u_z) the velocity vector, p the pressure, ρ the density, (g_x, g_y, g_z) the gravitational acceleration and μ is the viscosity of the fluid [LL66, p. 54]:

$$\left\{ \begin{array}{rcl} \rho \left(\dfrac{\partial u_x}{\partial t} + u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} + u_z \dfrac{\partial u_x}{\partial z} \right) & = & -\dfrac{\partial p}{\partial x} + \mu \left(\dfrac{\partial^2 u_x}{\partial x^2} + \dfrac{\partial^2 u_x}{\partial y^2} + \dfrac{\partial^2 u_x}{\partial z^2} \right) \\[2ex] & & -\mu\,\dfrac{\partial}{\partial x} \left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z} \right) + \rho\,g_x\,, \\[2ex] \rho \left(\dfrac{\partial u_y}{\partial t} + u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} + u_z \dfrac{\partial u_y}{\partial z} \right) & = & -\dfrac{\partial p}{\partial y} + \mu \left(\dfrac{\partial^2 u_y}{\partial x^2} + \dfrac{\partial^2 u_y}{\partial y^2} + \dfrac{\partial^2 u_y}{\partial z^2} \right) \\[2ex] & & -\mu\,\dfrac{\partial}{\partial y} \left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z} \right) + \rho\,g_y\,, \\[2ex] \rho \left(\dfrac{\partial u_z}{\partial t} + u_x \dfrac{\partial u_z}{\partial x} + u_y \dfrac{\partial u_z}{\partial y} + u_z \dfrac{\partial u_z}{\partial z} \right) & = & -\dfrac{\partial p}{\partial z} + \mu \left(\dfrac{\partial^2 u_z}{\partial x^2} + \dfrac{\partial^2 u_z}{\partial y^2} + \dfrac{\partial^2 u_z}{\partial z^2} \right) \\[2ex] & & -\mu\,\dfrac{\partial}{\partial z} \left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z} \right) + \rho\,g_z\,, \\[2ex] \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z} & = & 0\,. \end{array} \right.$$

The *consequences* of (3.1) are the partial differential equations for $u_1, \ldots, u_m$ which are obtained in a finite number steps from the following rules:

(a) The given equations $p_1 = 0$, $p_2 = 0$, $\ldots$, $p_s = 0$ are consequences of (3.1).

(b) If $p = 0$ is a consequence, then any partial derivative of $p = 0$ is a consequence.

(c) If $p \cdot q = 0$ is a consequence and q is a factor of some q_i, then $p = 0$ is a consequence.

(d) If $p = 0$ and $r = 0$ are consequences, then $a\,p + b\,r = 0$ are consequences for all polynomials a and b in $u_1, \ldots, u_m$ and their partial derivatives (of all orders).

Since this setup allows differential equations $p = 0$ to be differentiated, we are going to work with a polynomial ring in $u_1, \ldots, u_m$ which admits these differentiations.

Definition 3.3. A *differential ring* R with commuting derivations $\delta_1, \ldots, \delta_n$ is a commutative ring R endowed with maps $\delta_i \colon R \to R$, satisfying

$$\delta_i(r_1 + r_2) = \delta_i(r_1) + \delta_i(r_2), \quad \delta_i(r_1\, r_2) = \delta_i(r_1)\, r_2 + r_1\, \delta_i(r_2) \quad \text{for all } r_1, r_2 \in R,$$

$i = 1, \ldots, n$, and $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ for all $1 \leq i, j \leq n$. A differential ring which is a field is called a *differential field*, and similarly for a differential algebra over a differential field.

In what follows we only consider differential fields K of characteristic zero. Denote the derivations of K by $\partial_1, \ldots, \partial_n$.

Definition 3.4. The *differential polynomial ring* $K\{u_1, \ldots, u_m\}$ in the *differential indeterminates* $u_1, \ldots, u_m$ is the commutative polynomial algebra $K[(u_k)_J \mid 1 \leq k \leq m,\ J \in (\mathbb{Z}_{\geq 0})^n]$ with infinitely many, algebraically independent indeterminates $(u_k)_J$, also called *jet variables*, which represent the partial derivatives

$$\frac{\partial^{J_1 + \ldots + J_n} U_k}{\partial z_1^{J_1} \ldots \partial z_n^{J_n}}, \qquad k = 1, \ldots, m, \quad J \in (\mathbb{Z}_{\geq 0})^n,$$

of smooth functions $U_1, \ldots, U_m$ of independent variables $z_1, \ldots, z_n$. We use u_k as a synonym for $(u_k)_{(0,\ldots,0)}$, $k = 1, \ldots, m$. The ring $K\{u_1, \ldots, u_m\}$ is considered as differential ring with commuting derivations $\delta_1, \ldots, \delta_n$ defined by extending

$$\delta_i\, u_k \ := \ (u_k)_{1_i}, \qquad i = 1, \ldots, n, \quad k = 1, \ldots, m,$$

additively, respecting the product rule of differentiation, and restricting to the derivation ∂_i on K. (Here 1_i denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length n with 1 at position i.) More generally, the differential polynomial ring may be constructed with coefficients in a differential ring rather than in a differential field in the same way.

Recall that an open and connected subset Ω of $\mathbb{C}^n$ was considered. The set of (complex) meromorphic functions on Ω form a field K, and together with the partial differential operators $\delta_1, \ldots, \delta_n$ with respect to $z_1, \ldots, z_n$, respectively, K is a differential field.

A suitable choice of differential polynomial ring $R = K\{u_1, \ldots, u_m\}$ allows to consider the left hand sides $p_1, \ldots, p_s$, $q_1, \ldots, q_t$ in the system of nonlinear PDEs (3.1) as elements of R. Moreover, the left hand sides of all consequences of the system are elements of R as well. In fact, we may consider the *differential ideal* I of R which is generated by $p_1, \ldots, p_s$, i.e., the smallest ideal of R which contains $p_1, \ldots, p_s$ and all their derivatives. This is only a first step, because in general I does not contain all consequences of (3.1). Before we study these ideals further, we first deal with algebraic systems (e.g., in $(u_k)_J$, where $1 \leq k \leq m$, $J \in (\mathbb{Z}_{\geq 0})^n$).

3.1 Thomas decomposition of algebraic systems

Let K be a field of characteristic zero and $R = K[x_1, \ldots, x_n]$ the polynomial algebra with indeterminates $x_1, \ldots, x_n$ over K. We denote by $\overline{K}$ an algebraic closure of K.

