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Kähler immersions of homogeneous Kähler manifolds into complex space forms

Andrea Loi, University of Cagliari

joint works (2006-2018) with:

A. J. Di Scala, H. Hishi, R. Mossa, F. Salis, M. Zedda, F. Zuddas

Aim *Classify all homogeneous Kähler manifolds (h.K.m.) which admit a Kähler immersion into a given finite or infinite dimensional complex space form.*

Advertising for the book: -, M. Zedda, *Kähler immersions of Kähler manifolds into complex space forms*, https://arxiv.org/abs/1712.

1.General definitions: Kähler manifolds, Complex space forms and their classifications.

2.Kähler immersions into complex space forms, E. Calabi, Ann. Math. 1953.

3.Homogeneous Kähler manifolds and their classification (J. Dorfmeister, K. Nakajima, Acta Math. 1988).

4.Kähler immersions of h.K.m. into complex space forms (Theorem 1, 2, 3, 4).

5.Sketch of the proofs of Theorem 1, 2, 3, 4.

6.The Kähler-Einstein case

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1. General definitions: Kähler manifolds, complex space forms and their classification

Kähler manifolds

Let $(M, g) = (M, g, \omega, J)$ be a Kähler manifold of complex dimension n.

$\omega(X, Y) = g(X, JY)$, $X, Y \in \mathfrak{X}(M)$, $d\omega = 0$.

The form ω is called the *Kähler form associated to the metric* g.

On a contractible open set $U \subset M$

$$\omega = \frac{i}{2}\partial\bar{\partial}\Phi = \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}dz_j \wedge d\bar{z}_k,$$

where $\Phi : U \to \mathbb{R}$ is a strictly PSH function called a *Kähler potential* for the metric g.

Complex space forms

A *complex space form* $(S, g_S) = (S, g_S, \omega_S, J_S)$ is a finite or infinite dimensional Kähler manifold of constant holomorphic sectional curvature.

Classification of complex space forms

Complex Euclidean space $\mathbb{C}^{N\leq\infty} := (\mathbb{C}^{N\leq\infty}, g_0)$

$\mathbb{C}^{\infty} := \ell^2(\mathbb{C})$ $(z = \{z_j\} \in \ell^2(\mathbb{C})$ iff $\sum_{j=1}^{\infty} |z_j|^2 < \infty)$

$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}\sum_{j=1}^{N} dz_j \wedge d\bar{z}_j$, $|z|^2 = |z_1|^2 + \cdots + |z_N|^2$.

Complex hyperbolic space $\mathbb{C}H^{N\leq\infty} := (\{z \in \mathbb{C}^N \mid |z|^2 < 1\}, g_{hyp})$

$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2)$.

Complex projective space $\mathbb{C}P^{N\leq\infty} = (\mathbb{C}^{N+1} \setminus \{0\}/z \sim \lambda z, g_{FS})$

$\omega_{FS}|_{U_0} = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2)$, $z_j = \frac{Z_j}{Z_0}$, $j = 1, \ldots, N$, $U_0 = \{Z_0 \neq 0\}$.

2. Kähler immersions into complex space forms

(E. Calabi, Ann. Math. 1953)

Kähler immersions into complex space forms

Let (M, g) be a Kähler manifold. A Kähler immersion

$$f : (M, g) \to (S, g_S)$$

is a holomorphic map (i.e. $df \circ J = J_S \circ df$) which is isometric (i.e. $f^* g_S = g$).

Remark The "starting" manifold M will be always finite dimensional.

Terminology A Kähler metric g on a complex manifold M is *projectively induced* if (M, g) can be Kähler immersed into a finite or infinite dimensional complex projective space.

Calabi's results on Kähler immersions (1953)

Theorem (Calabi's rigidity) *Let* $f : (M, g) \to (S, g_S)$ *be a Kähler immersion. Then any other Kähler immersion of* (M, g) *into* (S, g_S) *is given by* $\mathcal{U} \circ f$ *where* $\mathcal{U}$ *is a unitary transformation, i.e.* $\mathcal{U} \in$ *Aut*$(S) \cap$ *Isom*(S, g_S).

Theorem (Calabi's extension theorem) *A simply-connected Kähler manifold* (M, g) *admits a Kähler immersion into a given complex space form* (S, g_S) *iff there exists an open set* $U \subset M$ *such that* $(U, g_{|U})$ *can be Kähler immersed into* (S, g_S).

Complex Euclidean spaces into complex space forms

$\mathbb{C}^n \nrightarrow \mathbb{C}H^{N\leq\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}^n \hookrightarrow \mathbb{C}^{N\leq\infty},\ n \leq N$

We have the following Calabi's immersion

$$\mathbb{C}^n \to \mathbb{C}P^\infty : z \mapsto (\ldots, \sqrt{\frac{1}{j!}}\, z^j, \ldots),\ |j| \geq 0$$

$$z^j = z_1^{j_1} \cdots z_n^{j_n}\ |j| = j_1 + \cdots + j_n,\ j! = j_1! \cdots j_n!$$

Complex hyperbolic spaces into complex space forms

Let $\mathbb{C}H^n_\lambda = (\mathbb{C}H^n, \lambda g_{hyp})$, $\lambda > 0$, $\mathbb{C}H^n := \mathbb{C}H^n_1 = (\mathbb{C}H^n, g_{hyp})$

$\mathbb{C}H^n_\lambda \not\to \mathbb{C}^{N<\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}H^n_\lambda \to \mathbb{C}H^{N\leq\infty} \;\Leftrightarrow\; \lambda = 1,\; n \leq N$

