Ribe's theorem and Krivine stabilization

Bünyamin Sarı

University of North Texas

CIRM–March 5–9, 2018

Ribe's theorem

Suppose $\phi : X \to Y$ is a bijection which is Lipschitz for large distances, i.e., There exist K_d such that

$$\|x - y\| \leq \|\phi(x) - \phi(y)\| \leq K_d \|x - y\|, \quad \|x - y\| \geq d$$

Then X and Y have the same local structure.

Ribe's theorem

Suppose $\phi : X \to Y$ is a bijection which is Lipschitz for large distances, i.e., There exist K_d such that

$$\|x - y\| \leq \|\phi(x) - \phi(y)\| \leq K_d \|x - y\|, \quad \|x - y\| \geq d$$

Then X and Y have the same local structure.

For all finite dimensional $E \subset X$ and $\varepsilon > 0$, E linearly embeds into Y with constant $(1 + \varepsilon)K_\infty$, and vice versa.

References/proofs

References/proofs

1. Ribe's original proof

References/proofs

1. Ribe's original proof
2. The proof in Benyamini-Lindenstrauss (based on a presentation by Enflo of the original proof)

References/proofs

1. Ribe's original proof
2. The proof in Benyamini-Lindenstrauss (based on a presentation by Enflo of the original proof)
3. Bourgain's proof (known as the discretization theorem, see Giladi, Naor, Schechtman, '12)

References/proofs

1. Ribe's original proof
2. The proof in Benyamini-Lindenstrauss (based on a presentation by Enflo of the original proof)
3. Bourgain's proof (known as the discretization theorem, see Giladi, Naor, Schechtman, '12)
4. Differentiation and ultrapower argument due to Heinrich and Mankiewicz

Ribe's argument

Ribe's theorem

Ribe's argument

Ribe's theorem

Start with $E = \overline{\text{span}}(x_i)_{i=1}^{n} \subset X$. A combinatorial argument that depends only on n.

Ribe's argument

Ribe's theorem

Start with $E = \overline{\text{span}}(x_i)_{i=1}^n \subset X$. A combinatorial argument that depends only on n.

$$\mathcal{M}_m = \left\{ \sum_{i=1}^n k_i x_i : k_i \in \mathbb{Z}, |k_i| \le m \right\} \setminus \{0\}$$

the set of lattice points of size $(2m+1)^n$

Ribe's argument

Ribe's theorem

Start with $E = \overline{\mathrm{span}}(x_i)_{i=1}^n \subset X$. A combinatorial argument that depends only on n.

$$\mathcal{M}_m = \left\{ \sum_{i=1}^n k_i x_i : k_i \in \mathbb{Z}, |k_i| \le m \right\} \setminus \{0\}$$

the set of lattice points of size $(2m+1)^n$

$s\mathcal{M}_m$: the scaled lattices with step size $s \in \mathbb{N}$

Ribe's argument

Ribe's theorem

Start with $E = \overline{\mathrm{span}}(x_i)_{i=1}^n \subset X$. A combinatorial argument that depends only on n.

$$\mathcal{M}_m = \left\{ \sum_{i=1}^n k_i x_i : k_i \in \mathbb{Z}, |k_i| \leq m \right\} \setminus \{0\}$$

the set of lattice points of size $(2m+1)^n$

$s\mathcal{M}_m$: the scaled lattices with step size $s \in \mathbb{N}$

$u + s\mathcal{M}_m$ scaled and translated by $u \in \mathcal{M}$

The formula
Ribe's theorem

Let $\varepsilon > 0$. Fix

$$m_0 = m_0(n, \varepsilon) \tag{1}$$

so that $\{x/\|x\| : 0 \neq x \in \mathcal{M}_{m_0}\}$ forms an ε/n-dense set in the unit sphere of E.

