Invariant solutions to the Yamabe Equation on the Koiso-Cao soliton

Jonatán Torres Orozco Román

Centre International de Rencontres Mathématiques

CIRM, Marseille

January 2018

Yamabe Problem (1960)

Given a Riemannian manifold (M, g) of dimension $n \geq 3$, up to conformal changes of g, there exist constant scalar curvature metrics.

Objective: Discuss uniqueness on Ricci solitons

Yamabe Problem (1960)

Given a Riemannian manifold (M, g) of dimension $n \geq 3$, up to conformal changes of g, there exist constant scalar curvature metrics.

Objective: Discuss uniqueness on Ricci solitons

Outline

1. Yamabe Problem
   - Yamabe Equation
   - Uniqueness
2. Ricci Solitons
   - Properties
   - Koiso-Cao soliton
3. Yamabe Equation of the Koiso-Cao soliton
   - Invariant solutions
   - Yamabe constant

Yamabe Equation

Let (M, g) be a closed Riemannian manifold. For $f \in C_+^\infty(M)$, the metric $f^{p-2} \cdot g$ has constant scalar curvature λ if and only if f satisfies the ***Yamabe Equation:***

$$-a_n \triangle_g f + S_g f = \lambda f^{p-1}$$

Theorem (Yamabe-Trudinger-Aubin-Schoen)

There exists a constant scalar curvature metric in every conformal class.

Theorem (Hebey-Vaugon, 1993)

If G is a Lie group acting on M by isometries. Then there exist $G-$invariant solutions to the Yamabe equation.

Yamabe Equation

Let (M, g) be a closed Riemannian manifold. For $f \in C^{\infty}_{+}(M)$, the metric $f^{p-2} \cdot g$ has constant scalar curvature λ if and only if f satisfies the ***Yamabe Equation:***

$$-a_n \triangle_g f + S_g f = \lambda f^{p-1}$$

Theorem (Yamabe-Trudinger-Aubin-Schoen)

There exists a constant scalar curvature metric in every conformal class.

Theorem (Hebey-Vaugon, 1993)

If G is a Lie group acting on M by isometries. Then there exist $G-$invariant solutions to the Yamabe equation.

Yamabe Equation

Let (M, g) be a closed Riemannian manifold. For $f \in C^\infty_+(M)$, the metric $f^{p-2} \cdot g$ has constant scalar curvature λ if and only if f satisfies the ***Yamabe Equation:***

$$-a_n \triangle_g f + S_g f = \lambda f^{p-1}$$

Theorem (Yamabe-Trudinger-Aubin-Schoen)

There exists a constant scalar curvature metric in every conformal class.

Theorem (Hebey-Vaugon, 1993)

If G is a Lie group acting on M by isometries. Then there exist $G-$invariant solutions to the Yamabe equation.

Yamabe Constant

The *Yamabe constant* is obtained by:

$$\mathrm{Y}(M,[g]) = \inf_{h\in[g]} \frac{\int_M S_h \mathrm{dv}_h}{\mathrm{Vol}(M,h)^{\frac{n-2}{n}}}$$

Theorem (Aubin, 1976)

Let (M,g) be a closed Riemannian manifold of dimension n. Then:

$$\mathrm{Y}(M,[g]) \leq \mathrm{Y}_n$$

where $\mathrm{Y}_n := \mathrm{Y}(S^n,[g_o^n]) = n(n-1)\mathrm{Vol}(S^n,g_o^n)^{\frac{2}{n}}$.

Yamabe Constant

The *Yamabe constant* is obtained by:

$$\mathrm{Y}(M,[g]) = \inf_{h\in[g]} \frac{\int_M S_h \mathrm{dv}_h}{\mathrm{Vol}(M,h)^{\frac{n-2}{n}}}$$

Theorem (Aubin, 1976)

Let (M,g) be a closed Riemannian manifold of dimension n. Then:

$$\mathrm{Y}(M,[g]) \leq \mathrm{Y}_n$$

where $\mathrm{Y}_n := \mathrm{Y}(S^n,[g_o^n]) = n(n-1)\mathrm{Vol}(S^n,g_o^n)^{\frac{2}{n}}$.

Uniqueness

There is uniqueness:

- If $\mathrm{Y}(M,[g]) \leq 0$.

- **Obata's Theorem**: If (M,g) is Einstein and not isometric to the round sphere (S^n, g_o^n).

