Complexity of distances between metric and Banach spaces

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Non Linear Functional Analysis, CIRM

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It is even known (Melleray 2007) that $\cong_L \sim_B \cong_i$.

Gromov-Hausdorff distance:
- for $A, B \subseteq M$ two non-empty subsets of a metric space M, the *Hausdorff distance* between A and B is

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The Lipschitz distance is analytic pseudometric on both $\mathcal{M}$ and $\mathcal{B}$.

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In this case we say that f is a *Borel-uniformly continuous reduction*.

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$$\forall x, y \in X : \quad \rho_X(x, y) < \delta_X \Rightarrow \rho_Y(f(x), f(y)) < \varepsilon$$

and

$$\forall x, y \in X : \quad \rho_Y(f(x), f(y)) < \delta_Y \Rightarrow \rho_X(x, y) < \varepsilon.$$

In this case we say that f is a *Borel-uniformly continuous reduction*. If $\rho_X \leq_{B,u} \rho_Y$ and $\rho_Y \leq_{B,u} \rho_X$, we say that ρ_X is *Borel-uniformly continuous bi-reducible* with ρ_Y and write $\rho_X \sim_{B,u} \rho_Y$.

Borel-uniformly continuous reduction

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- note that $\rho_X \leq_{B,u} \rho_Y$ implies the reducibility between the corresponding equivalence relations, i.e. $E_{\rho_X} \leq_B E_{\rho_Y}$.

A *correspondence* between A and B is a binary relation $\mathcal{R} \subseteq A \times B$ such that for every $a \in A$ there is $b \in B$ such that $a\mathcal{R}b$, and for every $b \in B$ there is $a \in A$ such that $a\mathcal{R}b$.

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Lemma

Let M and N be two metric spaces. For every $r > 0$ we have $\rho_{GH}(M, N) < r$ if and only if there exists a correspondence $\mathcal{R}$ between M and N such that $\sup |d_M(m, m') - d_N(n, n')| < 2r$, where the supremum is taken over all $m, m' \in M$ and $n, n' \in N$ with $m\mathcal{R}n$ and $m'\mathcal{R}n'$.

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For two metrics on natural numbers $f, g \in \mathcal{M}$ and $\varepsilon > 0$, we consider the relation

$$f \simeq_\varepsilon g \quad \Leftrightarrow \quad \exists \pi \in S_\infty \, \forall \{n, m\} \in [\mathbb{N}]^2 : |f(\pi(n), \pi(m)) - g(n, m)| \leq \varepsilon.$$

Reductions concerning GH distance

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Lemma

Let $p > 0$ be a real number. For any two metrics on natural numbers $f, g \in \mathcal{M}_p$ we have $\rho_{GH}(f, g) = \inf\{r : \ f \simeq_{2r} g\}$ provided that $\rho_{GH}(f, g) < p/2$.

Lemma

Let X and Y be separable Banach spaces, let $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be linearly independent and linearly dense sequences in X and Y, respectively, and put $V = \mathbb{Q}\,\mathrm{span}\{e_n : \ n \in \mathbb{N}\}$, $W = \mathbb{Q}\,\mathrm{span}\{f_n : \ n \in \mathbb{N}\}$.

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Then $\rho_{BM}(X, Y) < r$ if and only if there exists a surjective linear isomorphism $T : X \to Y$ with $\log \|T\|\|T^{-1}\| < r$ and $T(V) = W$.

Reducibility results

Theorem

- *The following pseudometrics are mutually Borel-uniformly continuous bi-reducible: the* **Gromov-Hausdorff distance** *when restricted to Polish metric spaces, to metric spaces bounded from above, from below, from both above and below, to Banach spaces; the* **Lipschitz distance** *when restricted to metric spaces bounded from below and above, and to uniformly discrete metric spaces; the* **Banach-Mazur distance***; the* **Kadets distance** *on Banach spaces; the* **Hausdorff-Lipschitz** *and* **net distances** *on Banach spaces.*
- *The pseudometrics above are Borel-uniformly continuous reducible to the* **uniform** *and* **Lipschitz** *distances on Banach spaces.*

Balls are not Borel

Theorem

Let ρ be any pseudometric to which the Kadets distance is Borel-uniformly continuous reducible (e.g. the Kadets distance itself, or the Gromov-Hausdorff distance). Then there are elements A from the domain of ρ such that the function $\rho(A, \cdot)$ is not Borel.

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Sketch of the proof:

$\mathcal{P}(\mathbb{N})$ = all subsets of $\mathbb{N}$ endowed with the coarsest topology for which $\{A \in \mathcal{P}(\mathbb{N}) : n \in A\}$ is clopen for every n (a copy of the Cantor space $2^{\mathbb{N}}$).

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Theorem (Hurewicz)

The set

$$\mathfrak{H} = \left\{\mathcal{A} \in K(\mathcal{P}(\mathbb{N})) : \mathcal{A} \text{ contains an infinite set}\right\}$$

is not Borel.

Proposition

Let us consider the space

$$X = \left(\bigoplus G_n \right)_{\ell_1},$$

where $G_1, G_2, \dots$ is a dense sequence of finite-dimensional spaces. Then, for every $\varepsilon > 0$, there exists a Borel mapping $\mathfrak{S} : K(\mathcal{P}(\mathbb{N})) \to \mathcal{B}$ such that
(a) *if $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ contains an infinite set, then $\rho_{BM}(\mathfrak{S}(\mathcal{A}), X) \leq \varepsilon$, and thus $\rho_K(\mathfrak{S}(\mathcal{A}), X) \leq \varepsilon$,*
(b) *if $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ consists of finite sets only, then $\mathfrak{S}(\mathcal{A})$ contains a normalized 1-separated shrinking basic sequence, and thus $\rho_K(\mathfrak{S}(\mathcal{A}), X) \geq 1/8$.*

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Given $\varepsilon > 0$, we put $\theta = e^{-\varepsilon}$ and

$$X_{\mathcal{A}} = T[\mathcal{A}_1, \theta] \oplus_1 X, \quad \mathcal{A} \in K(\mathcal{P}(\mathbb{N})),$$

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where $\mathcal{A}_1 = \{A \cup \{1\} : A \in \mathcal{A}\}$. We check that $X_{\mathcal{A}}$ satisfies the requirements (a) and (b) on $\mathfrak{S}(\mathcal{A})$.