Asymptotically Locally Euclidean Kähler manifolds

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Constant Scalar Curvature Metrics in Kähler and Sasaki Geometry
CIRM, Marseille
January 16, 2018

Set-up

- We study spaces (M, g) which generalize the space-time: complete manifolds with ends modeled by $((\mathbb{R}^{2n} \setminus Ball)/\Gamma, g_{Eucl})$ with Γ finite subgroup of $SO(2n)$, called asymptotically locally Euclidean (ALE)
- Require that the manifolds are endowed with a complex structure J such that the metric g is Kähler
- Preferred Kähler metrics:
  - ALE Ricci-flat Kähler if $c_1(K_M) = 0$ in $H^2(M, \mathbb{R})$, or hyperkähler if M simply connected (K_M=trivial line bundle)
  - ALE scalar flat Kähler metrics (scalar curvature $s_g = 0$) in the general case

Questions:

Can we classify the ALE Ricci-flat Kähler manifolds?
Can we classify the ALE scalar flat Kähler manifolds?
How does the classification depend on the complex dimension n?

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Complete Kähler manifolds with prescribed asymptotics

Definition

(M, J, g) is an **Asymptotically Locally Euclidean (ALE)** *Kähler manifold with asymptotics $\mathbb{C}^n/\Gamma_i$, with $\Gamma_i \subset U(n)$ is a finite group acting freely on $\mathbb{C}^{n*}$, if there exists a compact subset $K \subset M$ and for each connected component $U_i \subset M \setminus K$ there is a map $f : U_i \to \mathbb{C}^n/\Gamma_i$ which is a* **diffeomorphism** *between U_i and a subset $\{z \in \mathbb{C}^n/\Gamma_i \mid r(z) > R_i\}$ for some fixed $R_i \geq 0$, such that $f_*(g) - g_0 = O(r^{-2})$ and appropriate decay in the derivatives, where g_0 is the Euclidean metric on $\mathbb{C}^n/\Gamma_i$.*

Remark:

Hein-LeBrun (2016) show that an ALE Kähler manifold has only one end.

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Complex dimension 2: classification of ALE Ricci-flat Kähler surfaces

Theorem (Kronheimer 1989, S. 2012)

Let (M, J, g, ω_g) be a smooth ALE Ricci-flat Kähler surface, asymptotic to $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $U(2)$ acting freely on $\mathbb{C}^2 \setminus \{0\}$. Then

- *the complex manifold (M, J) can be obtained as the minimal resolution of a fiber of* **a one-parameter $\mathbb{Q}-$Gorenstein deformation** *of the quotient singularity $\mathbb{C}^2/\Gamma$,*
- *given the Kähler class $\Omega = [\omega_g] \in H^2(M, \mathbb{R})$, then g is the unique ALE Ricci-flat Kähler metric in this class,*

Any complex surface (M, J) obtained by the above construction admits a unique ALE Ricci-flat Kähler metric in any Kähler class Ω.

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Key ingredients

- The case of simply connected manifolds is due to Kronheimer and his proof is based on the twistor theory of hyperkähler 4-manifolds (holonomy $SU(2)$), and $\Gamma \subset SU(2)$ of type A, D, E.
- On A_{k-1}-surfaces the metric is of the form (Hitchin, Eguchi-Hanson, Gibbons-Hawking):

$$g = \gamma dz d\bar{z} + \gamma^{-1}(\frac{2dy}{y} + \bar{\delta} dz)(\frac{2d\bar{y}}{\bar{y}} + \delta d\bar{z})$$

$$\gamma = \sum_{i=1}^{k}((b-b_i)^2 + |\bar{z}+a_i|^2)^{-\frac{1}{2}}, \ \delta = \sum_{i=1}^{k} \frac{(b-b_i)-\Delta_i}{\Delta_i(\bar{z}+a_i)}$$

  - parameters $(a_i, b_i) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$, $i = 1, \ldots, k$
  - b and Δ_i are defined in terms of (a_i, b_i) and local coordinates y, z.
  - parameters $\{a_i\} \subset \mathbb{C}$ determine the complex on $(M, J) = (xy - \prod(z + \bar{a}_i) = 0)$,
  - the Kähler class is determined by the coefficients $8\pi(b_i - b_{i+1}) \in \mathbb{R}$ on a preferred basis of $H_2(M, \mathbb{R})$.