Ricci Solitons

Definition

Let (M, g) be a Riemannian manifold of dimension n such that

$$-2Ric(g) = \mathcal{L}_X g + 2\mu g$$

*for some $\mu \in \mathbb{R}$ and some complete vector field X on M. We say g is a **Ricci soliton**.*

- *If $\mu < 0$, g is said to be a shrinking Ricci soliton.*
- *$\mu = 0$, g is steady.*
- *$\mu > 0$, g is expanding.*

If $X = grad(u)$,

$$Ric(g) + Hess(u) + \mu g = 0$$

and g is called **gradient Ricci soliton.**

Ricci Solitons

Definition

Let (M, g) be a Riemannian manifold of dimension n such that

$$-2Ric(g) = \mathcal{L}_X g + 2\mu g$$

for some $\mu \in \mathbb{R}$ and some complete vector field X on M. We say g is a ***Ricci soliton****.*

- *If $\mu < 0$, g is said to be a shrinking Ricci soliton.*
- *$\mu = 0$, g is steady.*
- *$\mu > 0$, g is expanding.*

If $X = grad(u)$,

$$Ric(g) + Hess(u) + \mu g = 0$$

and g is called **gradient Ricci soliton.**

Ricci Solitons

Definition

Let (M, g) be a Riemannian manifold of dimension n such that

$$-2Ric(g) = \mathcal{L}_X g + 2\mu g$$

for some $\mu \in \mathbb{R}$ and some complete vector field X on M. We say g is a ***Ricci soliton****.*

- *If $\mu < 0$, g is said to be a shrinking Ricci soliton.*
- *$\mu = 0$, g is steady.*
- *$\mu > 0$, g is expanding.*

If $X = grad(u)$,

$$Ric(g) + Hess(u) + \mu g = 0$$

and g is called **gradient Ricci soliton.**

Ricci Solitons $-2Ric(g) = \mathcal{L}_X g + 2\mu g$

We are interested in compact non-Einstein Ricci solitons of positive scalar curvature.

Results from Hamilton, Ivey and Perelman give that compact non-trivial solitons have to be shrinking and gradient:

$$Ric(g) + Hess(u) - g = 0$$

Ricci Solitons $-2Ric(g) = \mathcal{L}_X g + 2\mu g$

We are interested in compact non-Einstein Ricci solitons of positive scalar curvature.

Results from Hamilton, Ivey and Perelman give that compact non-trivial solitons have to be shrinking and gradient:

$$Ric(g) + Hess(u) - g = 0$$

Koiso-Cao soliton $\mathbf{g}$ $\quad Ric(g) + Hess(u) - g = 0$

Koiso(1990)-Cao(1996)

- It is a compact shrinking Kähler-Ricci soliton on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.
- It has positive Ricci curvature.
- It admits an action of $\mathrm{U}(2)$ with cohomogeneity-one, and the principal orbits form an open dense subset

$$S^3 \times (\alpha, \beta) \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}.$$

- It has two singular orbits diffeomorphic to S^2.

Koiso-Cao soliton $\mathbf{g}$ $\quad Ric(g) + Hess(u) - g = 0$

Koiso(1990)-Cao(1996)

- It is a compact shrinking Kähler-Ricci soliton on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.
- It has positive Ricci curvature.
- It admits an action of $\mathrm{U}(2)$ with cohomogeneity-one, and the principal orbits form an open dense subset

$$S^3 \times (\alpha, \beta) \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}.$$

- It has two singular orbits diffeomorphic to S^2.

Koiso-Cao soliton $\mathbf{g}$ $\quad Ric(g) + Hess(u) - g = 0$

Koiso(1990)-Cao(1996)

- It is a compact shrinking Kähler-Ricci soliton on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.
- It has positive Ricci curvature.
- It admits an action of $\mathrm{U}(2)$ with cohomogeneity-one, and the principal orbits form an open dense subset

$$S^3 \times (\alpha, \beta) \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}.$$

- It has two singular orbits diffeomorphic to S^2.

Koiso-Cao soliton $\mathbf{g}$ $\quad Ric(g) + Hess(u) - g = 0$

Koiso(1990)-Cao(1996)

- It is a compact shrinking Kähler-Ricci soliton on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.
- It has positive Ricci curvature.
- It admits an action of $\mathrm{U}(2)$ with cohomogeneity-one, and the principal orbits form an open dense subset

$$S^3 \times (\alpha, \beta) \subset \mathbb{CP}^2\#\overline{\mathbb{CP}^2}.$$

- It has two singular orbits diffeomorphic to S^2.

