# Unlikely intersections in a family of semi-abelian surfaces.

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(Joint work with H. Schmidt)

# Abstract

[Joint with Harry Schmidt, see ArXiV:1803.04835v1).]

Let $E_0$ be an elliptic curve with CM, and let $G/S$ be a one-dimensional family of extensions of $E_0$ by $\mathbb{G}_m$. Then, $G$ contains a dense set $\mathcal{C}$ of special curves, known as Ribet curves. This set $\mathcal{C}$ contains all the torsion curves of $G$.

Let $W$ be an irreducible curve in $G$. With everything over $\overline{\mathbb{Q}}$, we show that $W$ meets $\mathcal{C}$ Zariski densely if and only if it is a translate of a Ribet curve by a multiplicative section, or it lies in a fiber of the family.

The proof involves the Pila-Zannier/Masser-Zannier strategy, a new type of bounded height, and the Mordell-Lang theorem.

See also D.B.-B. Edixhoven (ArXiv:1104.5178v1), and current work of Bas.

# Motivation

**Pink's conjecture** for an irreducible curve $Z/\overline{\mathbb{Q}}$ in a mixed Shimura variety $\mathfrak{S}$ says : let $\mathfrak{S}^{[2]}$ = union of all the special subvarieties of codimension $\geq 2$. *Assume that* $Z \cap \mathfrak{S}^{[2]}$ *is infinite. Then,* $Z$ *is contained in a proper special subvariety* $\mathfrak{S}'$ *of* $\mathfrak{S}$.

Special points are usually easy to describe, higher dim'l ones less so.

Examples with dim $\mathfrak{S} = 2$ ($\Rightarrow Z$ is a special curve) :

- pure $\mathfrak{S} = Y(2) \times Y(2)$ (André);
- mixed $\mathfrak{S}$ = the Legendre family $\mathcal{E}/Y(2)$ (André, Pila).

Examples with dim $\mathfrak{S} = 3$ ($\Rightarrow Z$ is special or lies in a special surface) :

- mixed $\mathfrak{S} = \mathcal{E} \times_{Y(2)} \mathcal{E}$. Two types of special curves (see Gao) :

- torsion curves (dominant over $Y(2)$) : see Masser-Zannier;
- torsion translates of elliptic curves in a CM fiber $E_0 \times E_0$: see Barroero.

- "doubly mixed" $\mathfrak{S} = \mathcal{P}_0$, the Poincaré $\mathbb{G}_m$-torsor over $E_0 \times \hat{E}_0$.

# Relative Manin-Mumford

A reduction in mixed cases : instead of $Z \subset \mathfrak{S}$, study a section $s$ of a group scheme $G/S$ over a curve $S/\overline{\mathbb{Q}}$, with $s(S) = Z' \subset G$.

**Theorem**

[Masser-Zannier] *Let $A = E \times_S E$, with $E/S$ non isoconstant and let $s = (p, q) \in A(S)$. The set*

$$S_s^{tor} := \{\lambda \in S(\overline{\mathbb{Q}}), s(\lambda) \in A_\lambda^{tor}\}$$

*is infinite if and only if one of the following conditions holds :*
**a)** *$s$ is a torsion section;*
**b)** *there exists an elliptic subscheme $E'/S$ of $A/S$ such that a multiple of $s$ factors through $E'$ [and is not a constant section if $E'/S$ is isoconstant].*

(**b**) = *relative dimension 1"* : if $s$ is a section of $E'/S$ which is not constant (and not torsion), it meets $E'^{tor}$ densely (but *with bounded height*).

In relative dimension 2, same for $G/S \in Ext_S(E, \mathbb{G}_a)$ (H. Schmidt), but for $G/S \in Ext_S(E_0, \mathbb{G}_m) \simeq \hat{E}_0(S) \simeq E_0(S)$,

## Theorem

[B-Masser-Pillay-Zannier] *Let $S/\overline{\mathbb{Q}}$, $G/S$, parametrized by a non constant $q \in E_0(S)$, and let $s \in G(S)$ be a section, with projection $p \in E_0(S)$. Then, the set*

$$S_s^{tor} := \{\lambda \in S(\overline{\mathbb{Q}}), s(\lambda) \in G_\lambda^{tor}\}$$

*is infinite if and only if one of the following conditions is satisfied :*
**a**) *s is a torsion section, or more generally :*
**a'**) *s is a Ribet section.*
**b**) *a multiple of s factors though $\mathbb{G}_{m/S}$ and is not constant.*

At the generic fiber of $G$, Ribet sections form a group *of rank 1*. They satisfy $p(\lambda) \in E_\lambda^{tor} \Rightarrow s(\lambda) \in G_\lambda^{tor}$, so their $S_s^{tor}$ is infinite. Their image (Ribet curves) form a set of special curves of $\mathcal{P}_0$, strictly containing the torsion curves.
Ribet points at any fiber $G_\lambda$ are defined in the same way. If $q(\lambda) \notin (\hat{E}_0)^{tor}$, they form a group of rank 1. Every Ribet point lies on a Ribet curve.