Definition 3.5. An *algebraic system* S, defined over R, is given by finitely many equations and inequations

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0, \tag{3.2}$$

where $p_1, \ldots, p_s, q_1, \ldots, q_t \in R$ and $s, t \in \mathbb{Z}_{\geq 0}$. The *solution set* of S in $\overline{K}^n$ is

$$\mathrm{Sol}_{\overline{K}}(S) \;:=\; \{\, a \in \overline{K}^n \mid p_i(a) = 0 \text{ and } q_j(a) \neq 0 \text{ for all } 1 \leq i \leq s,\, 1 \leq j \leq t \,\}.$$

We fix a total ordering $>$ on the set $\{x_1, \ldots, x_n\}$ allowing us to consider every non-constant element p of R as a univariate polynomial in the greatest variable with respect to $>$ which occurs in p, with coefficients which are themselves univariate polynomials in lower ranked variables, etc. Without loss of generality we may assume that

$$x_1 > x_2 > \ldots > x_n.$$

The choice of $>$ corresponds to a choice of projections

$$\begin{array}{rcllcr}
\pi_1\colon \overline{K}^n & \longrightarrow & \overline{K}^{n-1}\colon & (a_1, a_2, \ldots, a_n) & \longmapsto & (a_2, a_3, a_4, \ldots, a_n), \\
\pi_2\colon \overline{K}^n & \longrightarrow & \overline{K}^{n-2}\colon & (a_1, a_2, \ldots, a_n) & \longmapsto & (a_3, a_4, \ldots, a_n), \\
 & \vdots & & & \vdots & \\
\pi_{n-1}\colon \overline{K}^n & \longrightarrow & \overline{K}\colon & (a_1, a_2, \ldots, a_n) & \longmapsto & a_n.
\end{array}$$

Thus, the recursive representation of polynomials is motivated by considering the $(k-1)$-st projection $\pi_{k-1}(\mathrm{Sol}_{\overline{K}}(S))$ of the solution set as fibered over the k-th projection $\pi_k(\mathrm{Sol}_{\overline{K}}(S))$, for $k = 1, \ldots, n-1$, where we define $\pi_0 := \mathrm{id}_{\overline{K}^n}$ (cf. also [Ple09a]). The purpose of a Thomas decomposition of $\mathrm{Sol}_{\overline{K}}(S)$, to be defined below, is to clarify this fibration structure. The solution set $\mathrm{Sol}_{\overline{K}}(S)$ is partitioned into subsets $\mathrm{Sol}_{\overline{K}}(S_1), \ldots, \mathrm{Sol}_{\overline{K}}(S_r)$ in such a way that, for each $i = 1, \ldots, r$ and $k = 1, \ldots, n-1$, the fiber cardinality $|\pi_{k-1}^{-1}(\{\, a \,\})|$ does not depend on the choice of $a \in \pi_k(\mathrm{Sol}_{\overline{K}}(S_i))$. In terms of the defining equations and inequations in (3.2), the fundamental obstructions to this uniform behavior are zeros of the leading coefficients of p_i or q_j and zeros of p_i or q_j of multiplicity greater than one.

Definition 3.6. Let $p \in R \setminus K$.

a) The greatest indeterminate with respect to $>$ which occurs in p is referred to as the *leader* of p and is denoted by $\mathrm{ld}(p)$.

b) For $v = \mathrm{ld}(p)$ we denote by $\deg_v(p)$ the degree of p in v.

c) The coefficient of the highest power of $\mathrm{ld}(p)$ occurring in p is called the *initial* of p and is denoted by $\mathrm{init}(p)$.

d) The *discriminant* of p is defined as

$$\mathrm{disc}(p) \;:=\; (-1)^{d(d-1)/2}\,\mathrm{res}\left(p, \frac{\partial p}{\partial\, \mathrm{ld}(p)}, \mathrm{ld}(p)\right) / \ \mathrm{init}(p), \qquad d = \deg_{\mathrm{ld}(p)}(p),$$

where $\operatorname{res}(p, q, v)$ is the resultant of p and q with respect to the variable v.
(Note that $\operatorname{disc}(p)$ is a polynomial because $\operatorname{init}(p)$ divides $\operatorname{res}(p, \partial p/\partial \operatorname{ld}(p), \operatorname{ld}(p))$: for $p = c_d\, v^d + c_{d-1}\, v^{d-1} + \ldots + c_1\, v + c_0$ the determinant

$$\operatorname{res}(p, \partial p/\partial \operatorname{ld}(p), \operatorname{ld}(p)) \;:=\; \det \begin{pmatrix} c_d & \ldots & \ldots & c_1 & c_0 & & & \\ & c_d & \ldots & \ldots & c_1 & c_0 & & \\ & & \ddots & & & & \ddots & \\ & & & c_d & \ldots & \ldots & c_1 & c_0 \\ d\, c_d & \ldots & 2\, c_2 & c_1 & & & & \\ & d\, c_d & \ldots & 2\, c_2 & c_1 & & & \\ & & \ddots & & & \ddots & & \\ & & & d\, c_d & \ldots & 2\, c_2 & c_1 & \\ & & & & d\, c_d & \ldots & 2\, c_2 & c_1 \end{pmatrix}$$

has a column all of whose entries are divisible by $c_d = \operatorname{init}(p)$.)

Both $\operatorname{init}(p)$ and $\operatorname{disc}(p)$ are elements of the polynomial algebra $K[x \mid x < \operatorname{ld}(p)]$. The zeros of a univariate polynomial with multiplicity greater than one are the common zeros of the polynomial and its derivative. The solutions of $\operatorname{disc}(p) = 0$ in $\overline{K}^{n-k}$, where $\operatorname{ld}(p) = x_k$, are therefore those tuples $(a_{k+1}, a_{k+2}, \ldots, a_n)$ for which the substitution

$$x_{k+1} = a_{k+1}, \quad x_{k+2} = a_{k+2}, \quad \ldots, \quad x_n = a_n$$

in p results in a univariate polynomial with a zero of multiplicity greater than one.

Definition 3.7. An algebraic system S, defined over R, as in (3.2) is said to be *simple* (with respect to $>$) if the following three conditions hold.

a) For all $i = 1, \ldots, s$ and $j = 1, \ldots, t$ we have $p_i \notin K$ and $q_j \notin K$.

b) The leaders of the left hand sides of the equations and inequations in S are pairwise distinct, i.e., $|\{ \operatorname{ld}(p_1), \ldots, \operatorname{ld}(p_s), \operatorname{ld}(q_1), \ldots, \operatorname{ld}(q_t) \}| = s + t$.

c) For every $r \in \{ p_1, \ldots, p_s, q_1, \ldots, q_t \}$, if $\operatorname{ld}(r) = x_k$, then neither the equation $\operatorname{init}(r) = 0$ nor the equation $\operatorname{disc}(r) = 0$ has a solution $(a_{k+1}, a_{k+2}, \ldots, a_n)$ in $\pi_k(\operatorname{Sol}_{\overline{K}}(S))$.

Subsets of non-constant polynomials in R with pairwise distinct leaders (i.e., satisfying a) and b)) are also referred to as *triangular sets* (cf., e.g., [ALMM99], [Hub03a, Hub03b], [Wan01]).

Remark 3.8. A simple algebraic system S admits the following solution procedure, which also shows that its solution set is not empty. Let $S_{<k}$ be the subset of S consisting of the equations $p = 0$ and inequations $q \neq 0$ with $\operatorname{ld}(p) < x_k$ and $\operatorname{ld}(q) < x_k$. The fibration structure implied by c) ensures that, for every $k = 1, \ldots, n-1$, every solution

$$(a_{k+1}, a_{k+2}, \ldots, a_n) \in \overline{K}^{n-k}$$

in $\pi_k(\operatorname{Sol}_{\overline{K}}(S)) = \pi_k(\operatorname{Sol}_{\overline{K}}(S_{<k}))$ can be extended to a solution

$$(a_k, a_{k+1}, \ldots, a_n) \in \overline{K}^{n-(k-1)}$$

in $\pi_{k-1}(\operatorname{Sol}_{\overline{K}}(S))$. If S contains an equation $p = 0$ with leader x_k, then there exist exactly $\deg_{x_k}(p)$ such extensions (because zeros with multiplicity greater than one are excluded by the non-vanishing discriminant). If S contains an inequation $q \neq 0$ with leader x_k, all $a_k \in \overline{K}$ except $\deg_{x_k}(q)$ elements define a tuple $(a_k, a_{k+1}, \ldots, a_n)$ as above. If no equation and no inequation in S has leader x_k, then $a_k \in \overline{K}$ can be chosen arbitrarily.