Construction of the Koiso-Cao soliton

For every $t \in (\alpha, \beta)$, on S^3 consider a metric g_t such that:

$$(S^1,\ f^2(t)\cdot g_o) \hookrightarrow (S^3, g_t) \longrightarrow (S^2,\ h^2(t)\cdot g_o^2)$$

is a Riemannian submersion.

Lemma

On $S^3 \times (\alpha, \beta)$ the metric $g = g_t + dt^2$ **is Kähler if and only if**

$$f = -hh'.$$

We can extend g to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ provided:

$$h(\alpha)h''(\alpha) = -h(\beta)h''(\beta) = -1, \qquad h(\alpha) \neq h(\beta) \neq 0,$$

$$h^{2k+1}(\alpha) = h^{2k+1}(\beta) = 0.$$

Construction of the Koiso-Cao soliton

For every $t \in (\alpha, \beta)$, on S^3 consider a metric g_t such that:

$$(S^1,\ f^2(t) \cdot g_o) \hookrightarrow (S^3, g_t) \longrightarrow (S^2,\ h^2(t) \cdot g_o^2)$$

is a Riemannian submersion.

Lemma

On $S^3 \times (\alpha, \beta)$ *the metric* $g = g_t + dt^2$ **is Kähler if and only if**

$$f = -hh'.$$

We can extend g to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ provided:

$$h(\alpha)h''(\alpha) = -h(\beta)h''(\beta) = -1, \qquad h(\alpha) \neq h(\beta) \neq 0,$$

$$h^{2k+1}(\alpha) = h^{2k+1}(\beta) = 0.$$

Construction of the Koiso-Cao soliton

For every $t \in (\alpha, \beta)$, on S^3 consider a metric g_t such that:

$$(S^1,\ f^2(t) \cdot g_o) \hookrightarrow (S^3, g_t) \longrightarrow (S^2,\ h^2(t) \cdot g_o^2)$$

is a Riemannian submersion.

Lemma

On $S^3 \times (\alpha, \beta)$ *the metric* $g = g_t + dt^2$ **is Kähler if and only if**

$$f = -hh'.$$

We can extend g to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ provided:

$$h(\alpha)h''(\alpha) = -h(\beta)h''(\beta) = -1, \qquad h(\alpha) \neq h(\beta) \neq 0,$$

$$h^{2k+1}(\alpha) = h^{2k+1}(\beta) = 0.$$

Koiso-Cao soliton $\mathbf{g}$

Proposition

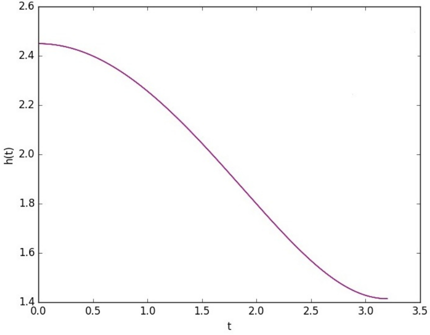
The $U(2)-$invariant Kähler metric $g = dt^2 + g_t$ defined on $S^3 \times (\alpha, \beta)$ by the function h, extends to a gradient shrinking Kähler-Ricci soliton $\mathbf{g}$ on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ if, for a constant $c \in \mathbb{R}$, h solves the ODE:

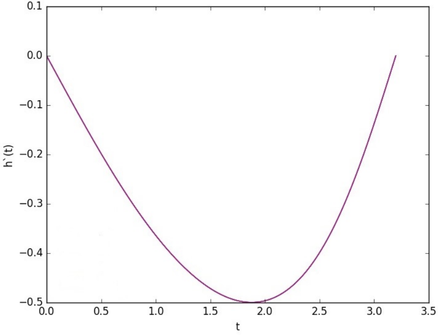
$$2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0$$

with $h(\alpha) = \sqrt{6}$ and $h'(\alpha) = 0$.

$$2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0$$

$c \approx -0.527619519896, h(\alpha) = \sqrt{6}$ and $h'(\alpha) = 0$





Koiso-Cao soliton $2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0$

$$c \approx -0.527619519896$$

Proposition

- *The scalar curvature is a decreasing function in* $[\alpha, \beta]$*,*

$$S_{\mathbf{g}} = 4ch'^2 + 2chh'' + 4.$$

- *The volume is* $\mathrm{Vol}(\mathbf{g}) = 16\pi^2$*.*
- *The total scalar curvature is* $\mathcal{S}(\mathbf{g}) = 16\pi$*.*

