Kähler metrics associated with Lorentzian metrics in dimension four

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Optical invariants (Sachs)

(M, g) Lorentzian manifold.

If X is a null vector field, $X \in \Gamma(X^\perp)$ and $\nabla X : \Gamma(X^\perp) \to \Gamma(X^\perp)$.

If X is also pre/geodesic ($\nabla_X X = \alpha X$),

$$DX : X^\perp/\langle X\rangle \to X^\perp/\langle X\rangle, \qquad DX([v]) := [\nabla_v X], \quad v \in X^\perp$$

is well-defined.

shear operator $D^o X :=$ trace-free symmetric part of DX.

twist operator $D^s X :=$ antisymmetric part of DX.

Relative (distribution-dependent) optical invariants

$$TM = \mathcal{V} \oplus \mathcal{H}, \qquad \mathcal{H} := \mathcal{V}^{\perp}, \qquad g\big|_{\mathcal{V}} \text{ non-degenerate.}$$

$\pi_{\mathcal{H}} : TM \to \mathcal{H}$ orthogonal projection.

$$X \in \Gamma(\mathcal{V}), \qquad \pi_{\mathcal{H}} \circ \nabla X|_{\mathcal{H}} : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$$

shear operator $\nabla^{o} X :=$ trace-free symmetric part of $\pi_{\mathcal{H}} \circ \nabla X|_{\mathcal{H}}$.

twist operator $\nabla^{s} X :=$ antisymmetric part of $\pi_{\mathcal{H}} \circ \nabla X|_{\mathcal{H}}$.

Frame representation of shear and twist

$\dim M = 4$, $\mathrm{rk}\ \mathcal{H} = 2$.

Ordered orthonormal frame $\mathbf{x}$, $\mathbf{y}$ for $\mathcal{H}$.

Shear operator:

$$[\nabla^o X]_{\mathbf{x},\mathbf{y}} = \begin{bmatrix} -\sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix},$$

with shear coefficients:

$$\begin{aligned} 2\sigma_1 &:= g(\nabla_{\mathbf{y}} X, \mathbf{y}) - g(\nabla_{\mathbf{x}} X, \mathbf{x}) = & g([X,\mathbf{x}],\mathbf{x}) - g([X,\mathbf{y}],\mathbf{y}), \\ 2\sigma_2 &:= g(\nabla_{\mathbf{y}} X, \mathbf{x}) + g(\nabla_{\mathbf{x}} X, \mathbf{y}) = - & g([X,\mathbf{x}],\mathbf{y}) - g([X,\mathbf{y}],\mathbf{x}). \end{aligned}$$

Twist operator:

$$[\nabla^s X]_{\mathbf{x},\mathbf{y}} = \begin{bmatrix} 0 & \iota/2 \\ -\iota/2 & 0 \end{bmatrix},$$

with twist function:

$$\iota := g(\nabla_{\mathbf{y}} X, \mathbf{x}) - g(\nabla_{\mathbf{x}} X, \mathbf{y}) = g(X, [\mathbf{x}, \mathbf{y}]).$$

Set-up:

(M, g) oriented Lorentzian manifold, $\dim M = 4$.

$\boldsymbol{k}_+$, $\boldsymbol{k}_-$ two pointwise linearly independent vector fields.

$$\mathcal{V} = \bigsqcup_{p \in M} V_p; \quad V_p := \mathrm{span}(\boldsymbol{k}_+|_p, \boldsymbol{k}_-|_p) \text{ is timelike:}$$

$$g\big|_{\mathcal{V}} \text{ has Lorentzian signature.}$$

$$G := \det \begin{bmatrix} g(\boldsymbol{k}_+, \boldsymbol{k}_+) & g(\boldsymbol{k}_+, \boldsymbol{k}_-) \\ g(\boldsymbol{k}_-, \boldsymbol{k}_+) & g(\boldsymbol{k}_-, \boldsymbol{k}_-) \end{bmatrix} \neq 0$$

The almost complex structure J determined by $(M, g, \boldsymbol{k}_\pm)$:

$$\textit{On } \mathcal{V}: \quad \textit{extends } J\boldsymbol{k}_+ = \boldsymbol{k}_-, \qquad J\boldsymbol{k}_- = -\boldsymbol{k}_+,$$

On $\mathcal{H} := \mathcal{V}^\perp$: $J|_\mathcal{H}$: makes $g|_\mathcal{H}$ hermitian; is orientation-compatible.

Theorem 1 (Aazami-M)

The almost complex structure J is integrable if,

$$\text{i) } [\boldsymbol{k}_\pm, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}) \quad \text{and} \quad \text{ii) } J\nabla^o \boldsymbol{k}_+ = \nabla^o J\boldsymbol{k}_+ \textit{ on } \mathcal{H}. \quad (1)$$

Definition

Given $TM = \mathcal{H} \oplus \mathcal{V}$, an almost complex structure J is called *split-adjoint* if $J\big|_{\mathcal{H}}$ is $g\big|_{\mathcal{H}}$-skew-adjoint and $J\big|_{\mathcal{V}}$ is $g\big|_{\mathcal{V}}$-self-adjoint.