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- The metrics are explicit, constructed by Eguchi-Hanson, Gibbons-Hawking, Hitchin and Kronheimer, in the simply connected case.
- The hyperkähler parameters $(g, J_1, J_2, J_3 = J_1 J_2)$ can be reinterpreted in terms of Kähler data (J, ω_g).
- The non simply connected case is proved via an equivariant twistor construction and uses a one point conformal compactification at infinity.
- In the non-simply connected case we have in addition only free quotients of A_*-surfaces with asymptotics given by

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Remarks:

- Given an asymptotic behavior the diffeotype of the ALE Ricci flat Kähler manifold is unique.
- the order of the decay of the metric is $O(r^{-4})$.
- If Γ cyclic, in each Kähler class there exists also an Asymptotically Locally Flat (ALF) Ricci-flat Kähler metric, which has cubic volume growth. (Hawking, Cherkis-Hitchin, S.)

$\mathbb{Q}$-Gorenstein smoothings of quotient singularities

Definition

A flat surjective map $\pi : \mathcal{X} \to \Delta$, *where* $\Delta \subset \mathbb{C}$ *is an open neighborhood of* 0, *is called a* **one-parameter** $\mathbb{Q}-$**Gorenstein smoothing** *of a normal variety* X_0 *if* $\pi^{-1}(0) = X_0$ *and the following conditions are satisfied:*

i) $\mathcal{X}$ *is* $\mathbb{Q}-$*Gorenstein,*

ii) $X_t = \pi^{-1}(t)$ *is smooth for every* $t \in \Delta \setminus \{0\}$.

$\mathbb{Q}-$Gorenstein: $\mathcal{X}$ is normal, Cohen-Macaulay and a multiple of the canonical divisor is Cartier.

Kollár and Shepherd-Barron's classification

Theorem (Kollár and Shepherd-Barron, 1988)

An isolated quotient surface singularity $(N_0 = \mathbb{C}^2/\Gamma, 0)$ *which admits a one-parameter* $\mathbb{Q}-$*Gorenstein smoothing is either a rational double point or a cyclic singularity of type* $\mathbb{C}^2/\frac{1}{dn^2}(1, dnm-1)$ *for* $d > 0, n \geq 2$ *and* n, m *relatively prime.*

Rational double points

- Rational double points correspond to the case when $\Gamma \subset SU(2)$. They are singularities of type A_k, D_k or E_6, E_7, E_8. All of them are isolated hypersurface singularities $(f(x,y,z) = 0)$, where $f \in \mathbb{C}[x,y,z]$.
- The total space $\mathcal{X} \to \Delta$ is smooth.
- The minimal resolution and the generic fiber of a deformation are diffeomorphic, but endowed with distinct complex structures. In particular the deformation X_t is simply connected.
- A complex complex structure on these spaces is given by a mixed construction: a $\mathbb{Q}$-Gorenstein deformation, possibly with isolated rational double points which can be resolve by considering the minimal resolutions. The singular fiber is the canonical model of the manifold.

Cyclic quotients of type $\mathbb{C}^2/\frac{1}{dn^2}(1, dnm-1)$

Remark: If $n = m = 1$ then this is an A_{d-1} singularity.

$$0 \to \mathbb{Z}_{dn} \to \mathbb{Z}_{dn^2} \to \mathbb{Z}_n \to 0 \text{ with } \mathbb{Z}_{dn} \subset SU(2)$$

$$\begin{array}{ccc} \mathbb{C}^2/\frac{1}{dn}(1,-1) & \hookrightarrow & M_0 = (xy = z^{dn}) \subset \mathbb{C}^3 \\ (z_1, z_2) & \to & (x,y,z) = (z_1^{dn}, z_2^{dn}, z_1 z_2) \in \mathbb{C}^3 \end{array}$$

Moduli space of deformations:

$$\begin{array}{ccl} \mathbb{C}^2/\mathbb{Z}_{dn} & \subset \mathcal{M} & = (xy = z^{dn} + e_1 z^{dn-1} + \cdots + e_{dn}) \subset \mathbb{C}^{3+dn} \\ \downarrow \pi & \downarrow \pi & \\ 0 & \in \mathbb{C}^{dn} & \ni (e_1, \ldots, e_{dn}) \end{array}$$

Then, $N_0 = M_0/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on $\mathbb{C}^3$ as follows:

$$\xi(x,y,z) = (\xi x, \xi^{-1} y, \xi^m z),\ \xi^n = 1.$$

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Let $\mathcal{M}' \subset \mathbb{C}^3 \times \mathbb{C}^d$, $\mathcal{M}' = (xy = z^{dn} + \sum_{j=1}^{d} e_{jn} z^{(d-j)n})$.