Koiso-Cao soliton $2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0$

$$c \approx -0.527619519896$$

Proposition

- *The scalar curvature is a decreasing function in* $[\alpha, \beta]$,

$$S_{\mathbf{g}} = 4ch'^2 + 2chh'' + 4.$$

- *The volume is* $\mathrm{Vol}(\mathbf{g}) = 16\pi^2$.
- *The total scalar curvature is* $S(\mathbf{g}) = 16\pi$.

Koiso-Cao soliton $2hh'' + 4h'^2 - 4 + h^2(1 + ch'^2) = 0$

$$c \approx -0.527619519896$$

Proposition

- *The scalar curvature is a decreasing function in* $[\alpha, \beta]$*,*

$$S_{\mathbf{g}} = 4ch'^2 + 2chh'' + 4.$$

- *The volume is* $\mathrm{Vol}(\mathbf{g}) = 16\pi^2$*.*
- *The total scalar curvature is* $\mathcal{S}(\mathbf{g}) = 16\pi$*.*

Yamabe Equation of the Koiso-Cao soliton

Proposition

Let ϕ be a positive $U(2)-$invariant function on $S^3 \times (\alpha, \beta)$ with the Koiso-Cao soliton $\mathbf{g}$. The metric $\phi^2 \cdot g$ extends to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ if

$$\phi'(\alpha) = \phi'(\beta) = 0.$$

The Yamabe equation is:

$$6\phi' \left(\frac{h''}{h'} + 3\frac{h'}{h} \right) + 6\phi'' + S_g \phi = \phi^3$$

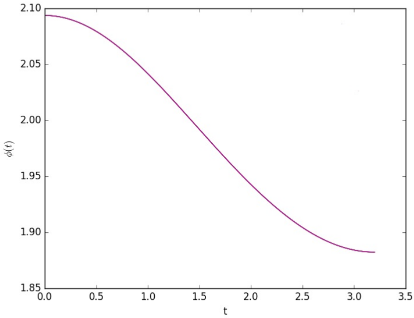
The solutions ϕ are decreasing on (α, β).

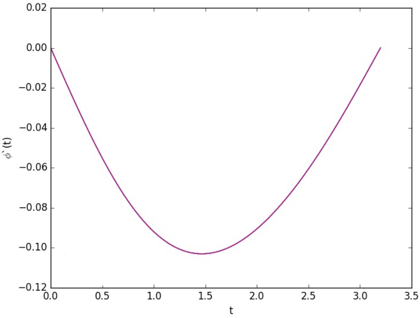
$$6\phi'\left(\frac{h''}{h'} + 3\frac{h'}{h}\right) + 6\phi'' + S_g\phi = \phi^3$$

$\phi(\alpha) > 0$ and $\phi'(\alpha) = 0$

Theorem

There exists a unique $\mathrm{U}(2)$*-invariant solution to the Yamabe equation on* $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ *with the Koiso-Cao* $\mathbf{g}$*.*





Yamabe constant

$$6\phi'\left(\frac{h''}{h'}+3\frac{h'}{h}\right)+6\phi''+S_g\phi=\phi^3$$

$\phi(\alpha)>0$ and $\phi'(\alpha)=0$

$$\mathrm{Y}(\mathbb{CP}^2\#\overline{\mathbb{CP}^2},[\mathbf{g}]) < \mathcal{S}(\mathbf{g})=16\pi\approx 50.26548<\mathrm{Y}_4=8\sqrt{6}\pi\approx 61.5623.$$

Yamabe constant

$$6\phi'\left(\frac{h''}{h'} + 3\frac{h'}{h}\right) + 6\phi'' + S_g\phi = \phi^3$$

$\phi(\alpha) > 0$ and $\phi'(\alpha) = 0$

$$\begin{aligned} \mathrm{Y}(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}, [\mathbf{g}]) &\approx 50.249772 \\ &< \mathcal{S}(\mathbf{g}) = 16\pi \approx 50.26548 < \mathrm{Y}_4 = 8\sqrt{6}\pi \approx 61.5623. \end{aligned}$$

Thank you!