Myers-Type Theorems, Diameter Bounds, and Gap Theorems for Sasaki Manifolds

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Constant Scalar Curvature Metrics in Kähler and Sasaki Geometry

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Aim & Plan

1. **Introduction**

   A Brief Review of **Sasaki Manifolds**.

   – Definition, Background, Properties, and Examples.

2. **Results**

   A Compactness Theorem for **Sasaki Manifolds**.

   – A **Cheeger-Gromov-Taylor Type Theorem**.

   Geometry of Gradient **Sasaki-Ricci Solitons**.

   – Some **Diameter Bounds** and **Gap Theorems**.

Sasaki Manifolds

Definition A Riemannian manifold (S, g) is a **Sasaki manifold** if

$$(C(S), \bar{g}) := (\mathbb{R}_+ \times S, dr^2 + r^2 g)$$

is a **Kähler manifold**. We identify S with the submanifold $\{r = 1\} \subset C(S)$.

Typical examples of Sasaki manifolds are **odd dimensional spheres**. For a $(2n+1)$-dimensional Sasaki manifold (S, g), we define a **Reeb vector field** ξ, a **contact form** η, and a $(1,1)$-tensor field Φ by

$$\xi := \left(J \frac{\partial}{\partial r} \right) \Bigg|_{r=1}, \quad \eta := \left(\sqrt{-1}(\bar{\partial} - \partial) \log r \right) \Big|_{r=1}, \quad \text{and} \quad \Phi(X) := \nabla_X \xi,$$

respectively. Here $X \in \mathfrak{X}(S)$. We see that

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0, \quad \eta \wedge (d\eta)^n \neq 0,$$

$$\boldsymbol{\Phi^2 = -\mathrm{id} + \xi \otimes \eta}, \quad \boldsymbol{g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)}, \quad X, Y \in \mathfrak{X}(S).$$

The 4-tuple (g, ξ, η, Φ) defines an **almost contact metric structure** on S.

From now on, (S, g) denotes a $(2n+1)$-dimensional Sasaki manifold.

Background and Motivation

- Introduced by Sasaki and Hatakeyama (1962).
- An **odd-dimensional counterpart** of a Kähler manifold.
  - A **Hodge decomposition** and a **Kähler identity** hold on Sasaki manifolds.
- Sasaki-Einstein manifolds play important roles in **Theoretical Physics**.
  - Sasaki-Einstein metrics are used to check the **AdS/CFT correspondence**.
  - The AdS/CFT correspondence stems from **String Theory**.
- Gauntlett et al discovered **irregular Sasaki-Einstein manifolds** (2004).

Definition A Sasaki manifold (S, g) is

- **quasi-regular** if all orbits of ξ are compact. Then the space of leaves are Kähler manifolds or Kähler orbifolds. (Example : Boyer-Galicki, Kollár)
- **irregular** if otherwise. Then the space of leaves never admit the structure of manifolds. (Example : Gauntlett-Martelli-Sparks-Waldram)

Transverse Geometry

Let (S, g) be a Sasaki manifold. We define a **contact bundle** by $D := \mathrm{Ker}\,\eta$. Then the tangent bundle of S splits as

$$TS = D \oplus \mathbb{R}\xi$$

and this induces a **transverse Riemannian metric** $g^T := g|_{D\times D}$ on D.

For $X \in \mathfrak{X}(S)$ and $Y \in \Gamma(D)$, we may define a **transverse Levi-Civita connection** ∇^T on D by

$$\nabla^T_X Y := \begin{cases} \pi(\nabla_X Y) & \text{if} \quad X \in \Gamma(D), \\ \pi([X,Y]) & \text{if} \quad X \in \Gamma(\mathbb{R}\xi), \end{cases}$$

where $\pi : TS \to D$ is the orthogonal projection.

Proposition ∇^T is the unique connection on D satisfying

$$Zg^T(X,Y) = g^T(\nabla^T_Z X, Y) + g^T(X, \nabla^T_Z Y), \quad \nabla^T_X Y - \nabla^T_Y X = \pi([X,Y]),$$

where $X, Y, Z \in \Gamma(D)$.