We extend trivially the $\mathbb{Z}_n$ action on $\mathcal{M}' \subset \mathbb{C}^{3+d}$. Let $\mathcal{N} = \mathcal{M}'/\mathbb{Z}_n$.

$$\textit{Then} \quad \begin{array}{ccc} \mathbb{C}^2/\mathbb{Z}_{dn^2} & \subset & \mathcal{N} \\ \downarrow\phi & & \downarrow\phi \\ 0 & \in & \mathbb{C}^d \quad \ni (e_n, \ldots, e_{dn}) \end{array}$$

Proposition (Kollár, Shepherd-Barron)

The map $\phi : \mathcal{N} \to \mathbb{C}^d$ is a $\mathbb{Q}-$Gorenstein deformation of the cyclic singularity of type $\frac{1}{dn^2}(1, dnm - 1)$. Moreover, every one-parameter $\mathbb{Q}-$Gorenstein deformation $\mathcal{X} \to \Delta$ of a singularity of type $\frac{1}{dn^2}(1, dnm - 1)$ is isomorphic to the pullback through ϕ of a germ of a holomorphic map $(\Delta, 0) \to (\mathbb{C}^d, 0)$.

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A complex analysis approach

Problem (Yau, ICM 1978)

Can the complete Ricci-flat Kähler metric be obtained via complex Monge-Ampère techniques?

Theorem (Tian-Yau, 1990, 2-dimensional simplified version)

Let $\overline{X}$ be a compact Kähler orbifold of complex dimension 2. Let D be an admissible, almost ample divisor in $\overline{X}$, such that $-K_{\overline{X}} = \beta D, \quad \beta > 1$. Assume that D admits a Kähler-Einstein metric with positive scalar curvature. Then $X = \overline{X} \setminus D$ admits a complete Ricci-flat Kähler metric g, which has Euclidean volume growth.

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Compactifications

Preferred compactification induced by the Kähler structure:

Theorem (Rasdeaconu-S. 2014)

Let (M, J, g) be an ALE Ricci-flat Kähler surface. Then there exists a complex compactification $(\overline{M}, \bar{J}, D)$, where $\overline{M}$ is an orbifold surface and $D = \overline{M} \setminus M$ is the divisor at infinity, such that D is admissible, almost ample, admits a Kähler-Einstein metric and $-K_{\overline{M}} = \beta[D], \beta > 1$. In particular, any ALE Ricci-flat Kähler metric g can be obtained as a Tian-Yau metric.

Corollary (Rasdeaconu-S. 2014)

The explicit ALE Ricci-flat Kähler metrics on complex surfaces constructed by Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer, and their finite free quotients can be obtained by the Tian-Yau construction.

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Other algebraic compactifications:

Theorem (Rasdeaconu-S. 2014)

The fiber of a one-parameter $\mathbb{Q}-$Gorenstein deformation of a quotient singularity embeds into a log del Pezzo surface as the complement of a smooth, rational curve, which is a rational multiple of the anticanonical divisor. The singularities along the divisor at infinity are all finite cyclic quotients. In the case of a finite cyclic singularity there are infinitely many minimal compactifications.

Example:

$$N = (xy = z^{dn} + 1)/\mathbb{Z}_n \subseteq \mathbb{C}^3/\frac{1}{n}(1, -1, m)$$

admits a compactification as a hypersurface in a weighted projective space:

$$\overline{N} = (xy = z^{dn} + w^{dc}) \subseteq \mathbb{P}^3(a, b, c, n)$$

where $a + b = dnc$, $am = c \mod n$, and $\gcd(c, n) = \gcd(a, c) = 1$.

- Infinitely many compactifications, parametrized by c.

Other algebraic compactifications:

Theorem (Rasdeaconu-S. 2014)

The fiber of a one-parameter $\mathbb{Q}-$Gorenstein deformation of a quotient singularity embeds into a log del Pezzo surface as the complement of a smooth, rational curve, which is a rational multiple of the anticanonical divisor. The singularities along the divisor at infinity are all finite cyclic quotients. In the case of a finite cyclic singularity there are infinitely many minimal compactifications.

Example:

$$N = (xy = z^{dn} + 1)/\mathbb{Z}_n \subseteq \mathbb{C}^3/\frac{1}{n}(1, -1, m)$$

admits a compactification as a hypersurface in a weighted projective space:

$$\overline{N} = (xy = z^{dn} + w^{dc}) \subseteq \mathbb{P}^3(a, b, c, n)$$

where $a + b = dnc, \ am = c \mod n$, and $\gcd(c, n) = \gcd(a, c) = 1$.

- Infinitely many compactifications, parametrized by c.

Using analytical compactifications, Conlon-Hein gave a new proof of the Kronheimer's classification.