Definition 3.9. Let S be an algebraic system, defined over R. A *Thomas decomposition* of S (or $\mathrm{Sol}_{\overline{K}}(S)$) with respect to $>$ is a collection of finitely many algebraic systems $S_1, \ldots, S_r$, each of which is defined over R and is simple, such that $\mathrm{Sol}_{\overline{K}}(S)$ is the disjoint union of the solution sets $\mathrm{Sol}_{\overline{K}}(S_1), \ldots, \mathrm{Sol}_{\overline{K}}(S_r)$.

We outline a method for computing a Thomas decomposition of algebraic systems.

Remark 3.10. Given S as in (3.2) and a total ordering $>$ on $\{x_1, \ldots, x_n\}$, a Thomas decomposition of S with respect to $>$ can be constructed by combining Euclid's algorithm with a splitting strategy.

First of all, if S contains an equation $c = 0$ with $0 \neq c \in K$ or the inequation $0 \neq 0$, then S is discarded because it has no solutions. Moreover, from now on the equation $0 = 0$ and inequations $c \neq 0$ with $0 \neq c \in K$ are supposed to be removed from S.

An elementary step of the algorithm applies a pseudo-division to a pair p_1, p_2 of non-constant polynomials in R with the same leader x_k and $\deg_{x_k}(p_1) \geq \deg_{x_k}(p_2)$. The result is a pseudo-remainder

$$r \;=\; c_1 \cdot p_1 - c_2 \cdot p_2 \,, \qquad \text{where } c_1, c_2 \in R \,, \tag{3.3}$$

and r is constant or has leader less than x_k or has leader x_k and $\deg_{x_k}(r) < \deg_{x_k}(p_1)$. Since the coefficients of p_1 and p_2 are polynomials in lower ranked variables, multiplication of p_1 by a non-constant polynomial c_1 may be necessary in general to perform the reduction in R (and not in its field of fractions). Choosing c_1 as a suitable power of $\mathrm{init}(p_2)$ always achieves this.

In order to turn S into a triangular set, the algorithm deals with three kinds of subsets of S of cardinality two. Firstly, each pair of equations $p_1 = 0$, $p_2 = 0$ in S with $\mathrm{ld}(p_1) = \mathrm{ld}(p_2)$ is replaced with the single equation $r = 0$, where r is the result of applying Euclid's algorithm to p_1 and p_2, considered as univariate polynomials in their leader, using the above pseudo-division. (If this computation was stable under substitution of values for lower ranked variables in p_1 and p_2, then r would be the greatest common divisor of the specialized polynomials.)

The solution set of the system is supposed not to change, when the equation $p_1 = 0$ is replaced with the equation $r = 0$ given by the pseudo-reduction (3.3). Therefore, we assume that the polynomial c_1, and hence $\mathrm{init}(p_2)$, does not vanish on the solution set of the system. In order to ensure this condition, a preparatory step splits the system into two, if necessary, and adds the inequation $\mathrm{init}(p_2) \neq 0$ to one of them and the equation $\mathrm{init}(p_2) = 0$ to the other. The algorithm then deals with both systems separately. These case distinctions also allow to arrange for the part of condition c) in Definition 3.7 which concerns initials.

Secondly, let $p = 0$, $q \neq 0$ be in S with $\mathrm{ld}(p) = \mathrm{ld}(q) = x_k$. If $\deg_{x_k}(p) \leq \deg_{x_k}(q)$, then $q \neq 0$ is replaced with $r \neq 0$, where r is the result of applying the pseudo-division (3.3) to q and p. Otherwise, Euclid's algorithm is applied to p and q, keeping track of the coefficients used for the reductions as in (3.3). Given the result r, the system is then split into two, adding the conditions $r \neq 0$ and $r = 0$, respectively. The inequation $q \neq 0$ is removed from the first new system, because $p = 0$ and $q \neq 0$ have no common solution in that case. The assumption $r = 0$ and the bookkeeping allows to divide p by the common factor of p and q (modulo left hand sides of equations with smaller leader). The left hand side of $p = 0$ is replaced with that quotient in the second new system. Not all of these cases need a closer inspection. For instance, if p divides q, then the solution set of S is empty and S is discarded.

Thirdly, for a pair $q_1 \neq 0$, $q_2 \neq 0$ in S with $\mathrm{ld}(q_1) = \mathrm{ld}(q_2)$, Euclid's algorithm is applied to q_1 and q_2 in the same way as above. Keeping track of the coefficients used in intermediate steps allows to determine the least common multiple m of q_1 and q_2, which again depends on distinguishing the cases whether the result of Euclid's algorithm vanishes or not. The pair $q_1 \neq 0$, $q_2 \neq 0$ is then replaced with $m \neq 0$.

The part of condition c) in Definition 3.7 regarding discriminants is taken care of by applying Euclid's algorithm as above to p and $\partial p / \partial \operatorname{ld}(p)$, where p is the left hand side of an equation or inequation. Bookkeeping allows to determine the square-free part of p, which depends again on case distinctions.

Expressions tend to grow very quickly when performing these reductions, so that an appropriate strategy is essential for dealing with non-trivial systems. Apart from dividing by the content (in K) of polynomials, in intermediate steps of Euclid's algorithm the coefficients should be reduced modulo equations in the system with lower ranked leaders. In practice, subresultant computations (cf., e.g., [Mis93]) allow to diminish the growth of coefficients significantly.

Termination of the procedure sketched above depends on the organization of its steps. One possible strategy is to maintain an intermediate triangular set, reduce new equations and inequations modulo the equations in the triangular set, and select among these results the one with smallest leader and least degree, preferably an equation, for insertion into the triangular set. If the set already contains an equation or inequation with the same leader, then the pair is treated as discussed above. Since equations are replaced with equations of smaller degree and inequations are replaced with equations if possible or with the least common multiple of inequations, this strategy terminates after finitely many steps.

For more details on the algebraic part of Thomas' algorithm, we refer to [BGL+12], [Bäc14], and [Rob14, Subsect. 2.2.1].

An implementation of Thomas' algorithm for algebraic systems was developed by T. Bächler at RWTH Aachen University as Maple package `AlgebraicThomas` [BLH].

In what follows, variables are underlined to emphasize that they are leaders of polynomials with respect to the fixed total ordering $>$.

Example 3.11. Let us compute a Thomas decomposition of the algebraic system

$$x^2 + y^2 - 1 \;=\; 0$$

consisting of one equation, defined over $R = \mathbb{Q}[x, y]$, with respect to $x > y$. First we set $p_1 := x^2 + y^2 - 1$. Then we have $\operatorname{ld}(p_1) = x$ and $\operatorname{init}(p_1) = 1$ and

$$\operatorname{disc}(p_1) \;=\; -4\, y^2 + 4\,.$$

We distinguish the cases whether or not $p_1 = 0$ has a solution which is also a zero of $\operatorname{disc}(p_1)$, or equivalently, of $y^2 - 1$. In other words, we replace the original algebraic system with two algebraic systems which are obtained by adding the inequation $y^2 - 1 \neq 0$ or the equation $y^2 - 1 = 0$. The first system is readily seen to be simple, whereas the second one is transformed into a simple system by taking the difference of the two equations and computing a square-free part. Clearly, the solution sets of the two resulting simple systems form a partition of the solution set of $p_1 = 0$. We obtain the Thomas decomposition

$$\boxed{\begin{array}{rcl} \underline{x}^2 + y^2 - 1 & = & 0 \\ \underline{y}^2 - 1 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} & = & 0 \\ \underline{y}^2 - 1 & = & 0 \end{array}}$$

In this example, all points of $\operatorname{Sol}_{\overline{K}}(\{\, p_1 = 0 \,\})$ for which the projection π_1 onto the y-axis has fibers of an exceptional cardinality have real coordinates, and the significance of the above case distinction can be confirmed graphically.