Examples: 1) if $\boldsymbol{k}_\pm$ are null; 2) $\boldsymbol{k}_\pm$ orthogonal with $g(\boldsymbol{k}_+, \boldsymbol{k}_+) = -g(\boldsymbol{k}_-, \boldsymbol{k}_-)$.

Theorem 2 (Aazami-M)

A split-adjoint almost complex structure J is integrable if and only if

$$\text{i) } g([\boldsymbol{k}_-, J\mathbf{x}], \boldsymbol{k}_+) - g([\boldsymbol{k}_+, J\mathbf{x}], \boldsymbol{k}_-) - g([\boldsymbol{k}_+, \mathbf{x}], \boldsymbol{k}_+) - g([\boldsymbol{k}_-, \mathbf{x}], \boldsymbol{k}_-) = 0, \quad \forall \mathbf{x} \in \Gamma(\mathcal{H}) \tag{2}$$

$$\text{and} \quad \text{ii) } J\nabla^o \boldsymbol{k}_+ = \nabla^o J\boldsymbol{k}_+ \text{ on } \mathcal{H}.$$

Flaherty gave a Newman-Penrose version of the theorem for case 1) above.

Geometric conditions guaranteeing (i) of the theorems:

$$g([\boldsymbol{k}_\pm, \Gamma(\mathcal{H})], \boldsymbol{k}_\pm) = 0 \quad \textit{if} \quad \begin{array}{l} \boldsymbol{k}_\pm \text{ is a geodesic of constant length, } \textit{or} \\ \boldsymbol{k}_\pm \text{ is a null pre-geodesic, } \textit{or} \\ \boldsymbol{k}_\pm \text{ is Killing.} \end{array}$$

$$g([\boldsymbol{k}_+, \Gamma(\mathcal{H})], \boldsymbol{k}_-) = 0 \quad \textit{if} \quad \begin{array}{l} \boldsymbol{k}_- = \ell \nabla \tau,\ d_{\mathcal{H}}\ell = 0, \textit{ and} \\ g(\boldsymbol{k}_+, \boldsymbol{k}_-) \text{ is constant.} \end{array} \qquad (*)$$

$$g([\boldsymbol{k}_-, \Gamma(\mathcal{H})], \boldsymbol{k}_+) = 0 \quad \textit{if} \quad g(\nabla_{\boldsymbol{k}_+}\boldsymbol{k}_- + \nabla_{\boldsymbol{k}_-}\boldsymbol{k}_+, \Gamma(\mathcal{H})) = 0. \quad (**)$$

geodesic: $\nabla_X X = 0$, pre-geodesic: $\nabla_X X = \alpha X$, $\alpha \neq 0$,

Killing: $\mathcal{L}_X g = 0$.

$(*)$ is called the near-gradient condition

$(**)$ is called the mixed condition for $\mathcal{V}$.

Definition

An oriented Lorentzian 4-manifold with two everywhere linearly independent vector fields $\boldsymbol{k} = \boldsymbol{k}_+$, $\boldsymbol{t} = \boldsymbol{k}_-$ and associated distribution $\mathcal{V}$ is called *pre/geodesic-admissible* (*Killing-admissible*) if

- $\mathcal{V}$ is timelike and mixed;
- $\boldsymbol{k}$ is a pre/geodesic vector field of constant length ($\boldsymbol{k}$ is Killing);
- $\boldsymbol{t}$ is a pre/geodesic vector field of constant length;
- $\boldsymbol{t}$ is near-gradient: $\boldsymbol{t} = \ell \nabla \tau$, where $d_{\mathcal{H}} \ell = 0$;
- $g(\boldsymbol{k}, \boldsymbol{t})$ is constant;
- $J \nabla^o \boldsymbol{k} = \nabla^o \boldsymbol{t}$ for the associated almost complex structure J.

Corollary

The almost complex structure associated to any admissible manifold is integrable.

symplectic forms for admissible manifolds

On an admissible manifold, The exact 2-form

$$\omega = d[f(\tau)\boldsymbol{k}^\flat], \qquad \boldsymbol{k}^\flat = g(\boldsymbol{k}, \cdot), \qquad \text{smooth } f : \mathrm{Im}(\tau) \to \mathbb{R},$$

is symplectic in the region where

$$\begin{cases} ff'\iota \neq 0 \text{ if } g \text{ is geodesic-admissible, or} \\ f(-f'G/\ell + f\alpha g(\boldsymbol{k}, \boldsymbol{t}))\iota \neq 0 \text{ if } g \text{ is pre-geodesic-admissible, or} \\ f(f'G/\ell + fd_{\boldsymbol{t}}(g(\boldsymbol{k}, \boldsymbol{k})))\iota \neq 0 \text{ if } g \text{ is Killing-admissible and } \boldsymbol{k} \text{ is proper.} \end{cases}$$

Here:

$\iota := \iota^{\boldsymbol{k}}$; $\boldsymbol{k}$ is proper if $\nabla(g(\boldsymbol{k}, \boldsymbol{k})) \in \Gamma(\mathcal{V})$;

α - pre-geodesic coeff.: $\nabla_{\boldsymbol{k}}\boldsymbol{k} = \alpha\boldsymbol{k}$.

