Mapping n grid points onto a square forces an arbitrarily large Lipschitz constant.

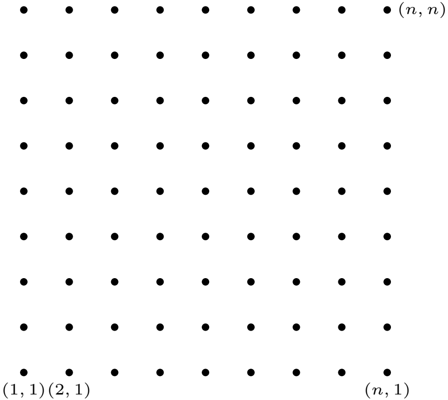
Michael Dymond

Universität Innsbruck

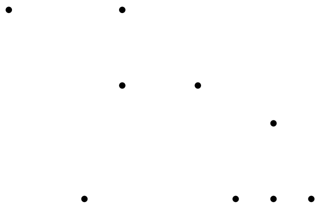
Nonlinear Functional Analysis, Luminy, 5-9 March 2018.

Joint work with Vojtěch Kaluža and Eva Kopecká.

‘The regular $n \times n$ grid’ Q_n.



Feige's Question.



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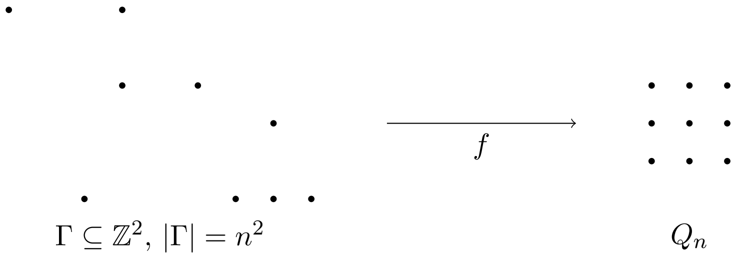

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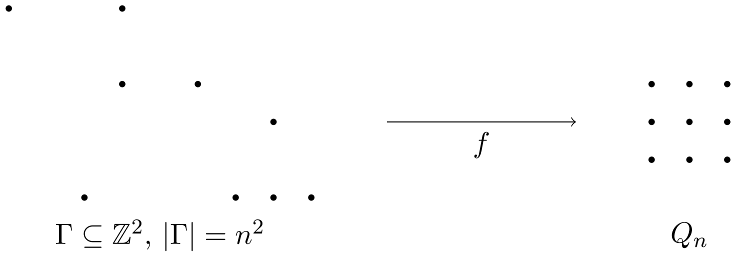

Problem Determine

(i) $L_{n,\Gamma} := \min\{\mathrm{Lip}(f) \colon f \colon \Gamma \to Q_n \text{ bijective}\}$,

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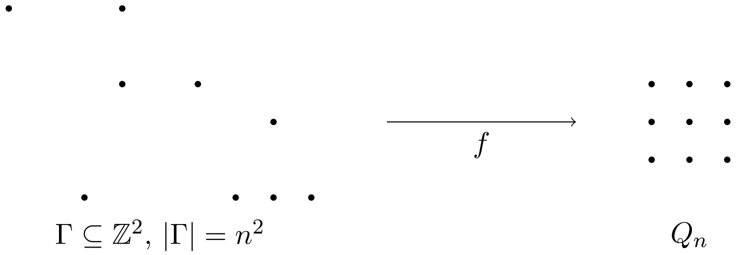

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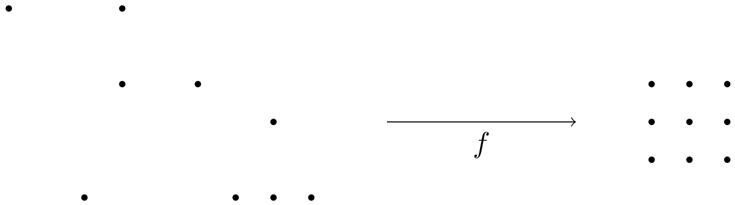

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Feige's Question

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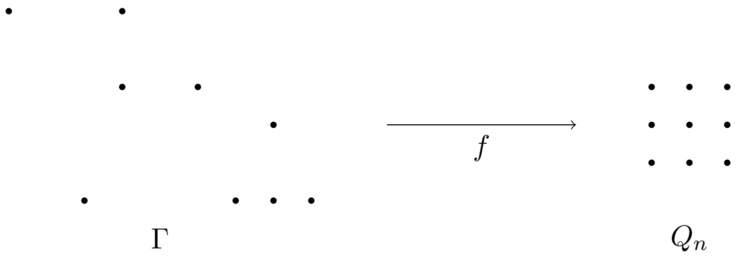
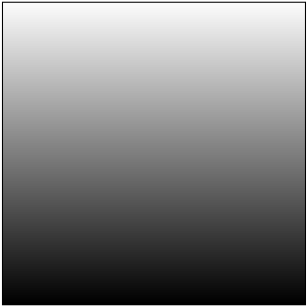


Does there exist $L > 0$ such that for any $n \in \mathbb{N}$ and any set $\Gamma \subseteq \mathbb{Z}^2$ with $|\Gamma| = n^2$ there exists a bijective mapping $f : \Gamma \to Q_n$ with $\mathrm{Lip}(f) \leq L$?

Densities as separated sets.



