Closed Ideals in $L(L_p)$

Bill Johnson

CIRM, March, 2018

Non(?) Linear Functional Analysis

Joint with G. Pisier and G. Schechtman [JPiS]

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Foundational work on $L(X)$: [Berkson-Porta 1969 JFA]

$L_p := L_p(0,1) \equiv L_p\{-1,1\}^{\mathbb{N}}$, $1 \le p < \infty$.

Motivation for studying $L(\ell_p)$ and $L(L_p)$:

After C^*-algebras, these are arguably the most natural non commutative Banach algebras.

The structure of $L(L_p)$ for $p \neq 2$ is very different from that of $L(\ell_p)$ and $L(L_2)$; more complicated and more interesting.

There are some connections to harmonic analysis.

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Background

For different p, the structure of $L(L_p(\mu))$ spaces are different. There is no non zero (always Banach algebra) homomorphism from $L(L_p(\mu))$ into $L(L_q(\nu))$ when $p \neq q$ and $p \neq 2$. [Phillips?]

Since L_2 is isomorphic to a complemented subspace of L_p, $1 < p < \infty$, there is a Banach algebra isomorphism from $L(L_2)$ into $L(L_p)$.

Proposition.

Suppose $\Pi : L(X) \to L(Y)$ is a non injective homomorphism, and let $1 \leq p < \infty$.

- *If $X = \ell_p$, then dens $Y \geq 2^{\aleph_0}$.* [known]
- *If $X = L_p$, then dens $Y \geq 2^{\aleph_0}$.* [JPhS], maybe new

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Note that since $\Pi^{-1}(0)$ is a non trivial closed ideal, it contains the compact operators $K(X)$ on X. ($K(X)$ is the closure of the finite rank operators since $L_p(\mu)$ spaces have the approximation property.) So the Calkin algebras $L(\ell_p)/K(\ell_p)$ and $L(L_p)/K(L_p)$ have no non zero representations in $L(Y)$ if Y is a separable Banach space.

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Lemma.

Let X *be* ℓ_p *or* L_p, $1 \le p < \infty$. *Then there are commuting contractive idempotents* $(P_\alpha)_{\alpha \in \mathbb{R}}$ *in* $L(X)$ *s.t.*

- *for* $\alpha \neq \beta$, $P_\alpha P_\beta$ *has finite rank*
- *for all* $\alpha \in \mathbb{R}$, P_α *is isometrically isomorphic to* X.

Assume Lemma. Then $\Pi(P_\alpha)$ are idempotents in $L(Y)$ and $\forall \alpha \neq \beta$, $\Pi(P_\alpha P_\beta) = 0$ since $P_\alpha P_\beta \in K(X) \subset \Pi^{-1}(0)$. So the unit spheres of $(\Pi(P_\alpha)(Y))_{\alpha \in \mathbb{R}}$ are disjoint separated subsets of Y.

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The Lemma is a good exercise for your students. Here are some hints to give them if they need help.

Hint 1. There are $2^{\aleph_0}$ infinite subsets of $\mathbb{N}$ s.t. the intersection of any two are finite.

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Algebra Isomorphisms into $L(L_p(\mu))$

For $1 \leq p < \infty$, there is an isometric Banach algebra isomorphism $L(\ell_p)/K(\ell_p) \to L(L_p(\mu))$ and a Banach algebra isomorphism $L(L_p)/K(L_p) \to L(L_p(\mu))$ [Calkin '41], [BoedinhardjoJ '15], [BlecherPhillips '18].

The 21st century proofs use the most natural left approximate identity for $K(\ell_p)$ and $K(L_p)$. [BJ] uses ultrapowers and [BP] uses the Arens multiplication on $K(\ell_p)^{**}$ and $K(L_p)^{**}$.

Problem. Are there other (always non trivial) closed ideals $\mathcal{I}$ of $L(L_p)$ s.t. $L(L_p)/\mathcal{I}$ is Banach algebra isomorphic to a subalgebra of $L_p(\mu)$? $K(\ell_p)$ is the only non trivial closed ideal in $L(\ell_p)$.

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This problem goes back at least to [Le Merdy '96], who proved that there there is an isometric algebra isomorphism $L(L_p)/\mathcal{I} \to L(X)$ for some X that is a subspace of a quotient of $L_p(\mu)$.

AFAIK, this problem is open for every $\mathcal{I} \neq K(L_p)$.

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Ideals in $L(X)$.

There are many classical closed ideals in $L(X)$. As long as X has the approximation property all $L_p(\mu)$ spaces do, $K(X)$ is the smallest one. Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. An especially important closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

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Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T. It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators [EnfloStarbird '79] for $p = 1$; [JMaureySchechtmanTzafriri '79] for $1 < p \neq 2 < \infty$.