Aazami considered earlier special cases.

Theorem 3 (Aazami-M)

Let (M, g) be an oriented Lorentzian 4-manifold with two pointwise linear independent vector fields $\boldsymbol{k}$, $\boldsymbol{t}$ such that $\boldsymbol{k}$ is proper, $\boldsymbol{t}$ is near-gradient, $g(\boldsymbol{k}, \boldsymbol{t})$ is constant, and their pointwise span is timelike. If the integrability conditions for the associated almost complex structure J are satisfied, then for any function f,

$$g_K := d(f(\tau)\boldsymbol{k}^\flat)(\cdot, J\cdot)$$

is a Kähler metric in the region where

$$f\iota < 0 \text{ and } f'G/\ell - f(g(\nabla_{\boldsymbol{k}}\boldsymbol{k}, t) - d_{\boldsymbol{t}}(g(\boldsymbol{k}, \boldsymbol{k}))/2) < 0,$$

where $\iota := g(\nabla_{J\mathbf{x}}\boldsymbol{k}, \mathbf{x}) - g(\nabla_{\mathbf{x}}\boldsymbol{k}, J\mathbf{x}) = g(\boldsymbol{k}, [\mathbf{x}, J\mathbf{x}])$, for a (local) unit vector field $\mathbf{x} \in \Gamma(\mathcal{H})$.

If (M, g) is an admissible Lorentzian 4-manifold, the second inequality simplifies to

$$\begin{cases} f' G/\ell < 0 \text{ if } g \text{ is geodesic-admissible, or} \\ f' G/\ell - f\alpha g(\boldsymbol{k}, \boldsymbol{t}) < 0 \text{ if } g \text{ is pre-geodesic-admissible, or} \\ f' G/\ell + f d_{\boldsymbol{t}}(g(\boldsymbol{k}, \boldsymbol{k})) < 0 \text{ if } g \text{ is Killing-admissible,} \end{cases}$$

and properness of $\boldsymbol{k}$ is only needed in the Killing-admissible case.

Basic properties of g_K:

For a Kähler metric induced from an admissible Lorentzian metric,

$$g_K(\mathcal{V},\mathcal{H}) = 0, \qquad g_K(\boldsymbol{k},\boldsymbol{t}) = 0, \qquad g_K(\boldsymbol{k},\boldsymbol{k}) = g_K(\boldsymbol{t},\boldsymbol{t}),$$
$$g_K\big|_{\mathcal{H}} = -f\iota g\big|_{\mathcal{H}}.$$

Hence: if $g(\boldsymbol{k},\boldsymbol{k}) = -g(\boldsymbol{t},\boldsymbol{t})$ and $g(\boldsymbol{k},\boldsymbol{t}) = 0$, then g_K is obtained from g via a biconformal change composed on a Wick rotation:

$$g_K\big|_{\mathcal{H}} = -f\iota g_{wick}\big|_{\mathcal{H}},$$
$$g_K\big|_{\mathcal{V}} = \beta g_{wick}\big|_{\mathcal{V}},$$

$$g_{wick}(\boldsymbol{t},\boldsymbol{t}) := -g(\boldsymbol{t},\boldsymbol{t}),$$
$$g_{wick}(a,b) := g(a,b), \qquad a,b \in \Gamma(\boldsymbol{t}^{\perp}).$$

"Admissibility" properties of g_K:

If $(M, g, \boldsymbol{k}, \boldsymbol{t})$ is geodesic-admissible with $\boldsymbol{t}$ geodesic, $f = \tau$, $\ell = 1$, then $\boldsymbol{k}$,$\boldsymbol{t}$ are also g_K-geodesic of constant g_K-length.

If $(M, g, \boldsymbol{k}, \boldsymbol{t})$ is Killing-admissible with $\iota = \iota(\tau)$, $\ell = \ell(\tau)$, $[\boldsymbol{k}, \boldsymbol{t}] = 0$, $g(\boldsymbol{k}, \boldsymbol{t}) = 0$ and $\boldsymbol{k}$ proper. Then $\boldsymbol{k}$ is also g_K-Killing.

Relations between shears of an admissible g and g_K:

$$\nabla^{o,K}\boldsymbol{k} = \nabla^{o}\boldsymbol{k}, \qquad \nabla^{o,K}\boldsymbol{t} = \nabla^{o}\boldsymbol{t}.$$

In particular, the shear condition holds for g_K:

$$J\nabla^{o,K}\boldsymbol{k} = \nabla^{o,K}\boldsymbol{t}.$$

If g is admissible, $\mathcal{V}$ is g_K-mixed and $g_K(\boldsymbol{k}, \boldsymbol{t}) = 0$.

