Extremal Metrics on Toric Manifolds

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1. Introduction

An important problem in complex geometry is to find canonical metrics on Kähler manifold.

Let (M, ω_0) be a compact Kähler manifold. For any Kähler metric $\omega \in [\omega_0]$, define

$$f(\omega) = \int_M \mathcal{S}(\omega)^2 \frac{\omega^n}{n!},$$

where $\mathcal{S}(\omega)$ is the scalar curvature.

The metric ω is called extremal if it is a critical point of this functional.

The Euler-Lagrange equation of the variational problem is that the gradient of the scalar curvature is a holomorphic vector field.

Example If $\mathcal{S}(\omega) \equiv constant$ then ω is extremal metric.

There are three aspects of the problem:

- (1) necessary condition of existence;
- (2) sufficient conditions of existence;
- (3) uniqueness.

The necessary conditions for the existence are conjectured to be related to certain stabilities. There are many works on this aspects.

[Yau,Tian,Donaldson] *Let M be a compact, complex manifold, and $L \to M$ be a positive line bundle. The manifold M admits a cscK metric in the class $c_1(L)$ if and only if (M, L) is K-stable.*

The uniqueness problem has been solved by **Mabuchi** in the algebraic case,
Chen-Tian,
Chen-Li-Paun, **Berman-Berndtsson** in general.

On the other hand, there has been not much progress on the existence of extremal metrics. One reason is that the equation is highly nonlinear and of 4th order.

It may become simpler if the manifold studied admits more symmetry. Donaldson initiated a program to study the extremal metrics on toric manifolds and solved the problem for cscK metrics on toric surfaces.

Following Donaldson, we study the prescribed scaler curvature problem on toric varieties.

1.1 Abreu's equation

Let $\Delta \subset \mathbb{R}^n$ be the Delzant polytope for a toric manifold M. We denote by M° the open dense subset of M defined by

$$M^\circ = \{p \in M : \mathbb{T}^n - action\ is\ free\ at\ p\}.$$

There are two natural types of local coordinates on a toric manifold: complex log affine coordinates

$$w_i = x_i + \sqrt{-1}y_i, \quad (x, y) \in M^\circ$$

and

symplectic (Darboux) coordinates

$$(\xi_i, y_i)$$

where $i = 1, ..., n$, $y = (y_1, ..., y_n)$ are the angular coordinates with periodic 4π.

We can describe M° in complex log affine coordinates as

$$M^\circ = \mathbb{R}^n \times 2i\mathbb{T}^n = \{x + iy : x \in \mathbb{R}^n, y \in \mathbb{R}^n/\mathbb{Z}^n\}.$$

In this $w = x + iy$ coordinates, the $\mathbb{T}^n$-action is given by

$$t \cdot (x + iy) = x + i(y + t),\ t \in \mathbb{T}^n.$$

The $\mathbb{T}^n$-invariant Kähler form in this coordinates is given by a potential potential f, which depends only on the x coordinates:

$$f = f(x) \in C^\infty(\mathbb{R}^n)$$

and is strictly convex.

The gradient of f defines a (normal) map ∇^f from $\mathbb{R}^n$ to Δ:

$$\xi = (\xi_1, ..., \xi_n) = \nabla^f(x) = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right).$$

The function u on Δ

$$u(\xi) = x \cdot \xi - f(x).$$

is called the Legendre transformation of f. We write $u = L(f)$. Conversely, $f = L(u)$.

The moment map, restricting to the on M°, is given by

$$(w_1, ..., w_n) \mapsto (x_1, ..., x_n) \mapsto (\xi_1, ..., \xi_n).$$

In terms of coordinates (x, y) the Ricci curvature and the scaler curvature are given respectively by

$$R_{ij} = -\frac{\partial^2}{\partial x_i \partial x_j} \left(\log \det \left(f_{kl}\right)\right),$$

$$R = -\sum_{i=1}^{n} \sum_{j=1}^{n} f^{ij} \frac{\partial^2 (\log \det(f_{kl}))}{\partial x_i \partial x_j}.$$

In terms of coordinates ξ and Legendre transform function u of f, the scaler curvature can be written as

$$R(u) = -\sum U^{ij} w_{ij},$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix (u_{ij}), $w = (\det(u_{ij}))^{-1}$.

Guillemin constructed a natural $\mathbb{T}^n$ invariant Kähler form ω_g. We denote the class by $[\omega_g]$ and call it the Guillemin metric. Let $v = L(g)$, where g is the potential function of the Guillemin metric.