ALE Ricci flat Kähler manifolds of dimension $n \geq 3$

Given an ALE Ricci flat Kähler manifolds (M, J, g) of dimension $dim_{\mathbb{C}} M = n \geq 3$, then:

- (M, J) is a crepant resolution of $\mathbb{C}^n/\Gamma$ with $\Gamma \subset SU(n)$ acting freely on $\mathbb{C}^n \setminus \{0\}$, consequence of Schlessinger's Rigidity Theorem and K_M =trivial
- on (M, J) there exists a unique ALE Ricci flat Kähler metric in each Kähler class, by Joyce.
- metrics also constructed by Calabi, Tian-Yau

ALE scalar flat Kähler surfaces ($n = 2$)

Existence known on:

- minimal resolution of $\mathbb{C}^2/\Gamma, \Gamma \subset U(2)$, by LeBrun, Joyce, Calderbank-Singer if Γ =cyclic group, Lock-Viaclovsky if Γ non-cyclic
- deformations of the minimal resolution by Honda (using twistor spaces) and Han-Viaclovsky

Deformations of $\mathbb{C}^2/\Gamma$, examples:

- The reduced deformation space of the singularities of the type $\frac{1}{n^2}(1, n-1),\ n \geq 2$ or $\frac{1}{4d}(1, 2d-1),\ d \geq 1$ has exactly two components, the $\mathbb{Q}-$Gorenstein component and the component corresponding to the (minimal) resolution.
- For the singularity $\frac{1}{(2k+1)^2}(1, 4k+1),\ k \geq 2$, in addition to the Artin and the $\mathbb{Q}-$Gorenstein components, the reduced deformation space has exactly one more component.

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ALE Kähler manifolds

General setting of ALE Kähler manifolds.

Open Questions:

Can we classify the ALE Kähler manifolds?

- *What is the underlying complex structure?*
- *Do they admit ALE scalar flat Kähler metrics?*
- *Are the metrics unique in a given Kähler class?*

A characterization of ALE Kähler manifolds

Theorem (Hein-Rasdeaconu-S. 2016)

Every ALE Kähler manifold asymptotic to $\mathbb{C}^n/\Gamma$ is isomorphic to a resolution of a deformation of the isolated quotient singularity $(\mathbb{C}^n/\Gamma, 0)$.

Corollary (Rigidity in higher dimensions, Hein-Rasdeaconu-S. 2016)

Every ALE Kähler manifold asymptotic to $\mathbb{C}^n/\Gamma, n \geq 3$, is the resolution of the quotient $\mathbb{C}^n/\Gamma$.

Consequence of the Schlessinger Rigidity Theorem.

Corollary (Finiteness in dimension two, Hein-Rasdeaconu-S. 2016)

For every finite subgroup $\Gamma \subset U(2)$ there exist only finitely many diffeomorphism types underlying **minimal** *ALE Kähler surfaces which are asymptotic to $\mathbb{C}^2/\Gamma$. All of them have finite fundamental group.*

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Main steps to prove the theorem:

- Hein-LeBrun: use the asymptotic behavior to compactify (M, J) to $(\overline{M}, J)$ with a divisor at infinity $D = D_\infty = \mathbb{P}(\mathbb{C}^n/\Gamma)$.
- the normal orbi-bundle $N_{D|\overline{M}} = \mathcal{O}_D(D)$ is ample and its ring of sections $R(D, \mathcal{O}_D(D))$ is isomorphic to the coordinate ring of the isolated quotient singularity $\mathbb{C}^n/\Gamma$.
- the $\mathbb{Q}$-Cartier divisor $\mathcal{O}_{\overline{M}}(D)$ is pseudo-ample
- $\overline{M}$ is projective variety, as the singularities of $\overline{M}$ are rational singularities

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- Consider $\phi : \overline{M} \to \mathbb{CP}^N$ the morphism defined by the linear system $|mD|$ for m large enough, and let $M' = \phi(\overline{M})$ and $D' = \phi(D)$.

Proposition

The complex variety M' is normal and the map ϕ is an isomorphism in a neighborhood of D. In particular, no ϕ-exceptional divisor intersects D.

Proposition

There exists a one-parameted family $\pi : \mathcal{M} \subset \mathbb{CP}^{N+1} \to \mathbb{C}$ such that $\pi^{-1}(1) = M'$ and $\pi^{-1}(0) = C_{D'}$ = cone over D', given by the sweeping of the cone construction.

Key ingredient: $R(M', \mathcal{O}(D')) \simeq R(D', \mathcal{O}_{D'}(D'))[S]$, where S is a degree 1 homogeneous element of $R(M', \mathcal{O}(D'))$.

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Thank you!