Further properties

If g is geodesic-admissible with g_K complete, then inextendible integral curves of $\boldsymbol{k}$ are defined on $\mathbb{R}$ if and only if f'/ℓ is bounded above. The same holds for $\boldsymbol{t}$ if it is geodesic.

This uses observations of Candela-Sánchez.

Assume (M, g) is a geodesic-admissible, $g(\boldsymbol{k}, \boldsymbol{k}) = -g(\boldsymbol{t}, \boldsymbol{t})$, $\ell = 1$, $f(\tau) = e^\tau$ and $[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\boldsymbol{t}^\perp)$ holds. Then g_K is isometric to a cone metric over a Sasaki 3-manifold.

Ricci form of g_K in special cases

Let $\pi : L \to N$ be a holomorphic line bundle over a Riemann surface with Kähler metric h. Suppose g a is geodesic-admissible metric on L, and $g\big|_{\mathcal{H}} = \pi^* h$. If $\boldsymbol{k}$, $\boldsymbol{t}$ are holomorphic, ℓ is a function of τ, $\boldsymbol{k}$ is not null and $\boldsymbol{k}^\flat$ is a connection one-form, then the Ricci form of g_K is given by

$$\rho_K = \frac{1}{2} dJd \log(ff' r\imath/\ell),$$

where h has Kähler form $r\, dx \wedge dy$ for a holomorphic coordinate $z = x + iy$ on N.

Examples

geodesic-admissible:

1] $M = P \times \mathbb{R}$ for certain Riemannian 3-manifolds P admitting a unit Killing field (e.g. $P = S^3$).

2] Gravitational plane-wave.

pre-geodesic-admissible:

3] Warped product generalizations of 1], e.g. de Sitter spacetime, or with P defined via a level set in a pp-wave.

4] A metric conformal to the Kerr metric.

Killing-admissible:

5] Metrics for which g_K is an SKR metric.

Related construction:

6] Some Petrov type D metrics: Kerr, NUT.

7] Lie group with solvable Lie algebra, non-shear-free example.

SKR metrics

A *Killing potential* τ on a Kähler manifold (M, J, g_K) is a smooth function τ such that $J\nabla^K\tau$ is a Killing vector field.

Set

$$v := \nabla^K\tau, \quad u := Jv, \quad \mathcal{V} := \mathrm{span}(v, u), \quad \mathcal{H} := \mathcal{V}^{\perp}.$$

$g_K = g_{SKR}$ is called an *SKR metric*, if τ is nonconstant, and at each regular point of τ, the nonzero tangent vectors in $\mathcal{H}$ are eigenvectors of both the Ricci endomorphism and the Hessian of τ.

SKR metrics include many Kähler conformally Einstein metrics in dimension four, such as the extremal metric conformal to Page's Einstein metric (and all of those, in higher dimensions).

SKR defining equation:

$$\alpha(\tau)\mathrm{Ric}_{SKR} + \nabla^{SKR}d\tau = \beta(\tau)g_{SKR}$$

Local classification of (irreducible) SKR metrics:

Theorem (Derdzinski-M)

every regular point of τ has a neighborhood U biholomorphically isometric to an open set in a holomorphic line bundle $\mathcal{L} \to (N, h)$ with h Kähler, equipped with a metric

$$g_{SKR} = \frac{1}{Q} d\tau^2 + Q\hat{u}^2 + 2|\tau_c|\pi^* h. \tag{3}$$

where $\tau_c = \tau - c$, c constant; Q is a positive function of τ; $\hat{u}$ a 1-form dual to u.

Known relations for SKR metrics with w, w' denoting horizontal lifts of vector fields on N:

$i)\, g_{SKR}(u, v) = 0,$ $ii)\, Q > 0$ if $v \neq 0$ or $u \neq 0,$ $iii)\, [u, v] = 0,$
$iv)\, [v, w] = 0,$ $v)\, [u, w] = 0,$ $vi)\, [w, w']^{\mathcal{V}} = -2\pi^* \omega^h(w, w')u.$

Ansatz:

Given: irreducible SKR metric g_{SKR} as in (3).

Set $\boldsymbol{k} := u$, $\boldsymbol{t} := -v$.

Set $g(\boldsymbol{k}, \boldsymbol{t}) := 0$, $g(\boldsymbol{t}, \boldsymbol{t}) := q < 0$, q constant.

Choose a function p of a variable τ such that $p > 0$ on $\{\tau > c\}$.

Set $g(\boldsymbol{k}, \boldsymbol{k}) := p$, with p now abusively denoting $p \circ \tau$.

Define $g\big|_{\mathcal{V}}$ by linear extension.

Define $g\big|_{\mathcal{H}} := \pi^* h$.

Declare $g(\mathcal{V}, \mathcal{H}) = 0$.