If $\mathcal{L}$ is a finite lattice of the form $u + s\mathcal{M}_m$, define a function $\phi_{\mathcal{L}}$ on $s\mathcal{M}_{m_0}$ by

$$\phi_{\mathcal{L}}(y) = \frac{1}{|\mathcal{L}|} \sum_{x \in \mathcal{L}} \big(\phi(x+y) - \phi(x)\big),$$

where $|\mathcal{L}| = (2m+1)^n$.

$\phi_{\mathcal{L}}$ is nearly linear

Ribe's theorem

Lemma

Let $\varepsilon > 0$, and $m_0 = m_0(n, \varepsilon)$ be as above. Then there exists $m' = m'(\varepsilon, m_0, K)$ such that for every s and every finite lattice $\mathcal{L}$ of the form $u + s\mathcal{M}_m$ with $m \geq m'$, $\phi_{\mathcal{L}}$ satisfies

$$\left\| \phi_{\mathcal{L}}(x) - \sum_{i=1}^{n} k_i \phi_{\mathcal{L}}(sx_i) \right\| \leq \varepsilon \|x\|$$

for every $x = s\sum_{i=1}^{n} k_i x_i \in s\mathcal{M}_{m_0}$. In particular, the linear operator $T_{\mathcal{L}} : E \to Y$ defined by $T_{\mathcal{L}}(sx_i) = \phi_{\mathcal{L}}(sx_i)$ for $1 \leq i \leq n$ satisfies

$$\|\phi_{\mathcal{L}}(x) - T_{\mathcal{L}}x\| \leq \varepsilon \|x\|$$

for every $x \in s\mathcal{M}_{m_0}$.

Proof. Given $x \in \mathcal{M}$, if $\mathcal{L}$ is large then $\mathcal{L}$ and its translate $x + \mathcal{L}$ coincide except for a very small percentage.

So for instance

$$
\begin{aligned}
& \|\phi_{\mathcal{L}}(sx_i + sx_j) - \phi_{\mathcal{L}}(sx_i) - \phi_{\mathcal{L}}(sx_j)\| \\
= \; & \|\frac{1}{|\mathcal{L}|}\sum_{x\in\mathcal{L}}\big(\phi(sx_i + sx_j + x) - \phi(x)\big) \\
& -\frac{1}{|\mathcal{L}|}\sum_{x\in\mathcal{L}}\big(\phi(sx_i + x) - \phi(x)\big) - \frac{1}{|\mathcal{L}|}\sum_{x\in\mathcal{L}}\big(\phi(sx_j + x) - \phi(x)\big)\| \\
= \; & \|\frac{1}{|\mathcal{L}|}\sum_{x\in\mathcal{L}}\big(\phi(sx_i + sx_j + x) - \phi(sx_i + x)\big) - \frac{1}{|\mathcal{L}|}\sum_{x\in\mathcal{L}}\big(\phi(sx_j + x) - \phi(x)\big)\| \\
= \; & \|\frac{1}{|\mathcal{L}|}\sum_{x\in sx_i+\mathcal{L}}\big(\phi(sx_j + x) - \phi(x)\big) - \frac{1}{|\mathcal{L}|}\sum_{x\in\mathcal{L}}\big(\phi(sx_j + x) - \phi(x)\big)\| \\
< \; & \frac{\varepsilon K\|sx_j\|}{2Knm_0}
\end{aligned}
$$

if $\sup_{x\in s\mathcal{M}_{m_0}} \frac{|(x+\mathcal{L})\triangle\mathcal{L}|}{|\mathcal{L}|} < \frac{\varepsilon}{2Knm_0}$.

Invertibility

Ribe's theorem

Lemma

Let $\varepsilon > 0$, $m_0 = m_0(n, \varepsilon)$ and $m' = m'(m_0, n, \varepsilon, K)$ be as above. Then for all $m \geq m'$ there is a step size s' and an $M = M(\mathcal{M}_{m_0}, m)$ such that for every finite lattice $\mathcal{L}'$ of size greater than $(2M + 1)^n$ there exists a sublattice $\mathcal{L} \subset \mathcal{L}'$ of size $(2m + 1)^n$ and step $s \leq s'$ such that for every $y \in \mathcal{M}_{m_0}$ there exists a norm one functional u^* in Y^* such that

$$\langle u^*, \phi(x + sy) - \phi(x) \rangle \geq s\|y\|/2$$

for every $x \in \mathcal{L}$.
In particular, $\|\phi_{\mathcal{L}}(sy)\| \geq s\|y\|/2$ for all $y \in \mathcal{M}_{m_0}$.