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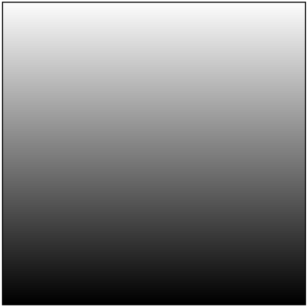

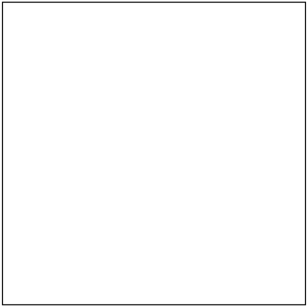

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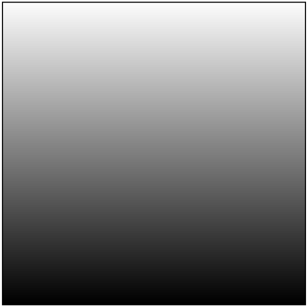

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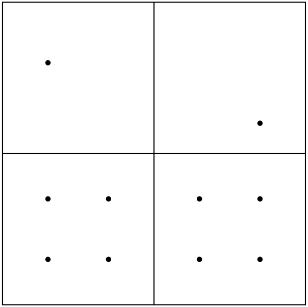

S_1

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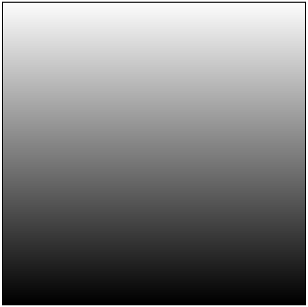

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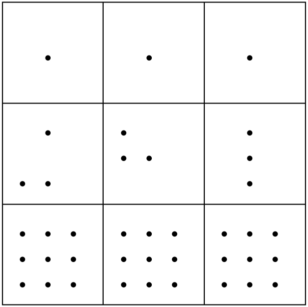

S_2

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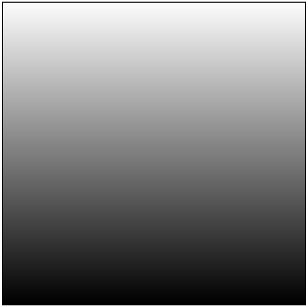

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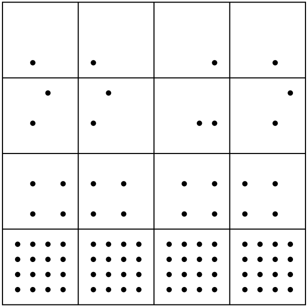

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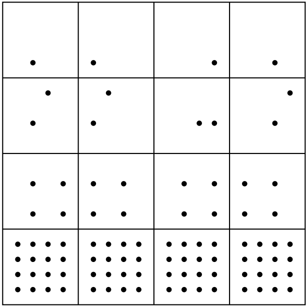

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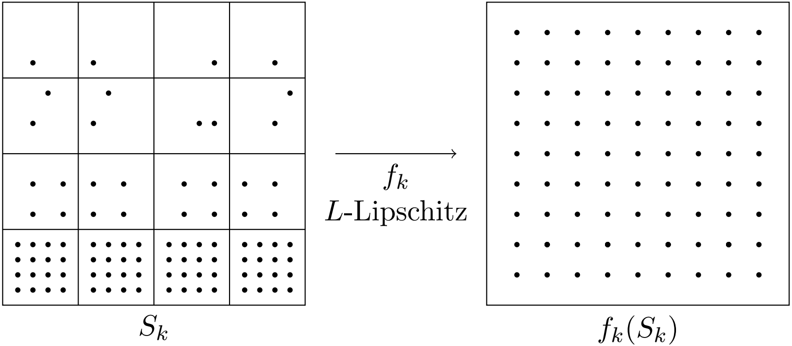

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Lemma (D., Kaluža, Kopecká (2017))

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$$f_\sharp \rho \mathcal{L} = \mathcal{L}|_{f([0,1]^2)}.$$

Non-bilipschitz equivalent separated nets.

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$$\rho = |\mathrm{Jac}(f)| \qquad a.e.$$

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We will call a Lipschitz mapping (C, L)-regular if it is L-Lipschitz and regular with constant C in the sense of 1.

Bilipschitz behaviour of regular mappings.

Let $B \subseteq \mathbb{R}^d$ be a non-empty open ball and $f\colon B \to \mathbb{R}^d$ be a (C, L)-Lipschitz regular mapping.

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Theorem (Bonk, Kleiner (2001))

There exists a non-empty, open ball $B' \subseteq B$ such that $f|_{B'}$ is bilipschitz with lower bilipschitz constant $b = b(C)$.

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- On the otherhand, any Lipschitz regular mapping $f\colon [0,1]^d \to \mathbb{R}^d$ with $C = 2$ must be locally injective almost everywhere.

Porous and σ-porous sets.

Definition

Let (M, d) be a complete metric space.

(i) A set $E \subseteq M$ is called *porous* if there exists $c > 0$ such that for every $\varepsilon > 0$ and every $x \in E$ there exists $y \in M$ with $d(x, y) < \varepsilon$ and $B(y, c\varepsilon) \cap E = \emptyset$.

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(ii) A set $F \subseteq M$ is called *σ-porous* if F can be written as a countable union of porous sets.

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Theorem (D., Kaluža, Kopecká (2017))

$\mathcal{E}$ is a σ-porous subset of $C([0,1]^2)$.

Thank you for your attention!