Suppose that Δ is defined by linear inequalities

$$\langle \xi, \nu_i \rangle - \lambda_i > 0,$$

where ν_i is the primitive inward pointing normal vector to the facet F_i of Δ, and $\langle \xi, \nu_i \rangle - \lambda_i = 0$ defines the facet. Write

$$l_i(\xi) = \langle \xi, \nu_i \rangle - \lambda_i.$$

Then

$$v(\xi) = \sum_i l_i \log l_i.$$

The prescribed scalar curvature problem reduces to finding a smooth convex solution in Δ for the 4-th order PDE

$$(*) \qquad \sum U^{ij} w_{ij} = -K, \quad K \in C^\infty(\bar{\Delta})$$

subject to the boundary condition

$$u - \sum_i l_i \log l_i \in C^\infty(\bar{\Delta}).$$

$(*)$ is called the Abreu equation. Set

$\mathcal{C}^\infty(\Delta, v) = \{u | u = v + \psi$ is strictly convex, $\psi \in C^\infty(\bar{\Delta})\}$

2. K-stability and Existence of Extremal Metrics

For any smooth function K on $\bar{\Delta}$, Donaldson defines a functional on $\mathcal{C}^\infty(\Delta)$:

$$\mathcal{F}_K(u) = -\int_\Delta \log\det(u_{ij})d\mu + \mathcal{L}_K(u),$$

where $\mathcal{L}_K$ is the linear functional

$$\mathcal{L}_K(u) = \int_{\partial\Delta} ud\sigma - \int_\Delta Kud\mu,$$

where $d\mu$ is the Lebesgue measure on $\mathbb{R}^n$ and on each face F, $d\sigma$ is a constant multiple of the standard $(n-1)$-dimensional Lebesgue measure.

$\mathcal{F}_K$ is known to be the Mabuchi functional and $\mathcal{L}_K$ is closely related to the Futaki invariants.

The Euler-Lagrange equation for $\mathcal{F}_K$ is

$$\sum U^{ij} w_{ij} = -K.$$

Donaldson proved that if v satisfies the above Eq., then v is an absolute minimizer for $\mathcal{F}_K$.

Definition Let $K \in C^\infty(\bar{\Delta})$ be a smooth function on $\bar{\Delta}$. (Δ, K) is called *relatively K-polystable* if $\mathcal{L}_K(u) \geq 0$ for all piecewise-linear convex functions u, and

$$\mathcal{L}_K(u) = 0$$

if and only if u is a linear function.

We will simply refer to relatively K-polystable as polystable.

Donaldson also introduced a stronger version of stability which we call uniform stability.

We fix a point $p \in \Delta$ and say u is normalized at p if

$$u \geq u(p) = 0.$$

Definition (Δ, K) is called uniformly stable if there is a constant $\lambda > 0$ such that for any normalized convex function $u \in \mathcal{C}^\infty(\Delta)$

$$\mathcal{L}_K(u) \geq \lambda \int_{\partial\Delta} u.$$

Sometimes, we say that Δ is (K, λ)-stable.

It is easy to show that the uniform stability implies the polystability. The reverse is unknown in general, however there are results for $dim\Delta = 2$: It is proved by Donaldson

Proposition When $n = 2$, if (Δ, K) is polystable and $K > 0$, then there exists a constant $\lambda > 0$ such that Δ is (K, λ)-stable.

3.1. On toric manifold

For any dimension, we have

Theorem 1(Bohui Chen, An-Min Li and Li Sheng) If the Abreu equation $(*)$ has a smooth solution in $\mathbb{C}^{\infty}(\Delta, v)$, then (Δ, K) is uniform stable.

Namely, we prove that the uniform K-polystability is a necessary condition for existing a solution of $(*)$ in $\mathbb{C}^{\infty}(\Delta, v)$.

Question Let (M, ω) be a toric manifold and Δ be the corresponding Delzant polytope. (M, ω) has a metric within the class $[\omega]$ that solves the Abreu equation if and only if (Δ, A) is uniformly K-stable?

Theorem 2.(Bohui Chen, Qing Han, An-Min Li, Li Sheng)

Let Δ be a Delzant polytope in $\mathbb{R}^n$. If (Δ, K) is uniformly stable, then for any solution u of the Abreu equation $(*)$ and any $\Omega_1 \subset\subset \Delta^o$, any constant $\alpha \in (0,1)$,

$$\|u\|_{C^{k+3,\alpha}(\Omega)} \leq C\|A\|_{C^k(\bar{\Delta})},$$

where C is a positive constant depending only on n, k, α, Ω, and λ in the uniform K-stability.