Diagonal lattices

For $\alpha > 0$ lattice $\mathcal{L}$ is **α-close to the diagonal** if for some $a \in \mathbb{N}$ every element of $\mathcal{L}$ is of the form $\sum_{i=1}^{n}(a + k_i)x_i$ where $|\frac{k_i}{a}| < \alpha$.

Lemma

Let $0 < \alpha < \frac{1}{4}$, and if $r > 2$ assume $\alpha < \frac{1}{3}(2^{\frac{1}{r-2}} - 1)$. Let $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ be of the form $a_i = a + k_i$ with $|\frac{k_i}{a}| < \alpha$, and let $k \in \mathbb{Z}$ with $|\frac{k}{a}| < \alpha$. If $y_j = (a_1, \ldots, a_j + k, \ldots, a_n)$ for $j = 1, \ldots, n$, then

$$\big|\|y_j\|_r - \|y_{j'}\|_r\big| < 4\alpha 3^{2r} k(r-1)n^{\frac{1}{r}-1}.$$

Eliminating F

Let

$$X \xrightarrow{\phi} Y \xrightarrow{\pi} F$$

where π is a projection onto finite dim F. Let $\mathcal{L}$ be a finite lattice of size $(2m+1)^n$ with step s which is α-close to the diagonal.
Suppose that ϕ additionally satisfies the following.

For every y, z in $\mathcal{L}$,

$$\|\pi\phi(\mathbf{y}) - \pi\phi(\mathbf{z})\| < \varepsilon' \mathbf{sm} \;\; \textit{whenever} \;\; \big|\|\mathbf{y}\| - \|\mathbf{z}\|\big| < \mathbf{4} \cdot \mathbf{3}^{\mathbf{2r}}(\mathbf{r}-\mathbf{1})\alpha \mathbf{ms}.$$

Then for every $1 \le i, j \le n$, $\phi_{\mathcal{L}}$ satisfies

$$\|\pi\phi_{\mathcal{L}}(sx_i) - \pi\phi_{\mathcal{L}}(sx_j)\| < \varepsilon' s.$$

Krivine stabilization

Theorem (Odell-Rosenthal-Schlumprecht, '93)

Let f be a real-valued uniformly continuous function defined on the unit sphere of a Banach space X with a basis $(e_i)_{i=1}^{\infty}$. Then there is a constant λ_0 such that for every $\varepsilon > 0$ and for every integer k there is a block sequence $(y_j)_{j=1}^{k}$ of $(e_i)_{i=1}^{\infty}$ with the property that $|f(y) - \lambda_0| < \varepsilon$ for every y in $\mathrm{span}(y_j)_{j=1}^{k}$ with $\|y\| = 1$.

Krivine stabilization

For a uniformly continous ϕ fix a function δ_ϕ such that for all $\varepsilon > 0$, $\|\phi(x) - \phi(y)\| < \varepsilon$ whenever $\|x - y\| < \delta_\phi(\varepsilon)$.

Theorem

Let ϕ be a uniformly continuous map from a ball $B_X(\rho)$ of radius $\rho > 0$ in a Banach space X with a basis $(e_i)_{i=1}^\infty$ into a finite dimensional real normed space F. Then for all $\varepsilon' > 0$ and for every integer k there is a block sequence $(y_j)_{j=1}^k$ of $(e_i)_{i=1}^\infty$ with the property that for all $y, z \in B_X(\rho)$ with $\big|\|z\| - \|y\|\big| < \delta_\phi(\varepsilon')$ we have $\|\phi(y) - \phi(z)\| < \varepsilon'$.