Joint spreading models and uniform approximation of bounded operators

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(joint work with S. A. Argyros - A. Georgiou - A.-R. Lagos)

Non Linear Functional Analysis - CIRM
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Answer: **never**, when $\dim(X) \geq 2$. (W. B. Johnson)

The Uniform approximation on large subspaces (UALS property)

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with $\|(T_0 - S)|_Y\|_{\mathcal{L}(Y,X)} \leq C\varepsilon$.

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  - *James space J and its dual J^*,*
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  - *in fact, every Asymptotic ℓ_p-space for $1 \leq p \leq \infty$,*
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- So: $\|\sum_{k=1}^m x_{i_k}^{(k)}\| = \max_{1 \leq k \leq m} \|x_{i_k}^{(k)}\| = 1$

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- If $*$ **fails**: we find **normalized block sequences** $(x_i^k)_i$, $1 \leq k \leq m$ with $\|(T_0 - S_k)x_i^k\| \geq 3\varepsilon$ for all $i \in \mathbb{N}$, $1 \leq k \leq m$.
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- **Absurd!** (W ε-pointwise approximates T_0)

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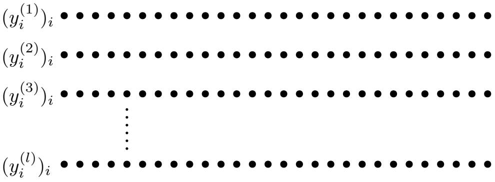

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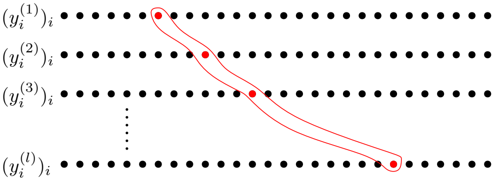

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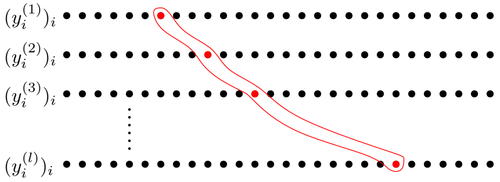

- Any finite collection of normalized block sequences in c_0 asymptotically **jointly** behaves like the uvb of c_0.

Plegma spreading sequences

Definition

A sequence $(e_i)_{i=1}^{\infty}$ *in a Banach space is called* spreading *if for any* $m \in \mathbb{N}$*, any*

$$i_1 < \cdots < i_m \text{ and } j_1 < \cdots < j_m$$

and any scalars $(a_n)_{n=1}^{m}$ *we have:*

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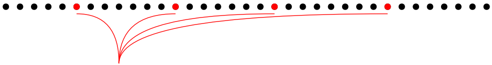

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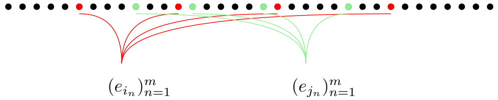

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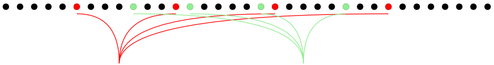

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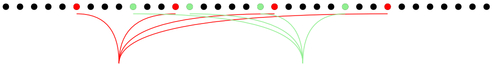

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- e.g. the unit vector basis of c_0, ℓ_p, $1 \leq p < \infty$.

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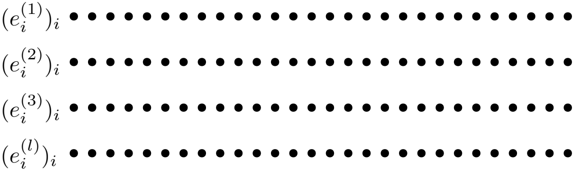

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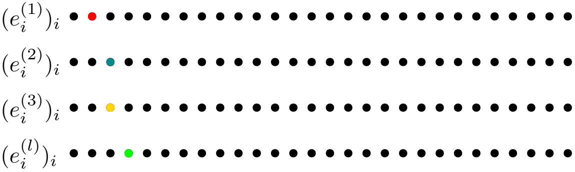

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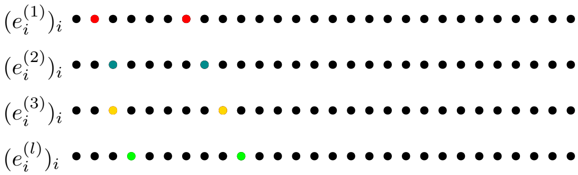

