On a difference between two methods of low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces

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Definition

Let M be a metric space, X be a Banach space, and $f : M \to X$.

We say that f is a bi-Lipschitz embedding of M into X if there exists a constant C so that for all $u, v \in M$

$$\|f(u) - f(v)\|_X \leq d_M(u, v) \leq C\|f(u) - f(v)\|_X.$$

The smallest possible C is called the distortion of f.

Methods of embedding finite metric spaces into non-superreflexive Banach spaces

The method used by Bourgain (1986) for trees and by Johnson and Schechtman (2009) for binary diamonds and Laakso graphs, uses the following classical cornerstone result

Theorem (Pták 59, Singer 62, Pełczyński 62, James 64, Milman-Milman 65)

A Banach space X is not reflexive if and only if

there exists θ with $0 < \theta < 1$ and a sequence (x_n) in X such that for any finitely supported scalar sequence (a_n) we have

$$\theta \sup_j \Big| \sum_{n \le j} a_n \Big| \le \Big\| \sum_n a_n x_n \Big\|_X \le \sum_n |a_n|$$

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$$\theta \underbrace{\sup_j \left| \sum_{n \le j} a_n \right|}_{\|a\|_s} \le \left\| \sum_n a_n x_n \right\|_X \le \underbrace{\sum_n |a_n|}_{\|a\|_1}$$

summing norm $\|a\|_s$ $\qquad$ $\|a\|_1$ **ℓ_1 norm**

Suppose that there exists an embedding $f : M \to c_{00}$ so that for all u, v in M

$$\|f(u) - f(v)\|_1 \leq d_M(u, v) \leq C\|f(u) - f(v)\|_s \tag{1}$$

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This method of construction of an embedding is based on the factorization between the summing basis and the unit vector basis of ℓ_1, and I will refer to it as the factorization method.

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We used to show that all finitely branching diamond and Laakso graphs embed into all non-superreflexive spaces with distortion not exceeding $8+\varepsilon$, independently of the number of branching.

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The first goal of the talk is to prove that embeddings of multibranching diamonds with distortions bounded independently of the number of branches cannot be constructed using the factorization method.

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For all $C > 1$ there exists $k(C) \in \mathbb{N}$ so that if for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$ there exists an embedding $f_n : D_{n,k} \to c_{00}$ so that $\forall u, v \in D_{n,k}$

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then $k \le k(C)$.

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Note that this result does not exclude the possibility that $\forall k \in \mathbb{N}\ \exists C = C(k) < \infty$ so that $\forall n \in \mathbb{N}\ \exists f_n : D_{n,k} \to c_{00}$ that satisfies condition $(s\text{-}\ell_1)$.

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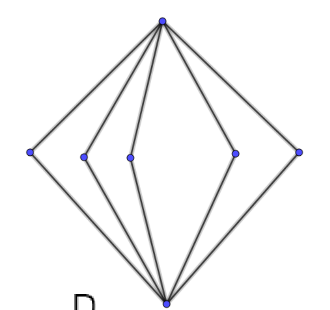
We do not know whether such numbers $C(k)$ exist for all $k \in \mathbb{N}$, or even whether $C(3)$ exists (Johnson and Schechtman (2009) proved that $C(2)$ exists).

Definition of multi-branching diamonds

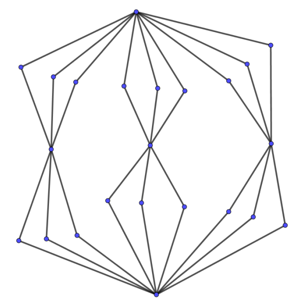
For any integer $k \geq 2$, we define $D_{1,k}$ to be the graph consisting of $(k+2)$ vertices, two of which are called top and bottom and are joined by k independent paths of length 2, i.e. $D_{1,k}$ is the complete bipartite graph $K_{2,k}$.

For any $n \in \mathbb{N}$, if $D_{n-1,k}$ is already defined, the graph $D_{n,k}$ is obtained from $D_{n-1,k}$ by replacing each edge in $D_{n-1,k}$ by a copy of $D_{1,k}$.

We equip $D_{n,k}$ with the shortest path distance.