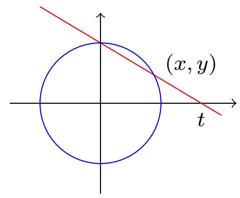

Figure 3: Stereographic projection from a circle

As a further illustration let us augment the original system by the equation which expresses the coordinate t of the point of intersection of the line through the two points $(0, 1)$ and (x, y) on the circle with the x-axis (stereographic projection, cf. Figure 3):

$$\begin{cases} x^2 + y^2 - 1 & = & 0 \\ (1 - y)\, t - x & = & 0 \end{cases}$$

A Thomas decomposition with respect to the ordering $x > y > t$ is obtained as follows. We set $p_2 := x + t\,y - t$. Since $\mathrm{ld}(p_1) = \mathrm{ld}(p_2)$, we apply polynomial division:

$$p_1 - (x - t\,y + t)\, p_2 \;=\; (1 + t^2)\, \underline{y}^2 - 2\,t^2\, \underline{y} + t^2 - 1 \;=\; (\underline{y} - 1)\,((1 + t^2)\, \underline{y} - t^2 + 1)\,.$$

Replacing p_1 with the remainder of this division does not alter the solution set of the algebraic system. It is convenient (but not necessary) to split the system into two systems according to the factorization of the remainder:

$$\begin{cases} \underline{x} + t\,y - t & = & 0 \\ (1 + t^2)\, \underline{y} - t^2 + 1 & = & 0 \\ \underline{y} - 1 & \neq & 0 \end{cases} \qquad \begin{cases} \underline{x} + t\,y - t & = & 0 \\ & & \\ \underline{y} - 1 & = & 0 \end{cases}$$

Another polynomial division reveals that the equation and the inequation with leader y in the first system have no common solutions. Therefore, the inequation can be omitted from that system. The initial of the equation has to be investigated. In fact, the assumption $1 + t^2 = 0$ leads to a contradiction. Finally, the equation with leader y can be used to eliminate y in the equation with leader x:

$$(1 + t^2)\,(\underline{x} + t\,y - t) - t\,((1 + t^2)\, \underline{y} - t^2 + 1) \;=\; (1 + t^2)\, \underline{x} - 2\,t\,.$$

A similar simplification can be applied to the second system above. We obtain the Thomas decomposition

$$\boxed{\begin{array}{rcl} (1 + t^2)\, \underline{x} - 2\,t & = & 0 \\ (1 + t^2)\, \underline{y} - t^2 + 1 & = & 0 \\ \underline{t}^2 + 1 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} & = & 0 \\ \underline{y} - 1 & = & 0 \\ & & \end{array}}$$

from which a rational parametrization of the circle can be read off.

Remark 3.12. A Thomas decomposition of an algebraic system is not uniquely determined. It depends on the chosen total ordering $>$, the order in which intermediate systems are dealt with and other choices, such as whether factorizations of left hand sides of equations are taken into account or not.

According to Hilbert's Nullstellensatz (cf., e.g., [Eis95]), the solution sets V of systems of polynomial equations in $x_1, \ldots, x_n$ in $\overline{K}^n$ are in one-to-one correspondence with their *vanishing ideals* in R

$$\mathcal{I}_R(V) \; := \; \{ \, p \in R \mid p(a) = 0 \text{ for all } a \in V \, \} \, ,$$

and these are the radical ideals of R, i.e., the ideals I of R which equal their radicals

$$\sqrt{I} \; := \; \{ \, p \in R \mid p^r \in I \text{ for some } r \in \mathbb{Z}_{\geq 0} \, \} \, .$$

The solution sets V can then be considered as the closed subsets of $\overline{K}^n$ with respect to the Zariski topology.

The fibration structure of a simple algebraic system S allows to deduce that the polynomials in R which vanish on $\mathrm{Sol}_{\overline{K}}(S)$ are precisely those polynomials in R whose pseudo-remainders modulo $p_1, \ldots, p_s$ are zero, where $p_1 = 0, \ldots, p_s = 0$ are the equations in S. If E is the ideal of R generated by $p_1, \ldots, p_s$ and q the product of all $\mathrm{init}(p_i)$, then these polynomials form the *saturation ideal*

$$E : q^{\infty} \; := \; \{ \, p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \, \} \, .$$

In particular, simple algebraic systems admit an effective way to decide membership of a polynomial to the associated radical ideal (cf. also Proposition 3.30 below).

Proposition 3.13 ([Rob14], Prop. 2.2.7)**.** *Let the algebraic system S given by*

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0,$$

be simple, let E be the ideal of R generated by $p_1, \ldots, p_s$, and q the product of all $\mathrm{init}(p_i)$*. Then $E : q^{\infty}$ consists of all polynomials in R which vanish on $\mathrm{Sol}_{\overline{K}}(S)$. In particular, $E : q^{\infty}$ is a radical ideal. Given $p \in R$, we have $p \in E : q^{\infty}$ if and only if the pseudo-remainder of p modulo $p_1, \ldots, p_s$ is zero.*

Example 3.14. Continuing Example 3.11, let E be the ideal of R generated by the left hand sides of the equations of the simple algebraic system

$$\boxed{\begin{array}{rcl} (1+t^2)\,\underline{x} - 2\,t & = & 0 \\ (1+t^2)\,\underline{y} - t^2 + 1 & = & 0 \\ \underline{t}^2 + 1 & \neq & 0 \end{array}}$$

and define $q = 1 + t^2$. Moreover, let $p = (1-t^2)\,\underline{x} + 2\,t\,y \in R$. The pseudo-remainder of p modulo the equations of the first simple algebraic system displayed at the end of Example 3.11 is computed as follows. First we have

$$p' \; := \; (1+t^2)\,p - (1-t^2)\left[(1+t^2)\,x - 2\,t\right] \; = \; 2\,(1+t^2)\,t\,\underline{y} + 2\,(1-t^2)\,t\,.$$

Then we have

$$r \; := \; p' - 2\,t\left[(1+t^2)\,y - t^2 + 1\right] \; = \; 0\,.$$

Since the pseudo-remainder r is zero, we conclude that $p \in E : q^{\infty}$.

3.2 Thomas decomposition of differential systems

Let K be the differential field of meromorphic functions on an open and connected subset Ω of $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. We define the differential polynomial ring $R = K\{u_1, \ldots, u_m\}$ and $\Delta := \{ \partial_1, \ldots, \partial_n \}$.