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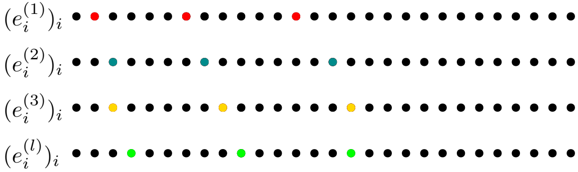

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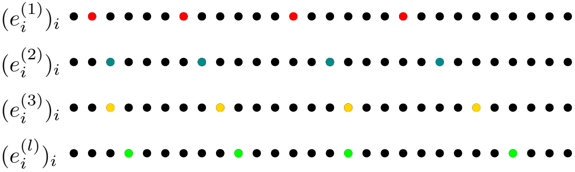

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Properties of a plegma spreading collection $((e_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$

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Remark: If each $(e_i^{(k)})_i$ is Schauder basic $((e_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ need not be Schauder basic.

ℓ-joint spreading models

Definition

Let $((x_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ *and* $((e_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ *be finite collections of Schauder basic sequences in Banach spaces* X *and* E *respectively.*

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We say that $((x_i^{(k)})_{i=1}^\infty)_{k=1}^l$ generates $((e_i^{(k)})_{i=1}^\infty)_{k=1}^l$ as an l-joint spreading model

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if there is $\delta_m \searrow 0$ *s.t. for any* $m \in \mathbb{N}$, *scalars* $((a_n^{(k)})_{k=1}^l)_{n=1}^m$, *and*

Definition

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We say that $((x_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ generates $((e_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ as an l-joint spreading model

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Remark: $((e_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ is plegma spreading.

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- If $((x_i^{(k)})_i$ is weakly null, for $1 \leq k \leq l$ then $((e_i^{(k)})_{i=1}^{\infty})_{k=1}^{l}$ is suppression unconditional.

Spaces with unique a l-joint spreading models

Spaces that satisfy the UALS

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  for any $l \in \mathbb{N}$, any two plegma spreading sequences

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Remark

There is a notion of an asymptotic model generated by an infinite array of sequences. (Halbeisen - Odell (2014)).
A space has a unique l-joint spreading model with respect to $\mathscr{F}$ if and only if it has a unique asymptotic model with respect to $\mathscr{F}$

Typical examples of families $\mathscr{F}$ in X:

- $\mathscr{F}(X)$ = all normalized Schauder basic sequences,
- $\mathscr{F}_C(X)$ = all normalized C-Schauder basic sequences,
- $\mathscr{F}_0(X)$ = all normalized weakly null Schauder basic sequences,
- $\mathscr{F}_{\mathrm{b}}(X)$ = all normalized block sequences if X has a basis.
- given a (countable) $\mathscr{A} \subset X^*$:

$$\mathscr{F}_{\mathscr{A},0} = \Big\{(x_k)_k \in \mathscr{F}(X) : \ f(x_k) \to 0 \text{ for all } f \in \mathscr{A}\Big\}.$$

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Theorem

If $X = JT$ (James tree space), then X admits a unique l-joint spreading model with respect to $\mathcal{F}_0(X)$ but not $\mathcal{F}(X)$.

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$$(y_k)_k = \left(\frac{x_{l_{2k}} - x_{l_{2k-1}}}{\|x_{l_{2k}} - x_{l_{2k-1}}\|} \right)_k \text{ is in } \mathscr{F}.$$

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Examples of difference including families $\mathscr{F}$:

- $\mathscr{F}(X)$ = all normalized basic sequences,
- $\mathscr{F}_C(X)$ = all normalized C-basic sequences,
- $\mathscr{F}_0(X)$ = all normalized w-null basic sequences, if $\ell_1 \not\subset X$,
- $\mathscr{F}_{\mathscr{A},0} = \left\{(x_k)_k \in \mathscr{F}(X) : \ f(x_k) \to 0 \text{ for all } f \in \mathscr{A}\right\}$, if $\mathscr{A}$ countable,
- $\mathscr{F}_{(e_i^*)_i(X)}$, if X has a basis $(e_i)_i$.

Theorem

Let X be a Banach space. If there exists a difference including family $\mathscr{F}$ so that X admits a unique l-joint spreading model with respect to $\mathscr{F}$ then X has the UALS property.

