Coarse Embeddings into $c_0(\Gamma)$

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(joint with Petr Hájek)

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$$d(x, y) \leq \|f(x) - f(y)\|_{c_0} \leq Kd(x, y), \text{ for all } x, y \in M.$$

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Question: Is there a non separable version of Aharoni's result? More precisely, which metric spaces or Banach spaces can be bi-Lipschitzly embedded into $c_0(\text{dens}(M))$?
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Answer: not many, at least if density is large enough.

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f is *coarse embedding*: $\forall t > 0 : w_f(t) < \infty$ and $\lim_{t\to\infty} \rho_f(t) = \infty$.

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We say it has Property (Q), if for $n \in \mathbb{N}$, any set Γ, and any $\sigma : [\aleph]^n \to \Gamma$ one of the following statements is true:

$$\exists (s_j) \in [\aleph]^n \quad (s_j) \text{ pw. disj. and } \sigma(s_i) = \sigma(s_j), i \neq j \qquad (Q_1(\aleph))$$

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If $P(\aleph)$ holds then $\ell_p(\aleph) \not\hookrightarrow_{unif} c_0(\Gamma)$.

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Theorem (Hájek & S, 2017)

Assume $P(\aleph)$ holds and X is a Banach space with nontrivial cotype and dens$(X) \geq \aleph$.
Then $X \not\hookrightarrow_{coarsely\,/\,unif}\ c_0(\Gamma)$.
Moreover, if X has a symmetric basis, dens$(X) \geq \aleph$, and $X \hookrightarrow_{coarsley} c_0(\Gamma)$ then X is isomorphic to $c_0(\text{dens}(X))$.

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Assume M has (CSP), and $\mathcal{U} = \{B_2(x) : x \in M\}$.

Then $\mathcal{U}$ refines a point finite covering $\mathcal{V}$ with $r = \sup_{V\in\mathcal{V}} \operatorname{diam}(V) < \infty$.

Choose $n \in \mathbb{N}$, with $\sqrt{n} > r$, pick for every $s \in [\aleph]^n$ a $V_s \in \mathcal{V}$, so that $x_s \in B_2(x_s) \subset V_s$, and consider

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- If $s \cap t = \emptyset \Rightarrow \sigma(s) \neq \sigma(t)$ $(\mathrm{diam}(V_s) \le r)$, thus P_1 not satisfied.
- If $s \in [\aleph]^{n-1}$ and $S = \sigma(\{s \cup \{\gamma\} : \gamma < \aleph\})$, then $x_s \in B_2(x_{s\cup\{\gamma\}}) \subset V_{s\cup\{\gamma\}}$, for all $\gamma < \aleph$, thus S finite. Thus P_2 is not satisfied.

$C(K)$ spaces, which are precluded from being coarsely embedded into $c_0(\Gamma)$

For a cardinal number $\aleph$, consider

$[\aleph]^{\le n} = \{F \subset \aleph : |F| \le n\} \equiv \{f : \aleph \to \{0,1\}, |\text{supp}(f)| \le n\} \subset \{0,1\}^\aleph$

$K_{\aleph,n} = [\aleph]^{\le n}$ with product topology, $K_\aleph :=$ Alex. compct.$\big(\oplus_{n=1}^\infty K_{\aleph,n}\big)$

Note: $\text{CB}(K_\aleph) = \omega_0 + 1$,

Deville, Godefroy, Zizler, 1990: $\text{CB}(K) < \omega_0 \Rightarrow C(K) \equiv_{Lip} c_0(\Gamma)$.

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If $\aleph$ satisfies $Q(\aleph)$, then $C(K_\aleph) \not\hookrightarrow_{uniformly\,/\,coarsely} c_0(\Gamma)$.

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Theorem (Hajek & S)

$P(\aleph)$ and $Q(\aleph)$ are equivalent, for any infinite cardinal number.
Thus, $Q(\aleph_\omega)$ holds and consequently $C(K_{\aleph_\omega}) \not\hookrightarrow_{uniformly\,/\,coarsely} c_0(\Gamma)$.

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b) For all $s \in [\aleph]^{n-1}$ the set $\sigma(\{s \cup \{\gamma\} : \gamma \in \aleph \setminus s\})$ is finite.

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b) For all $s \in [\aleph]^{n-1}$ the set $\sigma(\{s \cup \{\gamma\} : \gamma \in \aleph \setminus s\})$ is finite.

Then there exists a set $\tilde{\mathcal{C}}$ and a map $\tilde{\sigma} : [\aleph]^n \to \tilde{\mathcal{C}}$, which still satisfies (b) and moreover

c) For all $\tilde{c} \in \tilde{\mathcal{C}}$ there is a $\beta(\tilde{c}) \in \aleph$ so that

$$\tilde{\sigma}^{-1}(\tilde{c}) \subset H_{\beta(c)} = \{s \in [\aleph]^n : \beta(c) \in s\}.$$

Re-colouring argument

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Condition (c) now implies that any two disjoint $s, t \in [\aleph]^n$ must have different images under $\tilde{\sigma}$, and thus $\tilde{\sigma}$ witnesses that $P(\aleph)$ is not satisfied.

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3) Does ℓ_∞ coarsely embed into any $c_0(\Gamma)$? (Yes assuming $\aleph_\omega \leq c$)