We define a **transverse curvature**, a **transverse Riemannian curvature**, a **transverse Ricci curvature**, and a **transverse scalar curvature** by

$$R^T(X,Y)Z := \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z,$$

$$\mathrm{Rm}^T(X,Y,Z,W) := g^T(R^T(X,Y)Z, W),$$

$$\mathrm{Ric}^T(X,Y) := \sum_{i=1}^{2n} \mathrm{Rm}^T(e_i, X, Y, e_i), \quad \text{and} \quad R^T := \sum_{i=1}^{2n} \mathrm{Ric}^T(e_i, e_i),$$

respectively. Here $X, Y, Z, W \in \Gamma(D)$ and $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis of D.

Proposition (Bianchi identity) For all $X, Y, Z, W \in \Gamma(D)$,

- $R^T(X,Y)Z + R^T(Y,Z)X + R^T(Z,X)Y = 0$,
- $\mathrm{Rm}^T(Y,X,Z,W) = -\mathrm{Rm}^T(X,Y,Z,W)$,
- $\mathrm{Rm}^T(X,Y,Z,W) = \mathrm{Rm}^T(Z,W,X,Y)$,
- $(\nabla^T_X R^T)(Y,Z)W + (\nabla^T_Y R^T)(Z,X)W + (\nabla^T_Z R^T)(X,Y)W = 0$.

A Myers-Type Theorem for Complete Sasaki Manifolds

Theorem (Hasegawa and Seino 1981, Nitta 2009) Let (S, g) be a $(2n+1)$-dimensional complete Sasaki manifold. If there exists **a positive constant $\lambda > 0$** such that

$$\mathrm{Ric}^T(X, X) \geqslant \lambda g^T(X, X), \quad X \in \Gamma(D),$$

then (S, g) must be compact with finite fundamental group. Moreover, the diameter of (S, g) has the upper bound

$$\mathrm{diam}(S, g) \leqslant 2\pi \sqrt{\frac{2n-1}{\lambda}}.$$

Remark The key ingredient in proving Theorem above is the **Hopf-Rinow-type theorem**, which asserts that any two points on a complete Sasaki manifold (S, g) may be joined by a **length-minimizing normal geodesic** γ such that $\dot{\gamma} \in D$.

A Cheeger-Gromov-Taylor-Type Theorem for Complete Sasaki Manifolds

Theorem (— 2016) Let (S, g) be a $(2n+1)$-dimensional complete Sasaki manifold. Suppose that there exist **a point $p \in S$** and **positive constants $r_0 > 0$ and $\nu > 0$** such that

$$\mathrm{Ric}^T(x) \geqslant (2n-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{d^2(x,p)}$$

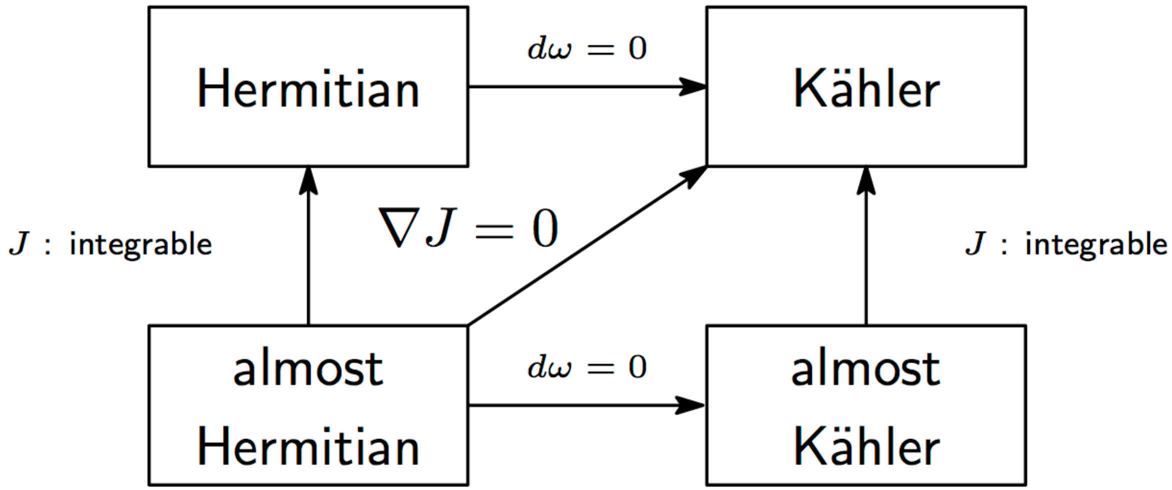
for all $x \in S$ satisfying $d(x,p) \geqslant r_0$, where $d(x,p)$ is the **transverse** distance between x and p. Then (S, g) must be compact. Moreover, the diameter from p satisfies