Theorem 4 (Aazami-M)

Let g_{SKR} be an irreducible SKR metric in dimension four. Then there exists a Killing-admissible Lorentzian metric g on the intersection $U = \{d\tau \neq 0\} \cap \{\tau > c\}$, isometric to that in the above ansatz, for which $\boldsymbol{t} = \ell \nabla \tau$ for $\ell := -q/Q$. If p is a positive constant, then g is also geodesic-admissible.

The metric g, along with the choice $f := \tau_c/p$ gives rise to an associated Kähler metric g_K such that

$$g_K = g_{SKR} \text{ on } U.$$

If g_{SKR} is defined on a compact manifold which is not $\mathbb{C}P^2$, and sign of τ is chosen appropriately, then the metric g is in fact defined on the open dense set $U = \{d\tau \neq 0\}$.

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Kähler metrics for Petrov Type D

A Petrov type D metric has a special structure of eigenbivectors of the Weyl tensor, corresponding to two principal null directions, which give rise to two null shear-free geodesic vector fields $\boldsymbol{k}_+$, $\boldsymbol{k}_-$.

Examples: Schwarzchild metric, Kerr metric, NUT metrics.

For the Schwarzchild metric $\boldsymbol{k}_\pm$ have zero twist, so it does not give rise to the above Kähler metrics.

For the other two types, $\boldsymbol{k}_\pm$ are not near-gradient, hence a different type of Kähler metric is needed for them.

Conditions for a Kähler metric

Proposition

Let g be a Petrov type D metric on an oriented 4-manifold, with null shear-free vector fields $\boldsymbol{k}_\pm$ spanning $\mathcal{V}$, with $\boldsymbol{k}_+$ geodesic, $\boldsymbol{k}_-$ pre/geodesic and $g(\boldsymbol{k}_+, \boldsymbol{k}_-) < 0$. Assume the Theorem 1's integrability conditions hold for $\boldsymbol{k}_\pm$. Set $p := 1/\sqrt{-g(\boldsymbol{k}_+, \boldsymbol{k}_-)}$.

Suppose u is a smooth function on M and f a smooth positive *function defined on the range of u, such that $\nabla(f(u)/p) \in \Gamma(\mathcal{V})$.*

If J is the complex structure determined by $\boldsymbol{k}_\pm$, then

$$g_K = d(f(u)p\boldsymbol{k}_+^\flat)(\cdot, J\cdot)$$

is Kähler in any region where $\iota^{\boldsymbol{k}_+} < 0$ and $d_{\boldsymbol{k}_+}(\log(f(u)p)) < 0$, with $\iota^{\boldsymbol{k}_+}$ computed as before.

Kerr metric

The Kerr metric is defined in an open subset of $M := \mathbb{R}^2 \times \mathbb{S}^2$. In coordinates $\{t, r, \vartheta, \varphi\}$, with $0 < \vartheta < \pi$ and $0 \le \varphi \le 2\pi$. In components

$$g_{tt} = -1 + \frac{2mr}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\vartheta\vartheta} = \rho^2,$$

$$g_{\varphi\varphi} = \left[r^2 + a^2 + \frac{2mra^2 \sin^2\vartheta}{\rho^2} \right] \sin^2\vartheta, \quad g_{\varphi t} = g_{t\varphi} = -\frac{2mra \sin^2\vartheta}{\rho^2},$$

a, m positive parameters

$$\rho^2 := r^2 + a^2 \cos^2\vartheta, \quad \Delta := r^2 - 2mr + a^2.$$

Rapidly rotating case: $a > m$ with only a naked singularity at $r = 0$.

$$\boldsymbol{k}_\pm := \pm\partial_r + \frac{r^2+a^2}{\Delta}\,\partial_t + \frac{a}{\Delta}\,\partial_\varphi,$$

$$g(\boldsymbol{k}_+, \boldsymbol{k}_-) = -2\rho^2/\Delta < 0, \quad p := \sqrt{\frac{\Delta}{2}\frac{1}{\rho}},$$

$$\iota = \frac{2a\cos\vartheta}{\rho^2}.$$

Let $u = e^{h(r)}p, \quad h(r) = \ln\frac{\sqrt{r^2+a^2+1}}{\Delta}, \quad f(u) = u.$

Associated Kähler metric: $g_K = d(e^{h(r)}p^2\boldsymbol{k}_+^\flat))(\cdot, J\cdot)$ defined on

$$M' := \{(t, r, \vartheta, \varphi) \in \mathbb{R}^2 \times \mathbb{S}^2 \,|\, \pi/2 < \vartheta < \pi\}.$$

Dixon: Wick rotation of the Kerr metric and a is Kähler, in fact ambitoric.

An example with non-trivial shear

The 4-dimensional solvable real Lie algebra $\mathfrak{g}$ with ordered basis $\boldsymbol{k}$, $\boldsymbol{t}$, $\mathbf{x}$, $\mathbf{y}$ defined by

$$[\boldsymbol{k},\mathbf{x}]=\mathbf{y}, \qquad [\boldsymbol{t},\mathbf{y}]=\mathbf{y}, \qquad [\boldsymbol{t},\boldsymbol{k}]=\boldsymbol{k}, \qquad [\mathbf{x},\mathbf{y}]=\mathbf{y}+r\boldsymbol{k},$$

with r a nonzero real constant.

