Some isometric properties of Lipschitz free spaces

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Non Linear Functional Analysis, CIRM

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- For every M: $\mathcal{F}(M) = \mathcal{F}(\mathit{completion}(M))$.

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$\Longrightarrow$ WLOG every M complete in what follows

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Theorem (L. C. García Lirola, A. Rueda Zoca, AP)

TFAE:

1. *M is a length space*
2. *$\mathcal{F}(M)$ has the Daugavet property*
3. *$Lip_0(M)$ has the Daugavet property*

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Lemma

X has the (DP) if and only if $\forall\ x \in B_X$ and
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X has the (DP) if and only if $\forall\ x \in B_X$ and $\forall\ S = S(x^*, \varepsilon)$ $\exists$ a slice $T \subset S$ such that $\forall\ y \in T : \|x - y\| > 2 - \varepsilon$.

Corollary

X has the (DP) if and only if $\forall\, x_1, \ldots, x_n \in B_X$ and $\forall\, S = S(x^*, \varepsilon)$ $\exists$ a slice $T \subset S$ such that $\forall\, y \in T$ $\forall\, i = 1, \ldots, n$: $\|x_i - y\| > 2 - \varepsilon$.

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In particular if X has the (DP) then $\forall\, x_1, \ldots, x_n \in B_X\ \forall\, \varepsilon > 0$ $\exists\, y \in B_X$ such that

$$\|x_i + y\| > 2 - \varepsilon,$$

i.e. X is octahedral.

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Theorem (A. Rueda Zoca, AP ’16)

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3. $\forall\, \varepsilon > 0\ \forall\, N \subset M$ *finite,* $\exists\, u \neq v \in M$ *such that*
$$(1-\varepsilon)(d(x,y)+d(u,v)) \leq d(x,u)+d(y,v)$$
for all $x, y \in N$.

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3. *$\forall\, \varepsilon > 0\ \forall\, N \subset M$ finite, $\exists\, u \neq v \in M$ such that*

$$(1-\varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$

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- We call the property in (3) the *long trapezoid property* (LTP).

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Proof of $\mathcal{F}(M)$ (DP) $\Longrightarrow$ M length

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M is called *local* if $\forall\, f \in \mathrm{Lip}_0(M)\ \forall\, \varepsilon > 0\ \exists\, u \neq v \in M$ such that $\frac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$ and $d(u,v) < \varepsilon$.

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$\Longrightarrow \frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$ and $d(u,v) < \frac{2\varepsilon}{(1-\varepsilon)^2} d(x,y)$. $\square$

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Key property of f_{xy}

$\frac{f_{xy}(u)-f_{xy}(v)}{d(u,v)} > 1-\varepsilon \implies$
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- The magic function is $f_{xy}(t) = \frac{d(t,y)}{d(t,y)+d(t,x)} d(x,y)$ (minus value at 0). It comes from Ivakhno-Kadets-Werner.

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If M not length $\implies \exists\, x \neq y$ and $\delta > 0$ such that $B(x, (1+2\delta)r) \cap B(y, (1+2\delta)r) = \emptyset$ where $r := \frac{d(x,y)}{2}$.

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Then $\frac{f(u)-f(v)}{d(u,v)} > \frac{1}{1+\delta} \implies d(u,v) \geq \delta r$. $\square$

Coming up next

Theorem (Ivakhno, Kadets, Werner, 2007)

If M is compact, then TFAE:

1. $\mathcal{F}(M)$ *has Daugavet property*
2. $Lip_0(M)$ *has Daugavet property*
3. *every pair* $x \neq y \in M$ *has property (Z), i.e. for every* $\varepsilon > 0$ *exists* $z \in M \setminus \{x, y\}$

$$d(x, z) + d(z, y) < d(x, y) + \varepsilon\, dist(z, \{x, y\}).$$

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- So if M is compact, $\text{str exp}(B_{\mathcal{F}(M)}) = \emptyset \implies \mathcal{F}(M)$ is Daugavet.
- We don't know if for M complete, global property (Z) implies that M is length.

But

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- Converse is not true: if $p \in (0, 1)$ then $\mathcal{F}([0, 1], |\cdot|^p)$ has the RNP.

Proof.

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$$\begin{aligned} d(x,y) &\le d(x,z) + \alpha d(y,z) \\ &= d(x,z) + d(y,z) - (1-\alpha) d(y,z) \\ &\le d(x,y) + \varepsilon \operatorname{dist}(z,\{x,y\}) - (1-\alpha) d(y,z) \end{aligned}$$

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WLOG $z \in V$; set $(u,v) = (x,z)$ above. Then

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Thus $d(y,z) \leq \frac{\varepsilon}{1-\alpha} d(z,\{x,y\}) < d(z,\{x,y\})$Contradiction! $\square$

Thank you for your attention!