Almost Kähler 4-manifolds of Constant Holomorphic Sectional Curvature are Kähler

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joint work with

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Preliminaries

Definition

An almost Kähler manifold (M, ω, g, J) is equipped with

$$\omega \in \Omega^2(M), \quad J \colon TM \to TM, \quad g \text{ metric}$$

such that

$$d\omega = 0, \quad J^2 = -1, \quad \omega = g(J\cdot, \cdot).$$

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Definition

The Hermitian connection is

$$\nabla_X Y := D^g_X Y \underbrace{-\frac{1}{2} J(D^g_X J) Y}_{A_X Y =}.$$

- $\nabla g = 0$, $\nabla J = 0$, but ∇ may have torsion.
- M Kähler $\iff \nabla = D^g$.

Statement of Result

The Hermitian holomorphic sectional curvature is

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Theorem (U.–Lejmi, 2017)

Let M be a closed almost Kähler 4-manifold of globally constant Hermitian holomorphic sectional curvature $k \geq 0$.

Then M is Kähler–Einstein, holomorphically isometric to:

- $(k > 0)$ $\mathbb{C}P^2$ *with the Fubini–Study metric.*
- $(k = 0)$ *a complex torus or a hyperelliptic curve with the Ricci-flat Kähler metric.*

Similar result for $k < 0$ under assumption that Ricci is J-invariant.

Background

Related Work

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Armstrong 1997 On four-dimensional almost Kähler manifolds.

Pointwise Implications (Algebraic Bianchi)

M^4 closed almost Kähler

Proposition

Pointwise constant holomorphic sectional curvature $H = k$ is equivalent to

1. $W^- = 0$
2. $*\rho = r$ *for two Ricci contractions of* R^∇.

Moreover,

$$v := \frac{\mathrm{Scal}^g}{12} \leq \frac{k}{2}$$

with equality if and only if M is Kähler.

Sketch of Proof for $W^- = 0$

Use

$$\begin{aligned} R^{\nabla}_{XYZW} =& R^{g}_{XYZW} \\ &+ \underbrace{g((\nabla_X A_Y - \nabla_Y A_X - A_{[X,Y]})Z, W)}_{\alpha \in \Lambda^2 \otimes \Lambda^{2,0+0,2}} - \underbrace{g([A_X, A_Y]Z, W)}_{\beta \in \Lambda^{1,1} \otimes \mathbb{C} \cdot F}. \end{aligned}$$

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Play off the symmetries of R^g against the assumption on R^{∇} (which gives it a special form).

$$R^g = \begin{array}{c} \\ \left[\begin{array}{cc|c} \mathbb{C}F & \Lambda^{2,0} \oplus \Lambda^{0,2} & \Lambda^{1,1}_0 \\ d \cdot g & W^+_F & R_F \\ (W^+_F)^T & W^+_{00} + \frac{c}{2} g & R_{00} \\ \hline R_F^T & R_{00}^T & W^- + \frac{\mathrm{Scal}^g}{12} g \end{array} \right] \end{array}$$

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 & \mathbb{C}F & \Lambda^{2,0} \oplus \Lambda^{0,2} & \Lambda^{1,1}_0 \\
 & \frac{s_C}{2} g & 0 & R_F \\
 & (W_F^+)^T & 0 & R_{00} \\
\hline
 & -R_F^T & 0 & \frac{s_g}{12} g
\end{array}
$$

Outline of Proof of Main Theorem II

Global Impliciations

From the differential Bianchi identity:

Proposition

Let M be a closed almost Kähler 4-manifold of pointwise constant holomorphic sectional curvature k. Then

$$\int_M |R_{00}|^2 = \int_M |W_F^+|^2 + |W_{00}^+|^2 + 4(5k - 7v)(k - 2v) \qquad (1)$$

$$\chi = \frac{-1}{8\pi^2} \int_M |W_{00}^+|^2 + (60v^2 - 72kv + 18k^2) \qquad (2)$$

$$\frac{3}{2}\sigma = \frac{1}{8\pi^2} \int_M 2|W_F^+|^2 + |W_{00}^+|^2 + 6(2k - 3v)^2 \geq 0 \qquad (3)$$

Recall: $v := \frac{\mathrm{Scal}^g}{12} = \frac{k}{2}$ implies Kähler.

Outline of Proof of Main Theorem III

Corollary (Signature zero case)

Let M be closed almost Kähler 4-manifold of pointwise constant holomorphic sectional curvature k. Suppose $\sigma = 0$.

Then $k = 0$ and M is Kähler, with a Ricci-flat metric.

Outline of Proof of Main Theorem III

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Then $k = 0$ and M is Kähler, with a Ricci-flat metric.

Corollary ('Reverse' Bogomolov–Miyaoka–Yau inequality)

If M is closed almost Kähler of globally constant holomorphic sectional curvature $k \geq 0$, then for the Euler characteristic

$$3\sigma \geq \chi.$$

Equality holds if and only if M is Kähler–Einstein.

End of the proof

Theorem

M^4 closed almost Kähler of constant holomorphic sectional curvature $k \geq 0$. Then M is Kähler.

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Proof.

- Suppose that M is <u>not</u> Kähler: $v < \frac{k}{2}$ somewhere.

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- Suppose that M is not Kähler: $v < \frac{k}{2}$ somewhere.
- $\int_M c_1(TM) \cup \omega = \int_M \frac{s_C}{2\pi} = \underbrace{\int_M \frac{3k}{2\pi}}_{\geq 0} + \int_M \frac{k-2v}{2\pi} > 0.$

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- M rational $\implies \sigma \leq 0 \implies \sigma = 0$.
- By previous propositions, '$=$' implies Kähler, contradiction!

□