Accurate pasting of bilipschitz embeddings of locally finite metric spaces from embeddings of their finite pieces

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- As far as I know the systematic investigation of the Pasting Problem started in 2006, although a very important for applications case was known much earlier.
- I mean the positive answer to the Pasting Problem in the case of Banach spaces X satisfying the condition: Each separable subset of an arbitrary ultrapower of X admits an isometric embedding into X (we shall discuss this result in more detail later).

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- It is not difficult to see that in all situations which we consider existence of pasting of bilipschitz embeddings implies existence of pasting of coarse embeddings, for this reason we shall consider only bilipschitz embeddings.

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- **Definition:** A map $f : A \to Y$ between metric spaces is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in A \quad r d_A(u, v) \le d_Y(f(u), f(v)) \le rC d_A(u, v) \qquad (*).$$

  A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $C < \infty$. The smallest constant C for which there exist $r > 0$ such that (*) is satisfied is called the *distortion* of f.

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- The most general qualitative pasting result:
- **Finite Determination Theorem (M.O., 2012)**: Let A be a locally finite metric space whose finite subsets admit bilipschitz embeddings with uniformly bounded distortions into a Banach space X. Then, A also admits a bilipschitz embedding into X.

An Application

- Combining the Finite Determination Theorem with the known results of I. Benjamini-O. Schramm (1997), Bourgain (1986), and S. Buyalo-A. Dranishnikov-V. Schroeder (2007), we get the following metric characterization of nonsuperreflexive Banach spaces.

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- **Theorem: (M.O., 2014)** Let G be an infinite finitely generated word hyperbolic group which does not have a finite index subgroup isomorphic to $\mathbb{Z}$. Then G admits a bilipschitz embedding into a Banach space X, if and only if X is nonsuperreflexive.

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- The estimate of D implied by the proof in M.O. (2012) is > 1000. In the pasting result of Baudier-Lancien (2008) for spaces X with no cotype they achieved $D(X) \leq 216$.

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- Recall the well-known **Observation:** $X \in (\mathbf{U})$ implies $D(X) = 1$.
- As I understand, the first version of this observation (for $L_p[0,1]$) goes back to J. Bretagnolle, D. Dacunha-Castelle, and J.-L. Krivine (1966). More general versions evolved together with the theory of ultraproducts.

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  - Since c_0 contains ℓ_∞^n isometrically for all n, Fréchet's observation implies that finite pieces of M embed into c_0 isometrically.
  - Using the triangle inequality one can show that an isometric embedding T of M into c_0 satisfying $T(0) = 0$ should be such that the image of e_0 should have infinitely many coordinates with an absolute value 1, a contradiction.

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- **1^+-Theorem (S.O. & M.O., 2017):** Let $1 \leq p < \infty$. If $\{X_n\}_{n=1}^{\infty}$ is a nested family of finite-dimensional Banach spaces, then $D\left((\oplus_{n=1}^{\infty} X_n)_p\right) \leq 1^+$.

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- **Corollary:** $D(\ell_p) = 1^+$ for $p \neq 2, \infty$.

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- **Corollary** (repeated) : $D(\ell_p) = 1^+$ for $p \neq 2, \infty$.
- This Corollary of the $\mathbf{1}^+$-**Theorem** gives the best possible answer to some of the questions on relations between embeddability into ℓ_p and L_p asked by A. Naor and Y. Peres (2011, Question 10.7, they mention that the subtlety between embeddings into L_p and ℓ_p was pointed out by M. Bourdon).

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- However this is known to be false even in the Euclidean plane.
- One of the standard examples is

$$\gamma(t) = t(\cos(\ln t), \ \sin(\ln t)), \quad t > 1,$$

which is (as is easy to check) a $\sqrt{2}$-bilipschitz embedding of the half-line $t > 1$.

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- To complete the proof of the 1^+-Theorem it remains to find:

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- Using such spirals we can construct a $(1+\varepsilon)$-bilipschitz embeddings of a half-axis into ℓ_2 which "stays" for an arbitrary long time on each of the coordinate axes and then travels along a spiral of the type described above to the next line.
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  - Let me sketch a figure for a special case: we prove that a locally finite subset A of $L_p[0,1]$, containing 0, admits a $(1+\varepsilon)$-bilipschitz embeddings into ℓ_p.

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- This modification allows to improve the constant in the Baudier-Lancien (2008) theorem. For Banach spaces X with no nontrivial cotype we get $D(X) \leq 4^+$ instead of $D(X) \leq 216$ proved by Baudier-Lancien.