Definition 3.15. A *differential system* S, defined over $R = K\{u_1, \ldots, u_m\}$, is given by finitely many equations and inequations

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0, \tag{3.4}$$

where $p_1, \ldots, p_s, q_1, \ldots, q_t \in R$ and $s, t \in \mathbb{Z}_{\geq 0}$. The *solution set* of S is

$$\begin{aligned} \mathrm{Sol}_\Omega(S) \; := \; \{ \, f = (f_1, \ldots, f_m) \quad | \quad & f_k \colon \Omega \to \mathbb{C} \text{ analytic}, \, k = 1, \ldots, m, \\ & p_i(f) = 0, \, q_j(f) \neq 0, \, i = 1, \ldots, s, \, j = 1, \ldots, t \, \}. \end{aligned}$$

Remark 3.16. Since each component f_k of a solution of (3.4) is assumed to be analytic, the equations $p_i = 0$ and inequations $q_j \neq 0$ (and their consequences) can be translated into algebraic conditions on the Taylor coefficients of power series expansions of $f_1, \ldots, f_m$ (around a point in Ω). An inequation $q \neq 0$ then turns into a disjunction of algebraic inequations for all coefficients which result from substitution of power series expansions for $u_1, \ldots, u_m$ in q.

An appropriate choice of $\Omega \subseteq \mathbb{C}^n$ can often only be made after the formal treatment of a given differential system by methods to be discussed below (as, e.g., singularities of coefficients in differential consequences will only be detected during that process). In general, we assume that Ω is chosen in such a way that the given systems have analytic solutions on Ω.

Clearly, by neglecting the derivations on $R = K\{u_1, \ldots, u_m\}$, a differential system can be considered as an algebraic system in the finitely many variables $(u_i)_J$ which occur in the equations and inequations. The same recursive representation of polynomials as in the algebraic case is employed, but the total ordering on the set of variables $(u_i)_J$ is supposed to respect the action of the derivations. Then the methods of the previous section on algebraic systems are applicable.

Definition 3.17. A *ranking* $>$ on $R = K\{u_1, \ldots, u_m\}$ is a total ordering on the set

$$\mathrm{Mon}(\Delta)\, u \; := \; \{ \, (u_k)_J \mid 1 \leq k \leq m, \, J \in (\mathbb{Z}_{\geq 0})^n \, \}$$

such that for all $j \in \{1, \ldots, n\}$, $k, k_1, k_2 \in \{1, \ldots, m\}$, $J_1, J_2 \in (\mathbb{Z}_{\geq 0})^n$ we have

a) $\partial_j \, u_k > u_k$ and

b) $(u_{k_1})_{J_1} > (u_{k_2})_{J_2}$ implies $\partial_j \, (u_{k_1})_{J_1} > \partial_j \, (u_{k_2})_{J_2}$.

Remark 3.18. Every ranking $>$ on R is a well-ordering (cf., e.g., [Kol73, Ch. 0, Sect. 17, Lemma 15]), i.e., every descending sequence of elements of $\mathrm{Mon}(\Delta)\, u$ terminates.

Example 3.19. On the differential polynomial ring $K\{u\}$ (i.e., with $m = 1$) with derivations $\partial_1, \ldots, \partial_n$ the *degree-reverse lexicographical ranking* (with $\partial_1 \, u > \partial_2 \, u > \ldots > \partial_n \, u$) is defined for $u_J, u_{J'}$, $J = (j_1, \ldots, j_n)$, $J' = (j'_1, \ldots, j'_n) \in (\mathbb{Z}_{\geq 0})^n$, by

$$u_J > u_{J'} \quad :\Longleftrightarrow \quad \begin{cases} j_1 + \ldots + j_n > j'_1 + \ldots + j'_n \quad \text{or} \\ \big(\; j_1 + \ldots + j_n = j'_1 + \ldots + j'_n \quad \text{and} \quad J \neq J' \quad \text{and} \\ \; j_i < j'_i \quad \text{for} \quad i = \max \{ \, 1 \leq k \leq n \mid j_k \neq j'_k \, \} \; \big). \end{cases}$$

For instance, if $n = 3$, we have $u_{(1,2,1)} > u_{(1,2,0)} > u_{(2,0,1)}$.

In what follows, we assume that a ranking $>$ on $R = K\{u_1, \ldots, u_m\}$ is fixed. For each $p \in R \setminus K$, the leader $\mathrm{ld}(p)$ and the initial $\mathrm{init}(p)$ are defined as in the previous section on algebraic systems. With the aim of introducing simple differential systems (Def. 3.24) we discuss pseudo-division for differential polynomials first.

Remark 3.20. Let $p_1, p_2 \in R$ be two non-constant differential polynomials. If p_1 and p_2 have the same leader $(u_k)_J$ and the degree of p_1 in $(u_k)_J$ is greater than or equal to the degree of p_2 in $(u_k)_J$, then the same pseudo-division as in (3.3) yields a remainder which is either zero, or has leader less than $(u_k)_J$, or has leader $(u_k)_J$ and smaller degree in $(u_k)_J$ than p_1.

More generally, if $\mathrm{ld}(p_1) = \theta\, \mathrm{ld}(p_2)$ for some $\theta \in \mathrm{Mon}(\Delta)$, then this pseudo-division can be applied with p_2 replaced with $\theta\, p_2$. Note that, by condition b) of the definition of a ranking, we have $\mathrm{ld}(\theta\, p_2) = \theta\, \mathrm{ld}(p_2)$, and that, if $\theta \neq 1$, the degree of $\theta\, p_2$ in $\theta\, \mathrm{ld}(p_2)$ is one, so that the reduction can be applied without assumption on the degree of p_2 in $\mathrm{ld}(p_2)$. Then c_1 in (3.3) is again chosen as a suitable power of $\mathrm{init}(\theta\, p_2)$. In case $\theta \neq 1$ we have

$$\mathrm{init}(\theta\, p_2) \;=\; \frac{\partial p_2}{\partial\, \mathrm{ld}(p_2)} \;=:\; \mathrm{sep}(p_2)\,,$$

and this differential polynomial is referred to as the *separant* of p_2.

In order not to change the solution set of a differential system, when $p_1 = 0$ is replaced with $r = 0$, where r is the result of a reduction of p_1 modulo p_2 or $\theta\, p_2$ as above, it is assumed that $\mathrm{init}(p_2)$ and $\mathrm{sep}(p_2)$ do not vanish on the solution set of the system. By definition of the separant and the discriminant (cf. Def. 3.6 d)), non-vanishing of $\mathrm{sep}(p_2)$ follows from non-vanishing of $\mathrm{disc}(p_2)$, as ensured by the algebraic part of Thomas' algorithm (cf. Rem. 3.10).

We assume now that the given differential system is simple as an algebraic system (cf. Def. 3.7); it could be one of the systems resulting from the algebraic part of Thomas' algorithm.

Remark 3.21. The symmetry of the second derivatives $\partial_i\, \partial_j\, u_k = \partial_j\, \partial_i\, u_k$ (and similarly for higher order derivatives) imposes necessary conditions on the solvability of a system of partial differential equations. Taking identities like these into account and forming linear combinations of (derivatives of) the given equations may produce differential consequences with lower ranked leaders. In order to obtain a complete set of algebraic conditions on the Taylor coefficients of an analytic solution, the system has to be augmented by these integrability conditions in general. If a system of partial differential equations admits a translation into algebraic conditions on the Taylor coefficients such that no further integrability conditions have to be taken into account, then it is said to be *formally integrable*.