A crucial step in the proof is a derivation of a new upper bound of determinants

$$det(u_{ij})$$

in terms of f.

Then we use the estimates for linearized Monge-Ampère equations due to Caffarelli and Gutiérrez and estimates for Monge-Ampère equations due to Caffarelli.

Definition Let K be a smooth function on $\bar{\Delta}$. It is called *edge-nonvanishing* if it does not vanish on any edge of Δ.

Our main result is following

Theorem 3(Bohui Chen, An-Min Li and Li Sheng) Let M be a compact toric surface and Δ be its Delzant polytope. Let $K \in C^{\infty}(\bar{\Delta})$ be an edge-nonvanishing function. If (M, K) is uniformly stable, then there is a smooth T^2-invariant metric on M that solves the Abreu equation.

For any $n \geq 2$, it is well known that ω_{f} gives an extremal metric if and only if $R(u)$ is a linear function of $\xi \in \Delta$. Let A be a linear function of ξ. As

$$\mathcal{L}_A(u) = 0$$

if and only if u is a linear function. This is set of $n+1$ linear constrains on A. So there is unique linear function A satisfies the above constrains.

For $n = 2$, we can use our Theorem to find extremal metrics.

3.2. On homogeneous toric bundles

The space $G \times_K M$ is a fiber bundle with fiber M, a toric manifold, and base space G/K, a generalized flag manifold.

The prescribed scalar curvature problem is reduced to study

$$\sum_{i,j} U^{ij}\mathbb{F}_{ij} = -\mathbb{D}A,$$

where

$$\mathbb{F} := \frac{\mathbb{D}}{\det(u_{ij})}, \quad U^{ij} = \det(u_{kl})u^{ij},$$

and $\mathbb{D} > 0$ and A are two given smooth functions on $\bar{\Delta}$. We call it a generalized Abreu Equation.

By the same method of Chen, Li, Sheng we can obtain:

Theorem 4.(Bohui Chen, Qing Han, An-Min Li, Lian Zhao, Li Sheng) Let (M, ω) be a compact toric surface and Δ be its Delzant polytope. Let G/K be a generalized flag manifold with $dim(Z(K)) = 2$ and $G \times_K M$ be the homogeneous toric bundle. Let $\mathbb{D} > 0$ and $A \in C^\infty(\bar{\Delta})$ be given smooth functions. Suppose that $\mathbb{D}$ is an edge-nonconstant function on $\bar{\Delta}$ and $(\Delta, \mathbb{D}, A)$ is uniformly stable. Then, there is a smooth (G, T^2)-invariant metric $\mathcal{G}$ on $G \times_K M$ that solves the generalized Abreu Equation.

Theorem 4 provides an affirmative answer to the Yau-Tian-Donaldson conjecture for the homogeneous toric bundle in the case $\mathbb{S} = constant$ and $n = 2$.

4.1. The continuity method

Let K be the scalar function on $\bar{\Delta}$ and suppose that there exists a constant $\lambda > 0$ such that Δ is (K, λ) stable.

Let $I = [0, 1]$ be the unit interval. At $t = 0$ we start with a known metric. Let K_0 be its scalar curvature on Δ. Then Δ must be (K_0, λ_0) stable for some constant $\lambda_0 > 0$. At $t = 1$, set $(K_1, \lambda_1) = (K, \lambda)$. On Δ, set

$$K_t = tK_1 + (1-t)K_0, \quad \lambda_t = t\lambda_1 + (1-t)\lambda_0.$$

Set

$$\Lambda = \{t | \mathcal{S}(u) = K_t \text{ has a solution in } \mathcal{C}^{\infty}(\Delta, v).\}$$

Then we should show that Λ is open and closed.

Openness is standard. It remains to get a priori estimates for solutions $u_t = v + \psi_t$ to show closedness.

For $n = 2$ Donaldson proved

- interior regularity
- C^0-estimate .

We need do more estimates:

- estimates on the boundary of polytopes.

This is most difficult part.

Since the boundary of polytopes corresponds to the interior of the complex manifold, it is then natural to extended the real affine techniques to the complex case.

The key is to get estimate on Ricci curvature.