$$\mathrm{diam}_p(S,g) \leqslant r_0 \exp\left(\frac{\pi}{\nu}\right).$$

Remark This theorem holds both for quasi-regular and **irregular** cases. A future work is to prove the sharpness of this theorem by constructing a complete non-compact Sasaki manifold satisfying the condition as in Theorem with $\nu = 0$.

Recall An almost Hermitian structure (g, J) of an almost Hermitian manifold (M, g, J) is called a **Kähler structure** if

$$\nabla J = 0.$$



Definition An almost contact metric structure (g, ξ, η, Φ) of an almost contact metric manifold (M, g, ξ, η, Φ) is called a **transverse Kähler structure** if

$$\nabla^T \Phi = 0.$$

Proposition The almost contact metric structure (g, ξ, η, Φ) of a Sasaki manifold (S, g) is a transverse Kähler structure.

Transverse Hodge Theory

Let (S, g) be a $(2n+1)$-dimensional **compact** Sasaki manifold.

Definition A real r-form $\alpha \in \Omega^r(S)$ on S is **basic** if

$$i_\xi \alpha = 0 \quad \text{and} \quad \mathcal{L}_\xi \alpha = 0.$$

A real function $f \in \mathcal{C}^\infty(S)$ on S is **basic** if $\xi f = 0$.

Definition

$$\Omega^r_B(S) := \{\text{all basic } r\text{-forms on } S\}, \quad \mathcal{C}^\infty_B(S) := \{\text{all basic functions on } S\},$$
$$d_B := d|_{\Omega^r_B(S)}.$$

The exterior derivative $d : \Omega^r(S) \to \Omega^{r+1}(S)$ preserves basic forms and induces the **basic de Rham complex**

$$0 \longrightarrow \mathcal{C}^\infty_B(S) \xrightarrow{d_B} \Omega^1_B(S) \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Omega^{2n}_B(S) \xrightarrow{d_B} 0.$$

We denote by $H^r_B(S)$ the cohomology group given by the complex above.

The almost complex structure $\Phi|_D$ on D induces the decomposition

$$D \otimes \mathbb{C} = D^{1,0} \oplus D^{0,1},$$

where

$$D^{1,0} := \{X \in D \otimes \mathbb{C} : \Phi(X) = \sqrt{-1}X\} \quad \text{and} \quad D^{0,1} := \overline{D^{1,0}}.$$

Then the set of all basic r-forms splits as

$$\Omega^r_B(S) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Omega^{p,q}_B(S), \quad \Omega^{p,q}_B(S) := \Gamma(\wedge^p (D^{1,0})^* \otimes \wedge^q (D^{0,1})^*).$$

We may define

$$\partial_B : \Omega^{p,q}_B(S) \to \Omega^{p+1,q}_B(S) \quad \text{and} \quad \bar{\partial}_B : \Omega^{p,q}_B(S) \to \Omega^{p,q+1}_B(S)$$

satisfying $d_B = \partial_B + \bar{\partial}_B$. The operators ∂_B and $\bar{\partial}_B$ preserve basic forms and induce the **basic Dolbeault complex**

$$0 \longrightarrow \Omega^{p,0}_B(S) \xrightarrow{\bar{\partial}} \Omega^{p,1}_B(S) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n}_B(S) \xrightarrow{\bar{\partial}} 0.$$

We denote by $H^{p,q}_B(S)$ the cohomology group given by the complex above.