$D_{1,k}$



$D_{2,k}$

Outline of a proof of the theorem

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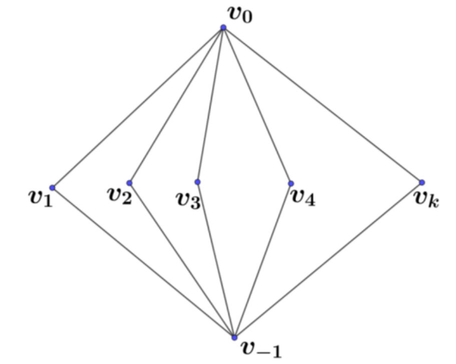
Then there exists a "vertically-faithful" embedding of $D_{1,k}$ into c_{00}, that is, $\exists g : D_{1,k} \to c_{00}$ so that $g(v_{-1}) = 0$, $g(v_i) = x_i$ and $\forall i, j \le k$



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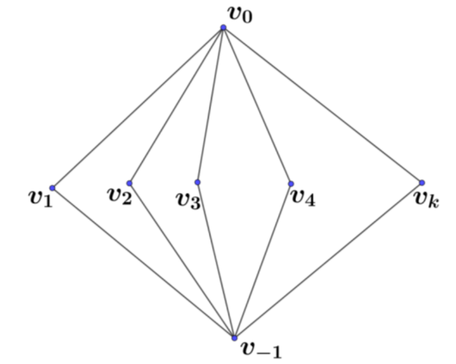

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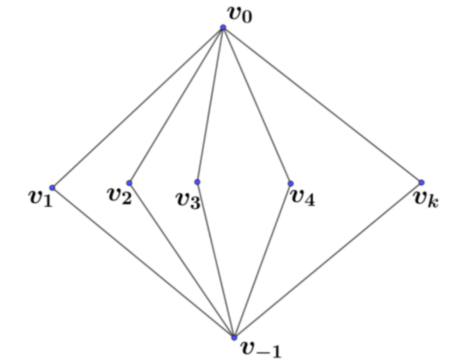

Idea/method goes back to Matoušek (1989,1999), Lee, Raghavendra (2010), Mendel, Naor (2013) "self-improvement argument", coarse differentiation (Eskin, Fisher, Whyte 2006)

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that is, for some $N \in \mathbb{N}$, the top vertex of $D_{1,k}$ is mapped onto a vector in ℓ_1^N whose every coordinate is 1 or -1,

and all "middle" vertices of $D_{1,k}$ are mapped onto elements of ℓ_1^N such that their pairwise ℓ_∞-distance does not exceed 1,

their pairwise summing norm distance is at least αN, where $\alpha = \frac{1}{2C^2}$,

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and the “vertical faithfulness” is preserved, so

There exist $N \in \mathbb{N}$ and elements $z_i \in c_{00}$, for $i \in \{1, \ldots, k\}$, so that

$$\begin{aligned}
&\forall i \in \{1, \ldots, k\} && \operatorname{supp}(z_i) \subseteq \{1, \ldots, N\}, \\
&\forall i, j \in \{1, \ldots, k\}\ \forall m \in \{1, \ldots, N\} && |z_{im} - z_{jm}| \leq 1, \\
&\forall i, j \in \{1, \ldots, k\},\ i \neq j, && \|z_i - z_j\|_s \geq \alpha N \geq 2.
\end{aligned}$$

Outline of a proof of the theorem

Main Step We use the Ramsey theorem to show that

For every $\alpha \in (0,1)$, there exists a natural number $k(\alpha)$, so that

if there exist $k, N \in \mathbb{N}$, and $\{z_i\}_{i=1}^k \subset c_{00}$ with

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then

$$k \le k(\alpha).$$

Sketch of proof of the Main Step

For every $i, j \in \{1, \ldots, k\}$ with $i \neq j$, we will denote by $r(i, j)$ the smallest index that witnesses the fact that $\|z_i - z_j\|_s \geq \alpha N$, that is, the smallest integer in $\{1, \ldots, N\}$ such that

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First, we prove that for every triple of pairwise distinct numbers $i, j, l \in \{1, \dots, k\}$ the values of indices $r(i,j), r(i,l), r(j,l)$ cannot stay together, and at least two of them are separated by a positive distance proportional to N, independent of the triple i, j, l.