An operator T is called Y-singular if T is not an isomorphism when restricted to any subspace that is isomorphic to Y. So an operator is strictly singular iff it is Y-singular for every infinite dimensional space Y.

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Constructing Ideals

A common way of constructing a (not necessarily closed) ideal in $L(X)$ is to take some operator $U : Y \to Z$ between Banach spaces and let $\mathcal{I}_U$ be the collection of all operators on X that factor through U, i.e., all $T \in L(X)$ s.t. $A \in L(X, Y)$ and $B \in L(Z, X)$ s.t. $T = BUA$.

$L(X)\mathcal{I}_U L(X) \subset \mathcal{I}_U$ is clear, so $\mathcal{I}_U$ is an ideal in $L(X)$ if $\mathcal{I}_U$ is closed under addition. One usually guarantees this by using a U s.t. $U \oplus U : Y \oplus Y \to Z \oplus Z$ factors through U, and these are the only U that I will mention. Then the closure $\overline{\mathcal{I}}_U$ will be a proper ideal in $L(X)$ as long as I_X does not factor through U.

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Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

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Ideals in $L(L_1)$

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Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$.
But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$.

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It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

I thought that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. Early last year Schechtman and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

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Theorem.

[JPiS] *There are at least $2^{\aleph_0}$ small closed ideals in $L(L_1)$.*

It remains open whether there are more than two large ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used (could be avoided by using B-space theory results from the 1970s).

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Small Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Much work this millenium, culminating in a paper that solved another problem from Pietsch's book:

Theorem. (SchlumprechtZsak '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; small closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

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Theorem. (JPiS)

There are at least $2^{\aleph_0}$ small closed ideals in $L(L_1)$.

Lemma. (JPiS)

Let $0 < p < q < \infty$. Suppose $v_1, \ldots, v_N$ in L_q satisfy

- $\max_{\epsilon_i = \pm 1} \|\sum_{i=1}^N \epsilon_i v_i\|_q \leq CN^{1/2}$
- $T : L_1 \to L_1^{N^{p/2}}$ *satisfies* $\min_{1 \leq i \leq N} \|Tv_i\|_1 \geq \epsilon$

Then

$$\|T\|_{L_1 \to L_1{}^{N^{p/2}}} \geq (\epsilon / C) N^{\frac{q-p}{2q}}$$

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Theorem. (JPhS)

If $\mathcal{I}$ is a closed ideal in $L(L_p)$, $1 \le p < \infty$, that has a left approximate identity, then $\mathcal{I} = K(L_p)$.

A left approximate identity for a Banach algebra $\mathcal{A}$ is a net (a_i) in $\mathcal{A}$ s.t. for all $x \in \mathcal{A}$, $\|a_i x - x\| \to 0$.

Proposition. (JPhS)

Let $\{e_{n,i} : 1 \le i \le 2^n;\ n = 1, 2, \dots\}$ be the natural basis for $(\sum_{n=1}^{\infty} \ell_2^{2^n})_p$ and let $\{h_{n,i} : 1 \le i \le 2^n;\ n = 1, 2, \dots\}$ be the L_p normalized Haar basis for L_p. Then for $2 < p < \infty$ the mapping $e_{n,i} \to h_{n,i}$ extends to a bounded linear operator from $(\sum_{n=1}^{\infty} \ell_2^{2^n})_p$ into L_p.

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Lemma

Let $1 \leq p < q < \infty$, $\{v_1, \ldots, v_N\} \subset L_q$, *and let* $T : L_1 \to L_1^{N^{\frac{p}{2}}}$ *be an operator. Suppose that* C *and* ϵ *satisfy*

1. $\max_{\epsilon_i = \pm 1} \| \sum_{i=1}^N \epsilon_i v_i \|_q \leq C N^{1/2}$, *and*
2. $\min_{1 \leq i \leq N} \| T v_i \|_1 \geq \epsilon$.

Then $\| T \| \geq (\epsilon / C) N^{\frac{q-p}{2q}}$.

Proof: Take u_i^* in $L_\infty^{N^{p/2}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, T v_i \rangle = \| T v_i \|_1 \geq \epsilon$. Then

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$$=: \int_0^1 \| \sum_{i=1}^{N} v_i(b) T^* u_i^* \|_{L_\infty[0,1]} \quad db$$

$$\leq \|T\| \int_0^1 \| \sum_{i=1}^{N} v_i(b) u_i^* \|_{L_\infty^{N^{p/2}}}\, db$$

$$\leq \|T\| N^{\frac{p}{2q}} \int_0^1 (\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q\, dc)^{\frac{1}{q}}\, db$$

$$\leq \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q\, db\, dc \Big)^{\frac{1}{q}}$$

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&\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q\, dc \Big)^{\frac{1}{q}}\, db \\
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Thank you for your attention