Definition 3.22. Each equation $p_i = 0$ in a differential system is assigned the set of *admissible derivations* $\mu(\theta_i, M_k)$, where $\mathrm{ld}(p_i) = \theta_i\, u_k$ and

$$M_k \;:=\; \{\, \theta \in \mathrm{Mon}(\Delta) \mid \theta\, u_k \in \{\, \mathrm{ld}(p_1), \ldots, \mathrm{ld}(p_s)\,\}\,\} \tag{3.5}$$

is the set of all monomials which define leaders $\mathrm{ld}(p_i)$ involving the same differential indeterminate u_k. We refer to $d\, p_i$ for $d \in \mathrm{Mon}(\mu(\theta_i, M_k))$ as the *admissible derivatives* of p_i.

Formal integrability of a differential system is then decided by applying to each equation $p_i = 0$ every of its non-admissible derivations $d \in \overline{\mu}(\theta_i, M_k)$ and computing the pseudo-remainder of $d\, p_i$ modulo $p_1, \ldots, p_s$ and their admissible derivatives. The restriction of the pseudo-division to admissible derivatives requires M_k to be Janet complete (cf. Def. 2.17). If one of these pseudo-remainders is non-zero, then it is added as a new equation to the system, and the augmented system has to be treated by the algebraic part of Thomas' algorithm again.

Definition 3.23. A system $\{\, p_1 = 0, \ldots, p_s = 0 \,\}$ of PDEs, where $p_1, \ldots, p_s \in R \setminus K$, is said to be *passive* if the following two conditions hold for $\mathrm{ld}(p_1) = \theta_1\, u_{k_1}, \ldots, \mathrm{ld}(p_s) = \theta_s\, u_{k_s}$, where $\theta_i \in \mathrm{Mon}(\Delta)$, $k_i \in \{1, \ldots, m\}$.

a) For all $k \in \{1, \ldots, m\}$, the set M_k defined in (3.5) is Janet complete.

b) For all $i \in \{1, \ldots, s\}$ and all non-admissible derivations $d \in \overline{\mu}(\theta_i, M_{k_i})$, the pseudo-remainder of $d\, p_i$ modulo $p_1, \ldots, p_s$ and their admissible derivatives is zero.

Definition 3.24. A differential system S, defined over R, as in (3.4) is said to be *simple* (with respect to $>$) if the following three conditions hold.

a) The system S is simple as an algebraic system (in the finitely many variables $(u_i)_J$ which occur in the equations and inequations of S, totally ordered by $>$).

b) The system $\{\, p_1 = 0, \ldots, p_s = 0 \,\}$ is passive.

c) The left hand sides of the inequations $q_1 \neq 0, \ldots, q_t \neq 0$ equal their pseudo-remainders modulo $p_1, \ldots, p_s$ and their derivatives.

Definition 3.25. Let S be a differential system, defined over R. A *Thomas decomposition* of S (or of $\mathrm{Sol}_\Omega(S)$) with respect to $>$ is a collection of finitely many simple differential systems $S_1, \ldots, S_r$, defined over R, such that the solution set $\mathrm{Sol}_\Omega(S)$ of S is the disjoint union of the solution sets $\mathrm{Sol}_\Omega(S_1), \ldots, \mathrm{Sol}_\Omega(S_r)$.

Remark 3.26. Given S as in (3.4) and a ranking on R, a Thomas decomposition of S with respect to $>$ can be computed by interweaving the algebraic part discussed in Subsection 3.1 and differential reduction and completion with respect to Janet division.

First of all, a Thomas decomposition of S, considered as an algebraic system, is computed. Each of the resulting simple algebraic systems is then treated as follows. Differential pseudo-division is applied to pairs of distinct equations with leaders $\theta_1\, u_k$ and $\theta_2\, u_k$ such that $\theta_1 \mid \theta_2$ until either a non-zero pseudo-remainder is obtained or no such further reductions are possible. Non-zero pseudo-remainders are added to the system, the algebraic part of Thomas' algorithm is applied again, and the process is repeated. Once the system is auto-reduced in this sense, then it is possibly augmented with certain derivatives of equations so that the sets M_k defined in (3.5) are Janet complete. Then it is checked whether the system is passive. If a non-zero remainder is obtained by a pseudo-division of a non-admissible derivative modulo the equations and their admissible derivatives, then the algebraic part of Thomas' algorithm is applied again to the augmented system. Otherwise, the system is passive. Finally, the left hand side of each inequation is replaced with its pseudo-remainder modulo the equations and their derivatives, in order to ensure condition c) of Definition 3.24. The main reason why this procedure terminates is Dickson's Lemma, which shows that the ascending sequence of ideals of the semigroup $\mathrm{Mon}(\Delta)$ formed by the monomials θ defining leaders of equations (for each differential indeterminate) becomes stationary after finitely many steps.

For more details on the differential part of Thomas' algorithm, we refer to [BGL+12], [LH14], and [Rob14, Subsect. 2.2.2].

An implementation of Thomas' algorithm for differential systems was developed by M. Lange-Hegermann at RWTH Aachen University as Maple package `DifferentialThomas` [BLH].

When displaying a simple differential system we indicate next to each equation its set of admissible derivations.

Example 3.27. Let us consider the ODE (discussed in [Inc56, Example in Sect. 4.7])

$$\left(\frac{\partial u}{\partial x}\right)^3 - 4\,x\,u(x)\,\frac{\partial u}{\partial x} + 8\,u(x)^2 \;=\; 0.$$

The left hand side is represented by the element $p := u_x^3 - 4\,x\,u\,u_x + 8\,u^2$ of the differential polynomial ring $R = K\{u\}$ with one derivation ∂_x, where $K = \mathbb{Q}(x)$ is the field of rational functions in x, endowed with differentiation with respect to x.

The initial of p is constant, the separant of p is $3\,u_x^2 - 4\,x\,u$. The algebraic part of Thomas' algorithm only distinguishes the cases whether the discriminant of p vanishes or not. We have

$$\mathrm{disc}(p) \;=\; -\,\mathrm{res}(p, \mathrm{sep}(p), u_x) \;=\; -64\,\underline{u}^3\,(27\,\underline{u} - 4\,x^3).$$

This case distinction leads to the Thomas decomposition

$$\boxed{\begin{array}{rcll} \underline{u_x}^3 - 4\,x\,u\,\underline{u_x} + 8\,u^2 & = & 0, & \{\partial_x\} \\ (27\,\underline{u} - 4\,x^3)\,\underline{u} & \neq & 0 & \end{array}} \qquad \boxed{\begin{array}{rcll} & & & \\ (27\,\underline{u} - 4\,x^3)\,\underline{u} & = & 0, & \{\partial_x\} \end{array}}$$

Since both systems contain only one equation, no differential reductions are necessary. The second simple system could be split into two with equations $27\,u - 4\,x^3 = 0$ and $u = 0$, respectively. The solutions of the first simple system are given by $u(x) = c\,(x - c)^2$, where c is an arbitrary non-zero constant. The solutions $u(x) = 0$ and $u(x) = \frac{4}{27}\,x^3$ of the second simple system are called *singular solutions*, the latter one being an envelope of the general solution.

More about singular solutions can be found, e.g., in [Dar73], [Ham93], [Rit36], [Hub97].