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Next, we prove that if, say, for all triples $1 \leq i < j < l \leq k$, the maximum of $r(i,j), r(i,l), r(j,l)$ is always attained at $r(j,l)$, then, for all i, j, the values of $r(i,j)$ have to grow by a fixed amount with every increase of i and j. Thus $r(k-1,k)$ is larger than $r(1,2)$ by an amount proportional to N and k.
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Similarly, k is bounded if the maximum of $r(i,j), r(i,l), r(j,l)$ is always equal to $r(i,j)$, or always equal to $r(i,l)$.
This leads to a 3-coloring of triples from $\{1, \ldots, k\}$, and by the Ramsey theorem, we can find a subset of $[k]$ with monochromatic triples.

For every $1 \leq i < j < l \leq k$ we define

$$\text{MAX}_{ijl} = \max\{r(i,j), r(i,l), r(j,l)\}$$

$$\text{min}_{ijl} = \min\{r(i,j), r(i,l), r(j,l)\}$$

Lemma

For every pairwise distinct triple of numbers $i, j, l \in \{1, \dots, k\}$ we have:

$$\mathit{MAX}_{ijl} - \mathit{min}_{ijl} \geq \frac{\alpha N - 1}{2} \geq \frac{\alpha N}{4}$$

Let $\tau(i)$, $\tau(j)$ and $\tau(l)$ be the sums of the coordinates of z_i, z_j, z_l, resp., up to the term number $r(i,j)$. That is, for example,

$$\tau(l) = \sum_{m=1}^{r(i,j)} z_{lm}$$

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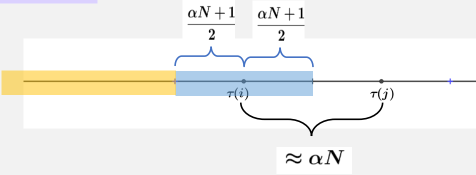
Say, $|\tau(l) - \tau(i)| \leq |\tau(l) - \tau(j)|$.



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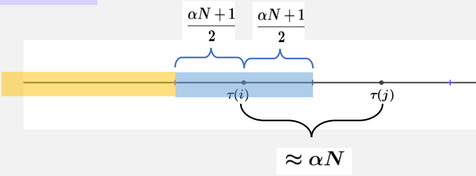

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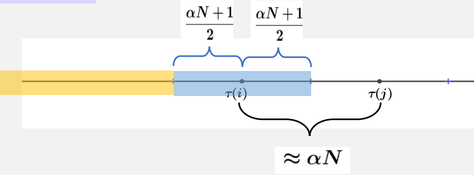

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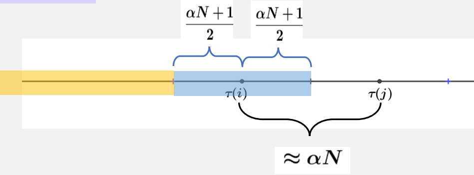

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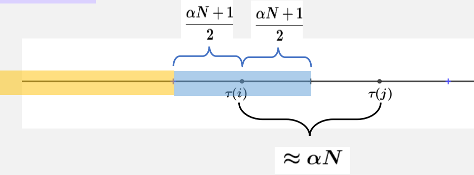

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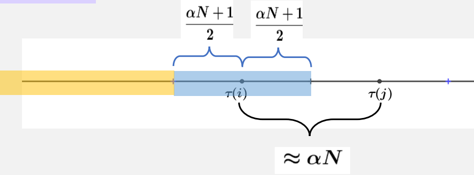

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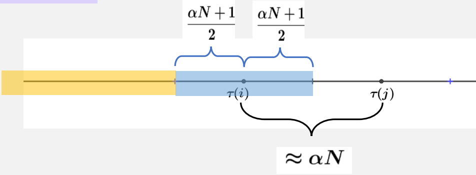

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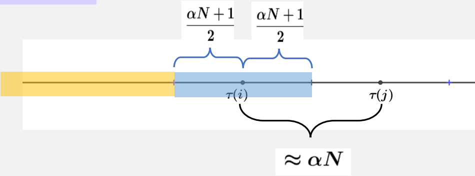

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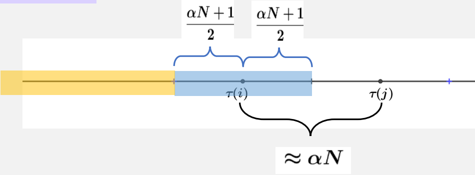