For the corresponding simply connected Lie group $\widehat{G}$ with corresponding left-invariant vector fields, define a Lorentzian metric g as a left-invariant metric extended from $\mathfrak{g}$ an inner product with the above four vectors orthonormal, with $g(\boldsymbol{k},\boldsymbol{k})=-g(\boldsymbol{t},\boldsymbol{t})=1$. Then, for the corresponding left invariant vector fields

$$g([\boldsymbol{k},\mathcal{H}],\boldsymbol{k})=g([\boldsymbol{t},\mathcal{H}],\boldsymbol{t})=g([\boldsymbol{k},\mathcal{H}],\boldsymbol{t})=g([\boldsymbol{t},\mathcal{H}],\boldsymbol{k})=0,$$

where $\mathcal{H}=\mathrm{span}(\mathbf{x},\mathbf{y})$. So J with $J\boldsymbol{k}=\boldsymbol{t}$, $J\mathbf{x}=\mathbf{y}$ is integrable.

Also

$$g([\boldsymbol{k},\mathbf{x}],\mathbf{y})+g([\boldsymbol{k},\mathbf{y}],\mathbf{x})=1, \qquad g([\boldsymbol{k},\mathbf{y}],\mathbf{y})-g([\boldsymbol{k},\mathbf{x}],\mathbf{x})=0,$$
$$g([\boldsymbol{t},\mathbf{x}],\mathbf{y})+g([\boldsymbol{t},\mathbf{y}],\mathbf{x})=0, \qquad g([\boldsymbol{t},\mathbf{y}],\mathbf{y})-g([\boldsymbol{t},\mathbf{x}],\mathbf{x})=1.$$

so the shear coefficients are

$$\sigma_1^{\boldsymbol{k}}=-\sigma_2^{\boldsymbol{t}}=0, \qquad \sigma_2^{\boldsymbol{k}}=\sigma_1^{\boldsymbol{t}}=-1\neq 0,$$

so that the shear operators of $\boldsymbol{k}$ and $\boldsymbol{t}$ are nonzero, while $J\nabla^o\boldsymbol{k}=\nabla^o\boldsymbol{t}$.
The twist function of $\boldsymbol{k}$ is

$$\iota=g(\boldsymbol{k},[\mathbf{x},\mathbf{y}])=r\neq 0,$$

so that the sign of ι is fixed by the choice of r.

As $\nabla_{\boldsymbol{t}}\boldsymbol{t} = -\mathrm{ad}^*_{\boldsymbol{t}}(\boldsymbol{t})$ and $g(\boldsymbol{t}, [\boldsymbol{t}, \cdot]) = 0$, $\boldsymbol{t}$ is geodesic and of constant length. Also $t^\perp$ is integrable. Hence $\boldsymbol{t}^\flat$ is closed, so $\boldsymbol{t}$ is locally gradient, and by simply connectedness in fact globally a gradient.

Since $\boldsymbol{k}$ is proper and $g(\boldsymbol{k}, \boldsymbol{t})$ is constant, $\hat{G}$ admits a Kähler metric by Theorem 3. Choosing $r < 0$, $f(\tau) = e^\tau$ we have $f\iota < 0$ and $f'G/\ell - f(g(\nabla_{\boldsymbol{k}}\boldsymbol{k}, \boldsymbol{t}) - d_t(g(k,k))/2) = -f' - f < 0$, so the Kähler metric

$$g_K = d(e^\tau \boldsymbol{k}^\flat)(, \cdot, J\cdot)$$

is defined on all of $\hat{G}$.

As $\boldsymbol{k}$ is not geodesic, pre-geodesic or Killing, g is not admissible.

Warped products:

Theorem 5 (Aazami-M)

Let $(N, \bar{g})$ be a Riemannian 3-manifold with a unit length vector field $\bar{\boldsymbol{k}}$, whose flow is geodesic, shear-free, and has a nowhere vanishing twist function. Suppose $w(t)$ is a smooth positive function on $\mathbb{R}$ satisfying $w'/w > -1$. Then $(\mathbb{R} \times_w N, g, \boldsymbol{k}, \nabla t)$ is pre/geodesic-admissible with respect to a chosen orientation, where g is the Lorentzian warped product

$$g := -dt^2 + w^2 \bar{g},$$

and $\boldsymbol{k} := \partial_t + \pi^* \bar{\boldsymbol{k}}/w$, with $\pi^* \bar{\boldsymbol{k}}$ is the lift of $\bar{\boldsymbol{k}}$ under the projection $\pi : \mathbb{R} \times N \to N$.
The metric g thus induces a Kähler metric on $\mathbb{R} \times N$.