Definition We define a **transverse Hodge star operator** $*_B$ by

$$*_B\alpha := *(\eta \wedge \alpha), \quad \alpha \in \Omega_B^r(S).$$

Definition We define operators δ_B, ϑ_B, and $\bar{\vartheta}_B$ by

$$\delta_B := - *_B \circ d_B \circ *_B, \quad \vartheta_B = - *_B \circ \bar{\partial}_B \circ *_B, \quad \text{and} \quad \bar{\vartheta}_B = - *_B \circ \partial_B \circ *_B,$$

respectively. Put

$$\Delta_B := d_B\delta_B + \delta_B d_B, \quad \Box_B := \partial_B\vartheta_B + \vartheta_B\partial_B, \quad \text{and} \quad \overline{\Box}_B := \bar{\partial}_B\bar{\vartheta}_B + \bar{\vartheta}_B\bar{\partial}_B.$$

Theorem (Boyer, Galicki, and Nakamaye 2003) Let (S, g) be a $(2n+1)$-dimensional compact Sasaki manifold. Then

$$H_B^r(S) \otimes \mathbb{C} = \bigoplus_{p+q=r} H_B^{p,q}(S), \quad H_B^{p,q}(S) \simeq H_B^{n-p,n-q}(S),$$

$$\frac{1}{2}\Delta_B = \Box_B = \overline{\Box}_B.$$

Definition We define a **transverse Ricci form** ρ^T by

$$\rho^T(X,Y) := \mathrm{Ric}^T(\Phi X, Y), \qquad X, Y \in \Gamma(D).$$

We define a **basic first Chern class** by $c_1^B(S) := \left[\frac{1}{2\pi}\rho^T\right]_B$. The basic first Chern class is called to be **positive** or **negative** if $c_1^B(S)$ is represented by a positive or negative basic closed form, respectively.

Definition A Riemannian metric g on a Sasaki manifold is called a **transverse Kähler-Einstein metric** if there exists some constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}^T = \lambda g^T.$$

Definition For a basic function $f \in \mathcal{C}_B^\infty(S)$, we define

- a **transverse gradient vector field** $\nabla^T f$ by

$$g^T(\nabla^T f, X) := d_B f(X), \quad X \in \Gamma(D).$$

- a **transverse Hessian** $\mathrm{Hess}^T f$ by

$$\mathrm{Hess}^T f(X,Y) := g^T(\nabla_X^T \nabla^T f, Y), \quad X, Y \in \Gamma(D).$$

Sasaki-Einstein Manifolds

Let (S, g) be a $(2n+1)$-dimensional Sasaki manifold.

Proposition The following are equivalent :

- (S, g) is Einstein. Then we have $\mathrm{Ric}_g = 2ng$.
- The cone manifold $(C(S), \bar{g})$ of (S, g) is Calabi-Yau, namely, $\mathrm{Ric}_{\bar{g}} = 0$.
- g^T satisfies the **transverse Kähler-Einstein equation**

$$\mathbf{Ric}^{T} = (2n+2)g^{T}.$$

Definition A $(2n+1)$-dimensional Sasaki manifold (S, g) is a **Sasaki-Einstein manifold** if one of the above conditions is satisfied. In this case, $c_1^B(S)$ is positive.

- An **obstruction** to the existence of Sasaki-Einstein metrics.
- A **uniqueness** of Sasaki-Einstein metrics.

Gradient Sasaki-Ricci Solitons

Definition (Futaki, Ono, and Wang 2006) A $(2n+1)$-dimensional compact Sasaki manifold (S, g) is a **gradient Sasaki-Ricci soliton** if

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2) g^T$$

for some **basic** function $f : S \to \mathbb{R}$.

- A **natural generalization** of a Sasaki-Einstein manifold.
- Corresponds to **self-similar solutions** to the **Sasaki-Ricci flow**.

$$\frac{\partial g^T}{\partial t} = -2\,\mathrm{Ric}^T$$

Riemannian Geometry $\longrightarrow$ Sasakian Geometry

Riemannian Geometry		Sasakian Geometry
Einstein Manifold	$\Longrightarrow$	Sasaki Einstein Manifold
trivial $\Uparrow$		trivial $\Uparrow$
Gradient Ricci Soliton	$\Longrightarrow$	**Gradient Sasaki Ricci Soliton**

A Lower Diameter Bound for Compact Gradient Sasaki-Ricci Solitons

A **lower diameter bound** for compact **shrinking** Ricci solitons was studied by

Fernández-López and **García-Río** 2008, **Futaki and Sano 2010**,

Andrews and **Ni** 2011, **Chu** and **Hu** 2011, **Futaki**, **Li**, and **Li** 2011.