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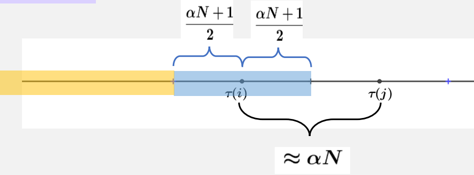

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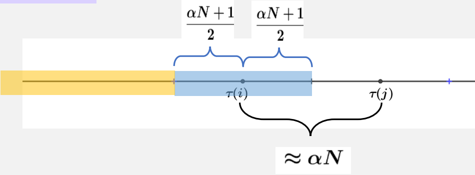

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Thus for every pairwise distinct triple of numbers $i, j, l \in \{1, \dots, k\}$ we have:

$$\mathsf{MAX}_{ijl} - \mathsf{min}_{ijl} \geq \frac{\alpha N - 1}{2} \geq \frac{\alpha N}{4}$$

We are now ready for the final step of the proof.

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- *red* - if $\text{MAX}_{ijl} = r(j, l)$,
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By the Ramsey Theorem, for every $s \in \mathbb{N}$, there exists a natural number $R_3(s, 3)$, so that for all $k \geq R_3(s, 3)$ the set $\{1, \dots, k\}$ contains a subset B with $\text{card}(B) \geq s$ such that every triple $(i, j, l) \in B^3$ is of the same color.

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We consider the three possible colors separately.

First suppose that every triple in B^3 is red.

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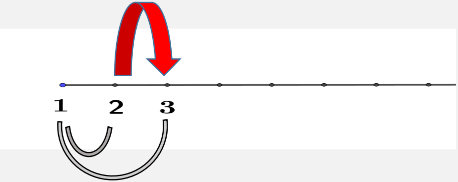

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$$r(2,3) \geq \alpha N + \frac{\alpha N}{4}$$

since both $r(1,2), r(1,3) \geq \alpha N$

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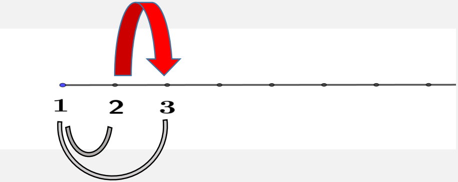

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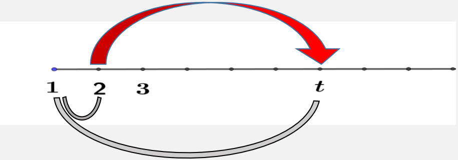

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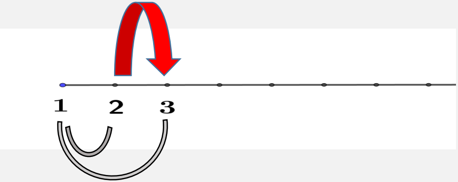

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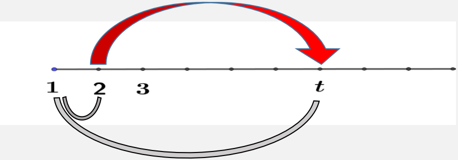

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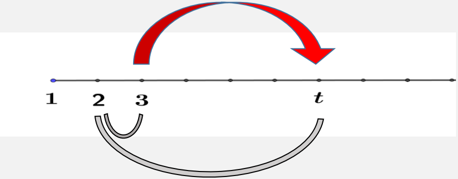

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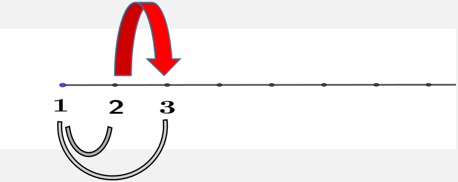

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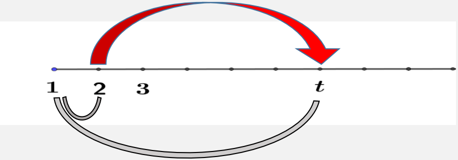

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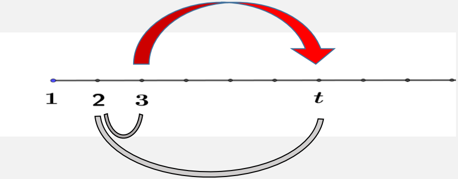

