Strong Orthogonal Arrays of Strength Two Plus and Second Order Saturated Designs

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1. Introduction

Designs for computer experiments

- distance or discrepancy criteria
- model-dependent criteria
- orthogonality
- orthogonal arrays or strong orthogonal arrays

  McKay, Beckman and Conover (1979); Owen (1992); Tang (1993); He and Tang (2013, 2014).

Strong orthogonal arrays (SOAs)
(when compared with ordinary orthogonal arrays)

- more space-filling
- very expensive for strength 4 or higher
- can be constructed at almost no cost for strength 3
- free but no more space-filling for strength 2

Inspired by (t, m, s)*-nets (Niederreiter 1992), SOAs were introduced and studied by He and Tang (2013, 2014)*

2. SOAs of strength $2+$ and their characterization

SOAs of strength $2+$

- almost as space-filling as SOAs of strength 3
- but much more economical than the latter

An $\mathrm{SOA}(n, m, s^2, 2+)$

- has n runs and m factors of s^2 levels
- any two columns are collapsible into an $\mathrm{OA}(n, 2, s^2 \times s, 2)$ and an $\mathrm{OA}(n, 2, s \times s^2, 2)$
- i.e., the design achieves stratifications on $(s^2) \times s$ and $s \times (s^2)$ grids in all two-dimensions.

An SOA$(16, 10, 4, 2+)$ is given below:

2	2	2	2	2	2	0	0	0	0
2	2	0	2	0	0	1	2	2	2
2	0	2	0	2	1	2	1	2	2
2	0	0	0	0	3	3	3	0	0
0	2	2	1	1	2	2	2	1	2
0	2	0	1	3	0	3	0	3	0
0	0	2	3	1	1	0	3	3	0
0	0	0	3	3	3	1	1	1	2
1	1	1	2	2	2	2	2	2	1
1	1	3	2	0	0	3	0	0	3
1	3	1	0	2	1	0	3	0	3
1	3	3	0	0	3	1	1	2	1
3	1	1	1	1	2	0	0	3	3
3	1	3	1	3	0	1	2	1	1
3	3	1	3	1	1	2	1	1	1
3	3	3	3	3	3	3	3	3	3

It achieves stratifications on 2×4 and 4×2 grids in every two-dimension.

To compare, in 16 runs,

- For $m \leq 7$ factors, an SOA of strength 3 can be constructed, which can do 2×4 and 4×2 grids in two-dimensions, and $2 \times 2 \times 2$ in three-dimensions.
- For $m \leq 5$ factors, an OA is available and can do 4×4 grids in two-dimensions.

An characterization of SOAs of strength $2+$

An $SOA(n, m, s^2, 2+)$, say D, exists if and only if there exist two arrays A and B where

- $A = (a_1, \ldots, a_m)$ *is an* $OA(n, m, s, 2)$ *and*
- $B = (b_1, \ldots, b_m)$ *is an* $OA(n, m, s, 1)$ *such that*
- (a_j, a_k, b_k) *is an orthogonal array of strength 3 for any* $j \neq k$.

The three arrays are linked through $D = sA + B$.

3. Construction using 2^{m-p} designs

Let S denote the saturated regular design of $n = 2^k$ runs with $m = n - 1$ factors.

Theorem 1. *If an SOA of strength $2+$ is to be constructed using regular A and B with their columns selected from S, then it is necessary and sufficient that $\bar{A} = S \setminus A$ is an SOS design.*

A design C is an SOS design if any $d \in \bar{C}$ can be written as $d = ab$ for some $a, b \in C$ (Block and Mee 2003).

The proof shows that an SOA$(2^k, m, 4, 2+)$ can be constructed from an SOS design C as follows:

1. Take $A = \bar{C}$. Write $A = (a_1, \ldots, a_m)$.
2. Since C is an SOS design, we must have $a_j = b_j b_j'$ for some $b_j, b_j' \in C$. Take $B = (b_1, \ldots, b_m)$.
3. Obtain D, an SOA$(2^k, m, 4, 2+)$, using

$$D = 2A + B.$$

The paper provides four constructions of SOS designs, based on which the bounds on the maximum number of factors in an SOA of strength 2+ have been obtained.

Table 1. Maximum numbers of factors for SOAs of strength 3 and $2+$.

k	$n = 2^k$	h_k (strength 3)	m_k (strength $2+$)
4	16	7	10*
5	32	15	22*
6	64	31	50*
7	128	63	106
8	256	127	226

4. Constructions using s^{m-p} designs

Theorem 4. *For any $k \geq 3$ and any prime power $s \geq 3$, an $SOA(s^k, m, s^2, 2+)$ can be constructed where*

$$m = (s^k - 1)/(s - 1) - ((s - 1)^k - 1)/(s - 2).$$

Table 2. A comparison of the number m' of factors for $SOA(s^k, m', s^3, 3)$ in He and Tang (2014) and the number m'' of factors for $SOA(s^k, m'', s^2, 2+)$ from Theorem 4.

k	s	$n = s^k$	m'	m''
3	3	27	4	6
3	4	64	5	8
3	5	125	6	10
4	3	81	10	25
4	4	256	17	45
4	5	625	26	71
3	s	s^3	$s+1$	$2s$
4	s	s^4	s^2+1	$3s^2-s+1$

5. A generalization

We only give one example here.

0	0	0	2	2	2	4	4	4	1	1	1	3	3	3	5	5	5
0	2	4	0	2	4	0	2	4	1	3	5	1	3	5	1	3	5
0	2	4	2	4	0	4	0	2	1	3	5	3	5	1	5	1	3
0	2	4	4	0	2	2	4	0	1	3	5	5	1	3	3	5	1

This array (transposed) achieves stratifications on 6×3 and 3×6 grids in all two-dimensions.

6. Discussion and further work

- further design selection by orthogonality, distance and discrepancy criteria
- minimal SOS designs
- construction using nonregular designs (Xu, Phoa and Wong 2009)

Minimal SOS designs

- concept and usefulness
- characterization using clear 2fi's
- 1-saturating sets and linear codes with covering radius 2: Davydov, Marcugini and Pambianco (2006).

Construction using nonregular designs

- similar results can be established
- an example is an SOA$(48, 33, 4, 2+)$

Thank you!