Theorem (Fukushima 2014) Let (S, g) be a $(2n+1)$-dimensional **non-trivial** compact gradient Sasaki-Ricci soliton satisfying

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2)g^T.$$

Then the soliton has the diameter bound

$$\mathrm{diam}(M, g) \geqslant \frac{10\pi}{13\sqrt{2n+2}}.$$

Remark Theorem above gives us a **gap phenomenon** between **non-trivial gradient Sasaki-Ricci solitons** and **Sasaki-Einstein manifolds**.

A Lower Diameter Bound for Compact Gradient Sasaki-Ricci Solitons

Theorem (— 2016) Let (S, g) be a $(2n+1)$-dimensional **non-trivial** compact gradient Sasaki-Ricci soliton satisfying

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2) g^T .$$

Then the diameter of (S, g) has the lower bound

$$\mathrm{diam}(S, g) \geqslant \frac{R^T_{\max} - 2n(2n+2)}{2(2n+2)\sqrt{R^T_{\max} - R^T_{\min}}},$$

where $R^T_{\max}$ and $R^T_{\min}$, respectively, denote the maximum and minimum values of the transverse scalar curvature.

Remark This theorem holds both for quasi-regular and **irregular** cases. When the soliton has positive transverse Ricci curvature, we have

$$\mathrm{diam}(S, g) \geqslant \frac{1}{2(2n+2)} \sqrt{R^T_{\max} - R^T_{\min}}.$$

A Myers-Type Compactness Theorem for Complete Gradient Sasaki-Ricci Solitons

Theorem (— 2018) Let (S, g) be a $(2n+1)$-dimensional complete gradient Sasaki-Ricci soliton satisfying

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2) g^T.$$

If $|\nabla f| \leqslant k$ for a non-negative constant $k < n$, then (S, g) must be compact. Moreover, the diameter of (S, g) has the upper bound

$$\mathrm{diam}(M, g) \leqslant \frac{k + \sqrt{k^2 + (n-k)n\pi^2}}{n-k}.$$

Remark Any compact gradient Sasaki-Ricci soliton satisfies

$$|\nabla f| \leqslant \sqrt{R^T_{\max} - R^T_{\min}}.$$

Hence, if $R^T_{\max} - R^T_{\min} < n^2$, then an upper diameter bound for the solitons may be obtained in terms of the range of the transverse scalar curvature.

A Gap Theorem for Gradient Sasaki-Ricci Solitons

Recall A $(2n+1)$-dimensional Sasaki manifold (S, g) is **Sasaki-Einstein** if

$$\mathrm{Ric}^T = (2n+2)g^T.$$

Theorem (— 2014) Let (S, g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2)g^T.$$

Then (S, g) is **Sasaki-Einstein** if and only if

$$|\,\mathrm{Ric}^T - (2n+2)g^T| \leqslant \frac{-n\mathcal{F} + \sqrt{n^2\mathcal{F}^2 + 4n(2n-1)(2n+2)\mathcal{F}}}{2(2n-1)},$$

where $\mathcal{F} := \frac{1}{\mathrm{vol}(S,g)} \int_S |\nabla^T f|^2$ is the Sasaki-Futaki invariant.

Remark This theorem holds both for quasi-regular and **irregular** cases.

Future Work

Riemannian Geometry	$\longrightarrow$	Sasakian Geometry
Einstein Manifold	$\Longrightarrow$	Sasaki Einstein Manifold
trivial $\Uparrow$		trivial $\Uparrow$
Gradient Ricci Soliton	$\Longrightarrow$	**Gradient Sasaki Ricci Soliton**

We **want** to establish

- **Upper diameter bounds** for compact gradient Sasaki-Ricci solitons.
  - **Nitta** 2009 : A Myers type theorem via transverse Ricci curvature.
- **Hitchin-Thorpe inequalities** for compact gradient Sasaki-Ricci solitons.
  - **Boyer** and **Galicki** 2002 : Hitchin-Thorpe inequalities for Sasaki-Einstein mfds.
- **Moduli spaces** of compact gradient Sasaki-Ricci solitons.
  - **Podestà** and **Spiro** 2013 : Moduli spaces of compact gradient Ricci solitons.

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Thank You for Your Attention !

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