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By Induction

$$r(s-1,s) \geq \alpha N + (s-2) \cdot \frac{\alpha N}{4}$$

Hence

$$N \geq r(s-1, s) \geq s \cdot \frac{\alpha N}{4}$$

Thus $s \leq \left\lfloor \frac{4}{\alpha} \right\rfloor$, i.e., if all triples in B^3 are red, we have

$$\operatorname{card}(B) \leq \left\lfloor \frac{4}{\alpha} \right\rfloor.$$

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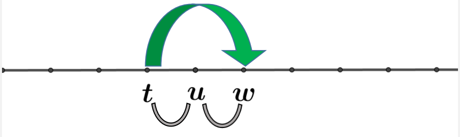
The case when all triples in B^3 are blue can be considered in the same way, we just list the elements in B in the decreasing order. Thus the same estimate for $\operatorname{card}(B)$ is valid also in this case.

When all triples in B^3 are green, we start in the middle of B.



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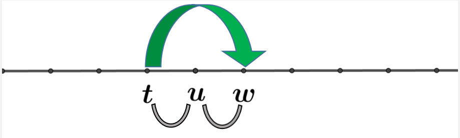

If $|w-t|>1$ then, by Lemma,

$$r(t,w)\geq \alpha N+\frac{\alpha N}{4}$$

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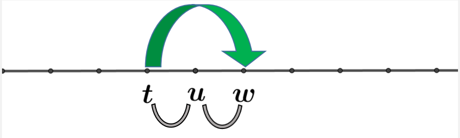

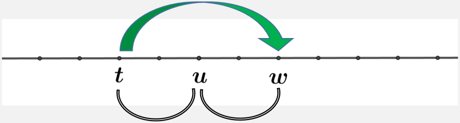

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Similarly, if $|w-t|>3$ then

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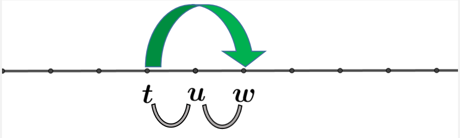

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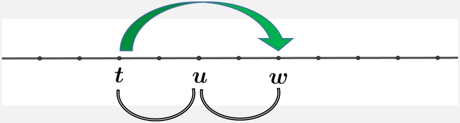

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By Induction we get that

If $|w-t|>2^q$ then

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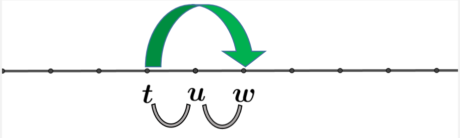

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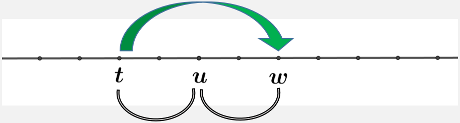

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$$r(b_t, b_u) \geq \alpha N + q \cdot \frac{\alpha N}{4}$$

Hence

$$N \geq r(1,s) \geq \left(\left\lfloor \log_2 |s-1| \right\rfloor + 2 \right) \frac{\alpha N}{4} \geq (\log_2 s) \cdot \frac{\alpha N}{4}.$$

Thus $\log_2 s \leq \left\lceil \frac{4}{\alpha} \right\rceil$, i.e. when all triples in B^3 are green we have

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which ends the proof of

Theorem

For all $C > 1$ there exists $k(C) \in \mathbb{N}$ so that if for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$ there exists an embedding $f_n : D_{n,k} \to c_{00}$ so that $\forall u, v \in D_{n,k}$

$$\|f_n(u) - f_n(v)\|_1 \le d_{D_{n,k}}(u, v) \le C\|f_n(u) - f_n(v)\|_s$$

then $k \le k(C)$, where $k(C) \stackrel{\text{def}}{=} R_3(2^{\lceil 8C^2 \rceil}, 3)$.

If time permits...

Theorem (Brunel, Sucheston 1975)

For each non-superreflexive space X there exists a Banach space E with an ESA basis such that E is finitely representable in X.