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Recall: X is Asymptotic-ℓ_p, $1 \leq p \leq \infty$ if $\exists\ C \geq 1$ so that $\forall n \in \mathbb{N}$

$\exists$ a finite codimensional $Y_1 \hookrightarrow X$ so that $\forall$ normalized $x_1 \in Y_1$

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$\exists$ a finite codimensional $Y_n \hookrightarrow X$ so that $\forall$ normalized $x_n \in Y_n$

$(x_k)_{k=1}^n$ is C-equivalent to the uvb of ℓ_p^n.

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E.g. c_0, ℓ_p, $1 \leq p < \infty$, James space J, Tsirelson space T, and T^*.

Theorem

Let X be a Banach space with an FDD. If there exists a difference including family $\mathscr{F}$ so that X^* admits a unique l-joint spreading model with respect to $\mathscr{F}$ then X has the UALS property.

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Corollary

Every $\mathscr{L}_\infty$-space with separable dual has the UALS-property.

Specifically, every $C(K)$ for K countable and compact has the UALS-property.

Theorem (Argyros - Georgiou - Lagos - M (2018))

- *The following Banach spaces satisfy the UALS property:*
  - *every X with the scalar-plus-compact property,*
  - ℓ_p, $1 \leq p < \infty$, *and* c_0,
  - *James space J and its dual J^*,*
  - *Tsirelson space T and its dual T^*,*
  - *in fact, every Asymptotic ℓ_p-space for $1 \leq p \leq \infty$,*
  - *James tree space JT ,*
  - *$C(K)$, for K countable and compact,*
  - *in fact, every $\mathscr{L}_\infty$-space with separable dual.*

Question (Halbeisen - Odell)

If X admits a unique l-joint spreading model with respect to a difference including family $\mathscr{F}$ does it contain an asymptotic ℓ_p subspace?

Answer (Freeman - Odell - Sari - Zheng (2016))

If the unique l-joint spreading model is isomorphic to c_0 then yes.

Spaces failing the UALS

Spaces that have a unique spreading model
but not a unique l-joint spreading model

- The space $\ell_1 \oplus \ell_2$ fails the UALS.

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$$X_n = (\underbrace{\ell_1 \oplus \cdots \oplus \ell_1}_{2n \text{ copies of } \ell_1})_{\ell_1} \equiv \ell_1, \quad Y_n = (\underbrace{\ell_2 \oplus \cdots \oplus \ell_2}_{2n \text{ copies of } \ell_2})_{\ell_2} \equiv \ell_2$$

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- Consider the formal identity $I : X_n \to Y_n$,

$$I(x_1, x_2, \ldots, x_{2n}) = (x_1, x_2, \ldots, x_{2n})$$

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- Define $W = \mathrm{co}\{P_F : \# \leq n\}$.
  - W $(1/\sqrt{n})$-pointwise approximates I.
  - For any $Z \overset{\text{finite codim}}{\hookrightarrow} Y$ and any $S \in W$

$$\|(I - S)|_Z\| \geq 1/2$$

Comment

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Question

- *If X has a unique l-joint spreading model then it has the UALS property.*
- *What if X just has a unique spreading model?*

- Consider the Banach spaces

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The space $X = (\sum_{k=1}^{\infty} X_n)_{\ell_2}$ fails the UALS property.

Remark

- *Every spreading model admitted by X is isometrically equivalent to the uvb of ℓ_2.*
- *The space X has no unique l-joint spreading model.*

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- *The following Banach spaces fail the UALS property:*
  - $\ell_p \oplus \ell_q$, *and* $\ell_p \oplus c_0$ *for* $1 \leq p \neq q \leq \infty$
  - $(\sum \ell_p)_{\ell_q}$, *for* $1 \leq p \neq q \leq \infty$,
  - $(\sum \ell_p)_{c_0}$, $(\sum c_0)_{\ell_p}$, *for* $1 \leq p \leq \infty$,
  - $L_p[0,1]$, $1 \leq p < \infty$, $p \neq 2$,
  - $C[0,1]$ *and its dual* $\mathcal{M}[0,1]$ *and* $L_\infty[0,1]$.

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- *$C(K)$, for countable compact K, satisfies the UALS property. It contains subspaces that fail the UALS property.*
- *If X is Asymptotic ℓ_p all of its subspaces are asymptotic ℓ_p.*

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- *If* X *is Asymptotic* ℓ_p *all of its subspaces have the UALS property.*

Question

Is there X *all subspaces of which fail the UALS property?*

Thank you!