Example 3.28. Let us compute a Thomas decomposition of the system of (nonlinear) PDEs

$$\left\{\begin{array}{rcl} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\[2ex] \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{array}\right.$$

for one unknown function $u(x, y)$. Define the elements $p_1 := u_{x,x} - u_{y,y}$ and $p_2 := u_x - u^2$ of the differential polynomial ring $R = \mathbb{Q}\{u\}$ with commuting derivations ∂_x, ∂_y. We choose the degree-reverse lexicographical ranking $>$ on R with $\partial_x\,u > \partial_y\,u$ (cf. Example 3.19).

Since the monomial ∂_x defining the leader of p_2 divides the monomial ∂_x^2 defining the leader of p_1, differential pseudo-division is applied and p_1 is replaced with

$$p_3 \;:=\; p_1 - \partial_x\,p_2 + 2\,u\,p_2 \;=\; u_{y,y} - 2\,u^3\,.$$

Janet division associates the sets of admissible derivations to the equations as follows:

$$\left\{\begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\[1ex] \underline{u_{y,y}} - 2\,u^3 & = & 0, & \{\,*\,, \partial_y\} \end{array}\right.$$

The set of monomials $\{\,\partial_x, \partial_y^2\,\}$ defining the leaders u_x and $u_{y,y}$ is Janet complete. The check whether the above system is passive involves the following reduction:

$$\partial_x\,p_3 - \partial_y^2\,p_2 + 6\,u^2\,p_2 - 2\,u\,p_3 \;=\; 2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,.$$

This non-zero remainder is a differential consequence which is added as an equation to the system. In fact, the system can be split into two systems according to the given factorization. For both systems a differential reduction of p_3 modulo the chosen factor is applied because the monomial ∂_y defining the new leader divides the monomial $\partial_{y,y}$ defining $\mathrm{ld}(p_3)$. In both cases the remainder is zero, the sets of monomials defining leaders are Janet complete, and the passivity check confirms formal integrability. We obtain the Thomas decomposition

$$\boxed{\begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\ \underline{u_y} + u^2 & = & 0, & \{\, *\,, \partial_y\} \end{array}} \qquad \boxed{\begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\ \underline{u_y} - u^2 & = & 0, & \{\, *\,, \partial_y\} \\ \underline{u} & \neq & 0. & \end{array}}$$

If the above factorization is ignored, then the discriminant of $p_4 := u_y^2 - u^4$ needs to be considered, which implies vanishing or non-vanishing of the separant $2\,u_y$. This case distinction leads to a different Thomas decomposition.

A Thomas decomposition of a differential system is not uniquely determined. In the special case of a system S of *linear* partial differential equations no case distinctions are necessary, and the single simple system in any Thomas decomposition of S is a Janet basis for S. Pseudo-reduction of a differential polynomial modulo the equations of a simple differential system and their derivatives decides membership to the corresponding saturation ideal.

Proposition 3.29 ([Rob14], Prop. 2.2.50)**.** *Let S be a simple differential system, defined over R, with equations $p_1 = 0, \ldots, p_s = 0$. Moreover, let E be the differential ideal of R generated by $p_1, \ldots, p_s$ and define the product q of the initials and separants of all $p_1, \ldots, p_s$. Then $E : q^\infty$ is a radical differential ideal. Given $p \in R$, we have $p \in E : q^\infty$ if and only if the pseudo-remainder of p modulo $p_1, \ldots, p_s$ and their derivatives is zero.*

Similarly to the algebraic case treated in the previous section, the Nullstellensatz for analytic functions (due to J. F. Ritt and H. W. Raudenbush, cf. [Rit50, Sects. II.7–11, IX.27]) establishes a one-to-one correspondence of solution sets $V := \mathrm{Sol}_\Omega(S)$ of systems of partial differential equations $S = \{\, p_1 = 0, \ldots, p_s = 0\,\}$ for m unknown functions, defined over R, and their vanishing ideals in $R = K\{u_1, \ldots, u_m\}$

$$\mathcal{I}_R(V) \;:=\; \{\, p \in R \mid p(f) = 0 \text{ for all } f \in V\,\}.$$

These are the radical differential ideals of R. The Nullstellensatz implies that, with the notation of Proposition 3.29, we have $\mathcal{I}_R(\mathrm{Sol}_\Omega(S)) = E : q^\infty$.

The following proposition allows to decide whether a given differential equation $p = 0$ is a consequence of a (not necessarily simple) differential system S by applying pseudo-division to p modulo each of the simple systems in a Thomas decomposition of S. It follows from the Nullstellensatz and it also applies to algebraic systems by ignoring the separants.

Proposition 3.30 ([Rob14], Prop. 2.2.72)**.** *Let a (not necessarily simple) differential system S be given by $p_1 = 0$, $p_2 = 0$, $\ldots$, $p_s = 0$, $q_1 \neq 0$, $q_2 \neq 0$, $\ldots$, $q_t \neq 0$, and let $S_1, \ldots, S_r$ be a Thomas decomposition of S with respect to any ranking on R. Moreover, let E be the differential ideal of R generated by $p_1, \ldots, p_s$ and define the product q of $q_1, \ldots, q_t$. For $i \in \{1, \ldots, r\}$, let $E^{(i)}$ be the differential ideal of R generated by the equations in S_i and define the product $q^{(i)}$ of the initials and separants of all these equations. Then we have*

$$\sqrt{E : q^\infty} \;=\; \left(E^{(1)} : (q^{(1)})^\infty\right) \cap \ldots \cap \left(E^{(r)} : (q^{(r)})^\infty\right).$$

An important class of rankings can be defined as follows. This definition can be traced back to C. Riquier, cf. [Riq10, no. 102].

Remark 3.31. Let the map $\varphi\colon \operatorname{Mon}(\Delta)u \to \mathbb{Q}^{(n+m)\times 1} = \mathbb{Q}^{n\times 1} \oplus \mathbb{Q}^{m\times 1}$ be defined by

$$\partial^I u_j \longmapsto (I, e_j)^\top, \qquad I \in (\mathbb{Z}_{\ge 0})^n, \quad 1 \le j \le m,$$

where $e_1, \ldots, e_m$ are the standard basis vectors of $\mathbb{Q}^{m\times 1}$. Then every matrix $M \in \mathbb{Q}^{r\times(n+m)}$ defines an irreflexive and transitive relation $>$ on $\operatorname{Mon}(\Delta)u$ by

$$v > w \quad :\Longleftrightarrow \quad M\,\varphi(v) > M\,\varphi(w), \qquad v, w \in \operatorname{Mon}(\Delta)u, \tag{3.6}$$

where vectors on the right hand side are compared lexicographically. Assume that M admits a left inverse (in particular, we have $r \ge n+m$). Then the linear map $\mathbb{Q}^{(n+m)\times 1} \to \mathbb{Q}^{r\times 1}$ induced by M is injective, and $>$ is a total ordering on $\operatorname{Mon}(\Delta)u$. Linearity of matrix multiplication implies that $>$ satisfies condition b) of Definition 3.17, p. 24, of a ranking. Moreover, condition a) of the same definition holds if and only if, for each $j = 1, \ldots, n$, the first non-zero entry of the j-th column of M is positive. Every ranking $>$ defined by (3.6) is a *Riquier ranking*, i.e.,

$$\theta_1\, u_i > \theta_2\, u_i \quad \Longleftrightarrow \quad \theta_1\, u_j > \theta_2\, u_j$$

holds for all $\theta_1, \theta_2 \in \operatorname{Mon}(\Delta)$, $1 \le i, j \le m$.