Definition (Brunel, Sucheston 1975)

A sequence $\{e_n\}$ is called

- equal signs additive (ESA) if for any finitely non-zero sequence $\{a_i\}$ of real numbers such that $\operatorname{sign} a_k = \operatorname{sign} a_{k+1}$,

$$\Big\|\sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1})e_k + \sum_{i=k+2}^{\infty} a_i e_i\Big\| = \Big\|\sum_{i=1}^{\infty} a_i e_i\Big\|.$$

- subadditive (SA) if for any finitely non-zero sequence $\{a_i\}$

$$\Big\|\sum_{i=1}^{k-1} a_i e_i + (a_k + a_{k+1})e_k + \sum_{i=k+2}^{\infty} a_i e_i\Big\| \leq \Big\|\sum_{i=1}^{\infty} a_i e_i\Big\|.$$

- invariant under spreading (IS) if for any finitely non-zero sequence $\{a_i\}$ and any increasing $(k_i)_i$

$$\Big\|\sum_{i=1}^{\infty} a_i e_i\Big\| = \Big\|\sum_{i=1}^{\infty} a_i e_{k_i}\Big\|.$$

Theorem (Brunel, Sucheston 1975) ESA $\iff$ (SA and IS)

An embedding of $D_{1,k}$ into a space with an ESA basis.

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Note that the element $(++--)$ has two metric midpoints $(0+-0)$ and $(+00-)$ whose distance from each other is

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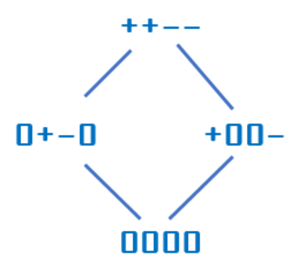
This gives a vertically faithful embedding of $D_{1,2}$ into E



To embed $D_{1,k}$ we map the bottom of $D_{1,k}$ to 0, and the top to

$$x_0 = (+ + - - \,|\, + + - - | + + - - | + + - - |....| + + - - |0000\ldots),$$

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Namely, for $1 \leq i \leq k$, we can define an element m_i by

$$\nu\text{-th block of } m_i = \begin{cases} 0 + -0 & \text{if } \; r_i(\nu) = 1, \\ +00- & \text{if } \; r_i(\nu) = -1, \end{cases}$$

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Thus for any k we have a vertically faithful embedding of D_{1k} into E.

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I will just show one iteration, i.e. how to embed D_{2k} into E.

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The top of $D_{2,k}$ will be mapped onto an element similar to

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but we will use more blocks and each block will be bigger.

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$$h = (+ + + + - - - -)$$

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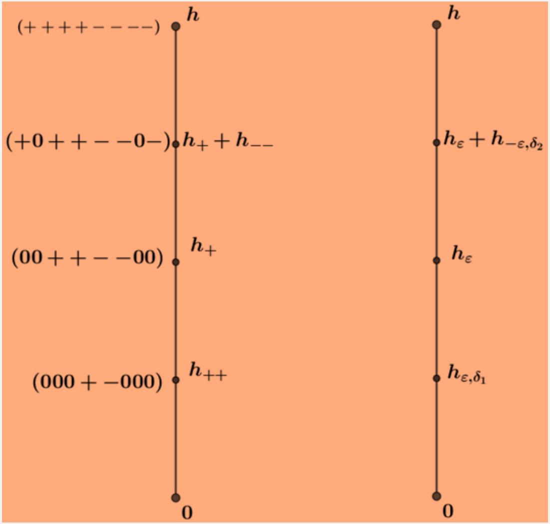
Thus one block gives multiple options for isometric embedding of vertical paths that connect the bottom and top of the graph.



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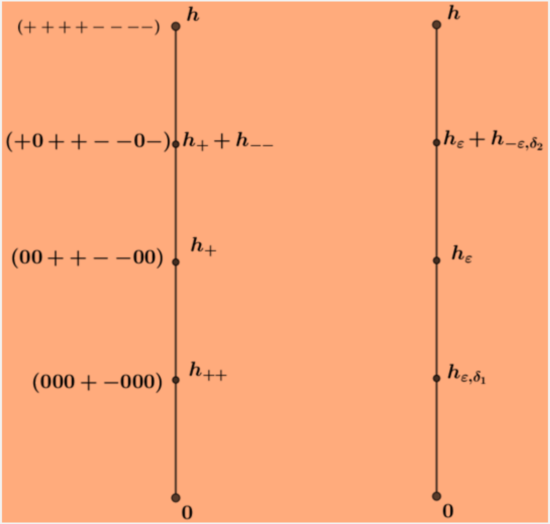

Multiple blocks give even more options

Thank you.