4.2. Complex differential inequalities

Let $\Omega \subset \mathbb{C}^n$, denote

$$\mathcal{R}^\infty(\Omega) := \{f \in C^\infty(\Omega) \mid$$

$$f \text{ is a real function and } (f_{i\bar{j}}) > 0\},$$

where $(f_{i\bar{j}}) = \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)$.

For $f \in \mathcal{R}^\infty(\Omega)$, (Ω, ω_f) is a Kähler manifold.
For the sake of notations, we set

$$W = \det(f_{s\bar{t}}), \quad V = \log\det(f_{s\bar{t}}),$$

Introduce the functions

$$\Psi = \|\nabla V\|_f^2, \quad P = \exp\left(\kappa W^{\alpha}\right)\sqrt{W}\Psi,$$

where κ, α are constants.

Denote

$$\|V_{,i\bar{j}}\|_f^2 = \sum f^{i\bar{j}} f^{k\bar{l}} V_{i\bar{l}} V_{k\bar{j}}, \quad \|V_{,ij}\|_f^2 = \sum f^{i\bar{j}} f^{k\bar{l}} V_{,ik} V_{,\bar{l}\bar{j}}.$$

Denote by $\Box = \sum f^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$ the Laplacian operator. Recall that

$$\mathcal{S}(f) = -\Box V = -\sum f^{i\bar{j}} V_{i\bar{j}}$$

is the scalar curvature of ω_f.

We have the following differential inequality

$$\frac{\Box P}{P} \geq \frac{\|V_{,i\bar{j}}\|_f^2}{2\Psi} + \alpha^2\kappa(1-2\kappa W^\alpha)W^\alpha\Psi - \frac{2|\langle\nabla\mathcal{S},\nabla V\rangle|}{\Psi} - \left(\alpha\kappa W^\alpha + \frac{1}{2}\right)\mathcal{S},$$

where $\langle , \rangle$ denotes the inner product with respect to the metric ω_f.

This inequality plays a key role in our paper. We use this inequality and use affine blow-up analysis to get a estimate for Ricci curvature in a geodesic ball $B_a(\mathfrak{z}_o)$:

$$\frac{W^{\frac{1}{2}}(\mathfrak{z})}{\max\limits_{B_a(\mathfrak{z}_o)} W^{\frac{1}{2}}}\|Ric\|_f^2 a^2 \leq \mathsf{C}_1,$$

for any $\mathfrak{z} \in B_{a/2}(\mathfrak{z}_o)$, where C_1 is a constant independent of f.

Denote $H = \frac{\det(g_{i\bar{j}})}{\det(f_{i\bar{j}})}$.

Using the above estimate on Ricci curvature we get estimate of H from below and above.

4.3

We introduce notations for $n = 2$.

Suppose that the vertices and edges of Δ are denoted by

$$\{\vartheta_0, \ldots, \vartheta_d = \vartheta_0\}, \quad \{\ell_0, \ell_1, \ldots, \ell_{d-1}, \ell_d = \ell_0\}.$$

Here $\vartheta_i = \ell_i \cap \ell_{i+1}$.

Suppose that the equation for ℓ_i is

$$l_i(\xi) := \langle \xi, \nu_i \rangle - \lambda_i = 0.$$

Then we have

$$\Delta = \{\xi | l_i(\xi) > 0, \quad 0 \le i \le d-1\}.$$

Let $\mathsf{U}_\Delta, \mathsf{U}_{\ell_i}$ and U_{ϑ_i} be the coordinate charts. Then

$$\mathsf{U}_\Delta \cong (\mathbb{C}^*)^2; \quad \mathsf{U}_{\ell_i} \cong \mathbb{C} \times \mathbb{C}^*; \quad \mathsf{U}_{\vartheta_i} \cong \mathbb{C}^2.$$

We have

$$(\mathbb{C}^*)^2 \cong \mathfrak{t} \times 2\sqrt{-1}\mathbb{T}^2.$$

Then (z_1, z_2) on the left hand side is the usual complex coordinate; while (w_1, w_2) on the right hand side is the log-affine coordinate.

Write $w_i = x_i + \sqrt{-1}y_i$, $y_i \in [0, 4\pi]$. Then (x_1, x_2) is the coordinate of $\mathfrak{t}$.