$\bar{\boldsymbol{k}}$ has nonzero twist function iff $\bar{\boldsymbol{k}}^{\perp}$ has nowhere integrable normal bundle, and this is implied by $\bar{\boldsymbol{k}}$ being complete with $Ric(\boldsymbol{k}, \boldsymbol{k}) > 0$ (Harris-Paternain and Aazami).

A similar result (Aazami) holds for a null pre/geodesic vector field on a Lorentzian 4-manifold. This can be shown via a the Bochner-type Newman-Penrose formulas

$$\boldsymbol{k}(\delta\boldsymbol{k}) = \iota^2/2 - 2((\sigma_1^{\boldsymbol{k}})^2 + (\sigma_2^{\boldsymbol{k}})^2) - (\delta\boldsymbol{k})^2/2 - \mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k}),$$
$$\boldsymbol{k}(\iota) = -(\delta\boldsymbol{k})\iota.$$

Examples:

de Sitter spacetime:
$(\mathbb{R} \times_w S^3_r, -dt^2 + w^2\bar{g})$, where $w(t) := r^2 \cosh^2(t/r)$. For $r \geq 2$, $w'/w > -1$ Kähler metric is globally defined.

$(\mathbb{R} \times_w \mathbb{R}^3, -dt^2 + w^2\bar{g}, \boldsymbol{k}, \nabla t)$, with $\bar{g}$ a metric on a slice of a $4d$ pp-wave.

Definition

A four-dimensional *standard pp-wave* is the Lorentzian manifold $(\mathbb{R}^4, g)$ with coordinates (u, v, x, y) and with g given by

$$g = H(u, x, y)du \otimes du + du \otimes dv + dv \otimes du + dx \otimes dx + dy \otimes dy.$$

If H is quadratic in x and y, then $(\mathbb{R}^4, g)$ is called a *gravitational plane wave*.

Let $k = k(x, y, u)$, $h = h(x, y, u)$ be smooth, and $\mathbf{z}$ the null vector field

$$\mathbf{z} := \frac{1}{2}(H + k^2 + h^2)\partial_v - \partial_u + k\partial_x + h\partial_y.$$

Proposition

Let $(\mathbb{R}^4, g)$ be the plane wave with $H(u, x, y) = -x^2 - y^2$ and the vector field $\mathbf{z}$ have $k(u, x, y) = -y$ and $h(u, x, y) = x$. Then $(\mathbb{R}^4, g, \mathbf{z}, \nabla u)$ is geodesic-admissible and induces a Kähler metric on $\mathbb{R}^4$.

THE END

Thanks!

Steps in the proof of Theorem 1:

For a frame $\mathbf{x}_\pm$ of $\mathcal{H}$, enough to check $N(\boldsymbol{k}_+, \mathbf{x}_+) = 0$ by symmetries of N.

$N(\boldsymbol{k}_+, \mathbf{x}_+)$ is horizontal by (1)i) (Lie bracket conditions).

Compute $g(N(\boldsymbol{k}_+, \mathbf{x}_+), \mathbf{x}_\pm)$ as linear combinations of shear coefficients using the hermitian property of $g\big|_{\mathcal{H}}$.

By linear algebra and $N(\boldsymbol{k}_+, \mathbf{x}_-) = -JN(\boldsymbol{k}_+, \mathbf{x}_+)$, these equations are equivalent to

$$\iota_{\boldsymbol{k}_+} N = 2(\nabla^o \boldsymbol{k}_+ - \nabla^o J\boldsymbol{k}_+ \circ J) \text{ on } \mathcal{H},$$

This equation is equivalent to (1)ii) since $J|_{\mathcal{H}}$ anticommutes with trace-free symmetric operators as $\mathcal{H}$ has rank two.

Proof sketch that the form is symplectic:

$$\omega = f' d\tau \wedge \boldsymbol{k}^\flat + f d\boldsymbol{k}^\flat. \qquad (\triangle)$$

For $\mathbf{x}' \in \Gamma(\mathcal{H})$, $\omega(\boldsymbol{k}, \mathbf{x}') = 0$: Clear for the first term. For the second:

$$\begin{aligned} d\boldsymbol{k}^\flat(\boldsymbol{k}, \mathbf{x}') &= (\nabla_{\boldsymbol{k}} \boldsymbol{k}^\flat)(\mathbf{x}') - (\nabla_{\mathbf{x}'} \boldsymbol{k}^\flat)(\boldsymbol{k}) \\ &= g(\nabla_{\boldsymbol{k}} \boldsymbol{k}, \mathbf{x}') - g(\nabla_{\mathbf{x}'} \boldsymbol{k}, \boldsymbol{k}) \\ &= \begin{cases} g(\nabla_{\boldsymbol{k}} \boldsymbol{k}, \mathbf{x}') - d_{\mathbf{x}'}(g(\boldsymbol{k}, \boldsymbol{k}))/2 = 0 \text{ for two admissible types,} \\ -2g(\nabla_{\mathbf{x}'} \boldsymbol{k}, \boldsymbol{k}) = 0 \text{ for Killing-adm. by properness.} \end{cases} \end{aligned}$$

$$\implies \text{For } \{v_i\} = \{\boldsymbol{k}, \mathbf{x}, \mathbf{y}, \boldsymbol{t}\}, \quad \det[\omega(v_i, v_j)] = (\omega(\mathbf{x}, \mathbf{y})\omega(\boldsymbol{k}, \boldsymbol{t}))^2.$$

$$\omega(\mathbf{x}, \mathbf{y}) = f d\boldsymbol{k}^\flat(\mathbf{x}, \mathbf{y}) = f(g(\nabla_{\mathbf{x}} \boldsymbol{k}, \mathbf{y}) - g(\nabla_{\mathbf{y}} \boldsymbol{k}, \mathbf{x})) = -f\iota,$$