In every equation $p = 0$ of a simple differential system S we can solve for the term containing the highest power of the leader $\operatorname{ld}(p)$ to obtain an equivalent equation

$$\operatorname{init}(p)\operatorname{ld}(p)^k \;=\; r\,,$$

where r consists of terms which are lower powers of $\operatorname{ld}(p)$ than the one on the left hand side or lower ranked than $\operatorname{ld}(p)$. Moreover, the differential polynomial $\operatorname{init}(p)$ does not vanish for any solution of the simple system S. We obtain a generalization of the Cauchy-Kovalevskaya Theorem (cf. Thm. 1.1 in the Introduction); cf. also [Tho28], [Tho34], [Ger09], [RRW99].

Corollary 3.32. *Let S be a simple differential system as in (3.4). Suppose that $(z_1^0, \ldots, z_n^0)$ is a point where all $p_1, \ldots, p_s$ and all $q_1, \ldots, q_t$ are defined and such that no initial or separant of any of these differential polynomials vanishes. Let formal power series around $(z_1^0, \ldots, z_n^0)$ be defined by*

$$f_k \;:=\; \sum_{J \in (\mathbb{Z}_{\ge 0})^n} c_{k,J}\, \frac{(z_1 - z_1^0)^{J_1}}{J_1!} \cdots \frac{(z_n - z_n^0)^{J_n}}{J_n!}\,, \qquad k = 1, \ldots, m\,,$$

with Taylor coefficients $c_{k,J}$. Then any assignment of values to $c_{k,J}$ for all $J \in (\mathbb{Z}_{\ge 0})^n$ such that $\partial^J u_k$ is not a principal derivative and

$$q_j(f_1, \ldots, f_m)|_{(z_1,\ldots,z_n)=(z_1^0,\ldots,z_n^0)} \;\neq\; 0\,, \qquad j = 1, \ldots, t\,,$$

gives rise to formal power series solutions

$$u_1(z_1, \ldots, z_n) \;=\; f_1\,, \quad \ldots, \quad u_m(z_1, \ldots, z_n) \;=\; f_m\,,$$

of S around $(z_1^0, \ldots, z_n^0)$ determined by the consistent system of algebraic equations for $c_{k,J}$

$$(\partial^J p_i)(f_1, \ldots, f_m)|_{(z_1,\ldots,z_n)=(z_1^0,\ldots,z_n^0)} \;=\; 0\,, \qquad J \in (\mathbb{Z}_{\ge 0})^n\,, \qquad i = 1, \ldots, s\,,$$

and conversely, every formal power series solution of S around $(z_1^0, \ldots, z_n^0)$ stems from such an assignment. If $>$ is a Riquier ranking, then (sufficiently generic) initial conditions determined by convergent power series yield convergent power series solutions.

3.3 Elimination

Thomas' algorithm can be used to solve various differential elimination problems. This section presents results on certain rankings on the differential polynomial ring $R = K\{u_1, \ldots, u_m\}$ which allow to compute all differential consequences of a given differential system involving only a specified subset of the differential indeterminates $u_1, \ldots, u_m$. In other words, this technique allows to determine all differential equations which are satisfied by certain components of the solution tuples. We adopt the notation from the previous section.

Definition 3.33. Let $I_1, I_2, \ldots, I_k$ form a partition of $\{1, 2, \ldots, m\}$ such that $i_1 \in I_{j_1}$, $i_2 \in I_{j_2}$, $i_1 \leq i_2$ implies $j_1 \leq j_2$. Let $B_j := \{u_i \mid i \in I_j\}$, $j = 1, \ldots, k$. Moreover, fix some degree-reverse lexicographical ordering $>$ on $\mathrm{Mon}(\Delta)$. Then the *block ranking* on R *with blocks* $B_1, \ldots, B_k$ (with $u_1 > u_2 > \ldots > u_m$) is defined for $\theta_1 u_{i_1}, \theta_2 u_{i_2} \in \mathrm{Mon}(\Delta)\, u$, where $u_{i_1} \in B_{j_1}$, $u_{i_2} \in B_{j_2}$, by

$$\theta_1 u_{i_1} > \theta_2 u_{i_2} \quad :\Longleftrightarrow \quad \left\{ \begin{array}{l} j_1 < j_2 \quad \text{or} \quad \Big(j_1 = j_2 \quad \text{and} \quad (\theta_1 > \theta_2 \quad \text{or} \\[1ex] \qquad (\theta_1 = \theta_2 \quad \text{and} \quad i_1 < i_2)) \Big). \end{array} \right.$$

Such a ranking is said to satisfy $B_1 \gg B_2 \gg \ldots \gg B_k$.

Example 3.34. With respect to the block ranking on $K\{u_1, u_2, u_3\}$ with blocks $\{u_1\}$, $\{u_2, u_3\}$ (and $u_1 > u_2 > u_3$) we have $(u_1)_{(0,1)} > u_1 > (u_2)_{(1,2)} > (u_3)_{(1,2)} > (u_2)_{(0,1)}$.

In the situation of the previous definition, for every $i \in \{1, \ldots, k\}$, we consider

$$K\{B_i, \ldots, B_k\} \;:=\; K\{u \mid u \in B_i \cup \ldots \cup B_k\}$$

as a differential subring of R, endowed with the restrictions of the derivations $\partial_1, \ldots, \partial_n$ to $K\{B_i, \ldots, B_k\}$.

For any algebraic or differential system S we denote by $S^=$ (resp. $S^{\neq}$) the set of the left hand sides of all equations (resp. inequations) in S.

Proposition 3.35 ([Rob14], Prop. 3.1.36)**.** *Let S be a simple differential system, defined over R, with respect to a block ranking with blocks $B_1, \ldots, B_k$. Moreover, let E be the differential ideal of R generated by $S^=$ and q the product of the initials and separants of all elements of $S^=$. For every $i \in \{1, \ldots, k\}$, let E_i be the differential ideal of $K\{B_i, \ldots, B_k\}$ generated by $P_i := S^= \cap K\{B_i, \ldots, B_k\}$ and let q_i be the product of the initials and separants of all elements of P_i. Then, for every $i \in \{1, \ldots, k\}$, we have*

$$(E : q^\infty) \cap K\{B_i, \ldots, B_k\} \;=\; E_i : q_i^\infty \,.$$

In other words, the differential equations implied by S which involve only the differential indeterminates in $B_i \cup \ldots \cup B_k$ are precisely those whose pseudo-remainders modulo the elements of $S^= \cap K\{B_i, \ldots, B_k\}$ and their derivatives are zero.

Corollary 3.36 ([Rob14], Cor. 3.1.37)**.** *Let S be a (not necessarily simple) differential system, defined over R, and $S_1, \ldots, S_r$ a Thomas decomposition of S with respect to a block ranking with blocks $B_1, \ldots, B_k$. Moreover, let E be the differential ideal of R generated by $S^=$ and q the product of all elements of $S^{\neq}$. Let $i \in \{1, \ldots, k\}$ be fixed. For every $j \in \{1, \ldots, r\}$, let $E^{(j)}$ be the differential ideal of $K\{B_i, \ldots, B_k\}$ generated by $P_j := S_j^= \cap K\{B_i, \ldots, B_k\}$ and let $q^{(j)}$ be the product of the initials and separants of all elements of P_j. Then we have*

$$\sqrt{E : q^\infty} \,\cap\, K\{B_i, \ldots, B_k\} \;=\; (E_1 : q_1^\infty) \cap \ldots \cap (E_r : q_r^\infty)\,.$$

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