On different types of coordinate chart we use different coordinate systems as follows:

- on $\mathrm{U}_{\vartheta} \cong \mathbb{C}^2$, we use the coordinate (z_1, z_2);
- on $\mathrm{U}_{\ell} \cong \mathbb{C} \times \mathbb{C}^*$, we use the coordinate (z_1, w_2);
- on $\mathrm{U}_{\Delta} \cong (\mathbb{C}^*)^2$, we use the coordinate (w_1, w_2),

where $z_i = e^{\frac{w_i}{2}}, i = 1, 2$.

We consider two cases.

Case 1. Regularity on edges

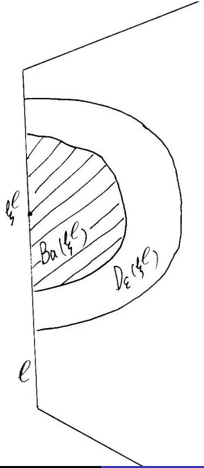
Let ℓ be any edge and $\xi^{(\ell)} \in \ell$ such that

$$D_b(\xi^{(\ell)}) \cap \partial\Delta \subset \ell,$$

$$\min_{D_b(p)\cap\bar{\Delta}} |\mathcal{S}(u)| \geq \delta > 0.$$

Then there is a constant $\epsilon > 0$ that is independent of k such that

$$B_\epsilon(\xi^{(\ell)}) \cap \Delta \subset D_a(\xi^{(\ell)}) \cap \Delta.$$



By the bound of H, we have, on $B_\epsilon(\mathfrak{z}^{(\ell)})$

$$C_1^{-1} \leq W \leq C_1.$$

By the estimate of $\mathcal{K}$ and the regularity theorem we obtain the regularity of f on $B_\epsilon(\mathfrak{z}^{(\ell)})$.

Case 2. Regularity at vertices

Let ϑ be any vertex. By **Case 1**, there is a bounded open set $\Omega_\vartheta \subset \mathsf{U}_\vartheta$, independent of k, such that $\vartheta \in \tau(\Omega_\vartheta)$ and the regularity of f_ϑ holds in a neighborhood of $\partial\Omega_\vartheta$. See Figure 2.

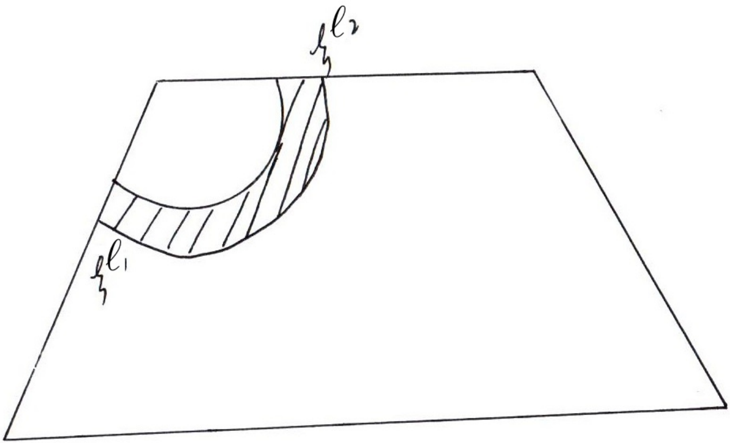


Figure: 2

To prove the regularity we need another differential inequality:

Let $0 \in \Omega \subset \mathbb{C}^2$, $f = f_\vartheta$ be the potential function on Ω. Let $T = \sum f^{i\bar{i}}$. Put

$$Q = e^{N_1(|z|^2 - A)}\sqrt{W}T,$$

where A, N_1 are constants. Suppose that

$$\max_{\tau_f(\bar{\Omega})}\left(|K| + \sum\left|\frac{\partial K}{\partial \xi_i}\right|\right) \leq \mathsf{N}_2,$$

$$\max_{\bar{\Omega}} W \leq \mathsf{N}_2, \qquad \max_{\bar{\Omega}} |z| \leq \mathsf{N}_2$$

for some constant $\mathsf{N}_2 > 0$.

Then we may choose

$$A = \mathsf{N}_2^2 + 1, N_1 = 100, \alpha = \frac{1}{3}, \kappa = [4\mathsf{N}_2^{\frac{1}{3}}]^{-1}$$

such that

$$\Box(P + Q + \mathsf{C}_1 f) \geq \mathsf{C}_2(P + Q)^2 > 0$$

for some positive constants C_1 and C_2 that depend only on N_2 and n.

By the Inequality, P and Q are bounded above. Then we obtain the regularity of f on Ω_ϑ.

Finally we got closedness.

We are written a notes to explain the papers of Chen-Li-Sheng, and simplify some parts.