$\omega(\boldsymbol{k}, \boldsymbol{t})$: first term in $(\triangle)$ is $-f' G/\ell$, second depends on type.

<u>J-invariance of ω</u>:

J-invariance of ω requires, as $\omega(\boldsymbol{k}, \mathbf{x}') = 0$, $\mathbf{x}' \in \Gamma(\mathcal{H})$, that $\omega(\boldsymbol{t}, \mathbf{x}') = 0$ as well, i.e. $d\boldsymbol{k}^\flat(\boldsymbol{t}, \mathbf{x}') = 0$. But

$$
\begin{aligned}
&d\boldsymbol{k}^\flat(\boldsymbol{t}, \mathbf{x}') \\
&\qquad = g(\nabla_{\boldsymbol{t}}\boldsymbol{k}, \mathbf{x}') - g(\nabla_{\mathbf{x}'}\boldsymbol{k}, \boldsymbol{t}) \\
&\qquad = -g(\boldsymbol{k}, \nabla_{\boldsymbol{t}}\mathbf{x}') + g(\boldsymbol{k}, \nabla_{\mathbf{x}'}\boldsymbol{t}) \text{ as } g(\boldsymbol{k}, \boldsymbol{t}) \text{ is constant,} \\
&\qquad = -g(\boldsymbol{k}, [\boldsymbol{t}, \mathbf{x}']) = 0 \text{ as } [\boldsymbol{t}, \mathbf{x}'] \in \Gamma(\mathcal{H}).
\end{aligned}
$$

Some pointers to the proof of Theorem 3:

Verifications that g is admissible are based on properties of g_{SKR}. For example, to check $\boldsymbol{k}$ is g-Killing, verify

$$d_{\boldsymbol{k}} g(a,b) = g([\boldsymbol{k},a],b) + g(a,[\boldsymbol{k},b])$$

for a, b in the frame $\boldsymbol{k}$, $\boldsymbol{t}$, $\mathbf{x}$, $\mathbf{y}$, e.g. $d_{\boldsymbol{k}}(g(\boldsymbol{k},\boldsymbol{k})) = d_{\boldsymbol{k}} p = p' d_{\boldsymbol{k}} \tau = p' d_u \tau = p' g_{SKR}(u,v) = 0 = 2g([\boldsymbol{k},\boldsymbol{k}],\boldsymbol{k})$.

To check the domain of definition inequalities for g_K, for example we have $f\imath < 0$ on U, since for an ordered h-orthonormal frame $\mathbf{x}$, $\mathbf{y}$=$J\mathbf{x}$,
$\iota = g(\boldsymbol{k},[\mathbf{x},\mathbf{y}]) = -2\pi^*\omega^h(\mathbf{x},\mathbf{y})p = -2\pi^* h(\mathbf{y},\mathbf{y})p = -2p < 0$.

The global claim:

if M is compact and not biholomorphic to $\mathbb{C}P^2$, by SKR theory on $\mathcal{L}$, the non-critical set of τ is $\mathcal{L} \setminus \{0 - \mathrm{section}\}$, and $c \notin \mathrm{Image}(\tau)$, and one can choose $\tau > c$. So

$$U = \tau\text{-non-critical set.}$$

As τ is a Killing potential, U is open and dense.

On the proof of existence of type D Kähler metric:

Set $\tilde{\boldsymbol{k}}_\pm = p\boldsymbol{k}_\pm$. They define the same complex structure as $\boldsymbol{k}_\pm$, though satisfy Theorem 2 rather than Theorem 1.

$$\omega = d(f\tilde{\boldsymbol{k}}_+) = f' du \wedge \tilde{\boldsymbol{k}}_+ + fd\tilde{\boldsymbol{k}}_+.$$

On $\tilde{\boldsymbol{k}}_+$, $\mathbf{x}' \in \Gamma(\mathcal{H})$ the first term vanishes since $\tilde{\boldsymbol{k}}_+$ is null. The second term vanishes as before since $\tilde{\boldsymbol{k}}_+$ is null geodesic.

On $\tilde{\boldsymbol{k}}_-$, $\mathbf{x}' \in \Gamma(\mathcal{H})$ neither term vanishes separately, but they cancel each other out if $\nabla(f(u)/p) \in \Gamma(\mathcal{V})$.