# Ribet curves

Let $\mathcal{S} = Pic^0(E_0) = \hat{E}_0$ be the moduli space of extensions of $E_0$ by $\mathbb{G}_m$, and let $\pi : \mathcal{G} \to \mathcal{S}$ be the universal semi-abelian scheme over $\mathcal{S}$ : dimension 3, relative dim. 2.
This $\mathcal{G}$ can also be viewed as the Poincaré biextension $\mathcal{P}_0$ of $E_0 \times \hat{E}_0$ by $\mathbb{G}_m$.

Let $B$ be an elliptic curve in $E_0 \times \hat{E}_0 \simeq E_0 \times E_0$, so there is an isogeny : $(a, b) : E_0 \to B$, with $a, b \in End(E_0)$. Then, the $\mathbb{G}_m$-torsor $\mathcal{P}_{0|B}$, or at least $(\mathcal{P}_{0|B})^{\otimes 2}$, is trivial iff $N(a-b) = N(a) + N(b)$, i.e. iff $\alpha := a/b$ (or $(b/a)$) is purely imaginary. If so, the "isotropic"

$$B \subset \{(p, q), bp - aq = 0\}$$

has a canonical lift to $\mathcal{P}_0$, which we call a basic Ribet curve of $\mathcal{P}_0$. More generally, any curve $C$ in $\mathcal{G}$ such that $[n]_{\mathcal{G}}.C$ is a basic Ribet curve for some $n \in \mathbb{Z}_{>0}$ is called *a Ribet curve (in the sense of* $\mathcal{G}$*)*. So, these include all the (local) torsion curves of $\mathcal{G}$.

Duality $\rightsquigarrow$ Ribet curves in the sense of $\mathcal{G}' \in Ext_{E_0}(\hat{E}_0, \mathbb{G}_m)$.

# "Relative Mordell-Lang"

Let $G \in Ext_S(E_0, \mathbb{G}_m)$ as above, so $G = G_q$ for a non constant $q \in E_0(S)$.

**Theorem**

[D.B.-H. Schmidt] *Let $s \in G(S)$. Assume that the set*

$$S_s^{Rib} = \{\lambda \in S(\overline{\mathbb{Q}}), s(\lambda) \text{ is a Ribet point of its fiber } G_\lambda.\}$$

*is infinite. Then, $s = s' + s"$, where $s'$ is a Ribet section, and $s"$ factors through $\mathbb{G}_m$. (So $s$ factors trough $\mathbb{G}_m\times$ some isotropic $B$.)*

List of special curves of $\mathfrak{S} = \mathcal{P}_0$ : if $Z$ is special, then :
- if $Z \to \hat{E}_0$ is dominant, $Z$ is a Ribet curves in the sense of $\mathcal{G}$;
- if $Z \to E_0$ is dominant, $Z$ is a Ribet curves in the sense of $\mathcal{G}'$;
- if $Z \sim \mathbb{G}_m$ projects to a point $(p_0, q_0) \in E_0 \times \hat{E}_0$, then $(p_0, q_0)$ is torsion and $Z$ is the fibre of $\mathcal{P}_0$ above $(p_0, q_0)$.

[WiP] : check that the list is complete, and conclude the proof that Pink's conjecture holds [for a curve] over $\overline{\mathbb{Q}}$ in the MSV $\mathcal{P}_0$.

# Sketch of proof

As a first case, assume that $p$ and $q$ are linearly independent over $End(E_0)$, modulo constants. Then,

- $\lambda \in S_s^{Rib} \Rightarrow$ bounded height , since $b_\lambda p(\lambda) - a_\lambda q(\lambda) = 0$.
- height of relations controled by degrees (Masser-type estimate for $ns(\lambda) = ms_R(\lambda)$)
- $o$-minimal count on an incidence variety (Habegger-Pila) ;
- logarithmic Ax $\rightsquigarrow$ no algebraic subset under the “non-weakly" assumption. NB : the Betti coordinates of a Ribet section $s_R$ generate the same field as the torsion sections.

2nd case: same argument if $p = rq + p_0$ with $r \in \mathbb{Z}$, reducing finally to

3rd case : $p = p_0$. Using duality, transfer the problem into a special case of Mordell-Lang for the semi-abelian variety $G'_{p_0} \in Ext(\hat{E}_0, \mathbb{G}_m)$, and the rank 1 subgroup of $G'_{p_0}(\overline{\mathbb{Q}})$ formed by its Ribet points.