We have the following Calabi's immersions

$$\mathbb{C}H^n_\lambda \to \ell^2(\mathbb{C}) : z \mapsto \sqrt{\lambda}(\dots, \sqrt{\frac{(|j|-1)!}{j!}}\, z^j, \dots),\;\; |j| \geq 1$$

$$\mathbb{C}H^n_\lambda \to \mathbb{C}P^\infty : z \mapsto (\dots, \sqrt{\frac{\lambda(\lambda+1)\cdots(\lambda-1+|j|)}{j!}}\, z^j, \dots),\;\; |j| \geq 0$$

$z^j = z_1^{j_1}\cdots z_n^{j_n}$, $|j| = j_1 + \cdots + j_n$, $j! = j_1!\cdots j_n!$

Complex projective spaces into complex space forms

Let $\mathbb{C}P^n_\lambda = (\mathbb{C}P^n, \lambda g_{FS}),\ \lambda > 0$, $\mathbb{C}P^n := \mathbb{C}P^n_1 = (\mathbb{C}P^n, g_{FS})$

$$\boxed{\mathbb{C}P^n_\lambda \nrightarrow \mathbb{C}^{N\leq\infty}, \mathbb{C}H^{N\leq\infty}}$$

Let $k \in \mathbb{Z}$ and $N_k := \frac{(n+k)!}{n!k!} - 1$. Then the map

$$\mathbb{C}P^n_k \xrightarrow{V_k} \mathbb{C}P^{N_k} : [Z] \longmapsto [\dots, \sqrt{\frac{|j|!}{j!}}\ Z^j, \dots],\ |j| \geq 0$$

$Z^j = Z_0^{j_0} \cdots Z_n^{j_n}$, $|j| = j_0 + \cdots + j_n$, $j! = j_0! \cdots j_n!$ satisfies

$$V_k^* g_{FS} = k g_{FS}$$

3. H.K.m. and their classification

(J. Dorfmeister, K. Nakajima, 1988)

Homogeneous Kähler manifolds

A homogeneous Kähler manifold (h.K.m.) is a Kähler manifold (M, g) *such that the Lie group* $G = \mathit{Aut}(M) \cap \mathit{Isom}(M, g)$ *acts transitively on* M.

Remark. The metric g is not uniquely determined by G. There exist different (neither homothetic or isometric) G-invariant homogeneous metrics.

Examples: Complex space forms are h.K.m.

Homogeneous bounded domains

Let $\Omega \subset \mathbb{C}^n$, Ω bounded domain endowed with a homogeneous Kähler metric g_Ω. Then (Ω, g_Ω) is called a *homogeneous bounded domain* (h.b.d.).

If Aut(Ω) acts transitively on $\Omega \subset \mathbb{C}^n$ then $(\Omega, g_\Omega = g_B)$ is a bounded symmetric domain *, g_B is is the Bergman metric whose associated Kähler form $\omega_B = \frac{i}{2}\partial\bar{\partial} \log K$, where K is the reproducing kernel for the Hilbert space of holomorphic L^2-functions on Ω.

Remark. Every bounded symmetric domain (Ω, g_B) is a h.b.d. but there exist (Pyatetskii-Shapiro, 1969) h.b.d. (Ω, g_B) which are not bounded symmetric domains.

*A bounded symmetric domain $\Omega \subset \mathbb{C}^n$ is a domain where the geodesic symmetry $\exp_x(v) \mapsto \exp_x(-v)$ is a Kähler map.

Other examples of h.K.m.

Flat h.K.m. $\mathcal{E} = \mathbb{C}^k \times F$ where F is a non simply-connected (either compact or non compact) flat Kähler manifold.

Compact simply-connected h.K.m. These are also called *Kähler* C*-spaces* or *rational homogeneous varieties*.

Compact h.K.m. $(M, g) = \mathcal{C} \times T_1 \times \cdots \times T_l$, C-space, T_j flat torus.

Products of homogeneous Kähler manifolds The products of h.K.m. is a h.K.m.

Solution of the fundamental conjecture (FC) for h.K.m.

Theorem FC (J. Dorfmeister, K. Nakajima, 1988) *A h.K.m.* (M, g) *is the total space of a holomorphic fiber bundle over a h.b.d.* (Ω, g_Ω)*. Moreover the fiber* $\mathcal{F} = \mathcal{E} \times \mathcal{C}$ *is (with the induced Kähler metric) the Kähler product of a flat homogeneous Kähler manifold* $\mathcal{E} = \mathbb{C}^k \times F$ *and a* C*-space* $\mathcal{C}$*.*

$$\mathcal{F} = \mathcal{E} \times \mathcal{C} \overset{\text{Kähler}}{\hookrightarrow} \begin{array}{c} (M, g) \\ \pi\downarrow \\ (\Omega, g_\Omega) \end{array}$$

Remark. $M \overset{top}{=} \Omega \times \mathcal{F}$ as a complex manifold.

4. Kähler immersions of h.K.m.
into complex space forms (Theorems 1, 2, 3, 4)

Homogeneous Kähler manifolds into $\mathbb{C}^{N\leq\infty}$

Theorem 1 (-, A. J. Di Scala, H Hishi, 2012) *Let (M, g) be a n-dimensonal h.K.m. which can be Kähler immersed into $\mathbb{C}^{N\leq\infty}$. Then $(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$. Moreover, the immersion is given, up to a unitary transformation of $\mathbb{C}^N$ by*

$$f_0 \times f_1 \times \cdots \times f_l,$$

where f_0 is the linear inclusion $\mathbb{C}^k \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}^N$ and each f_r : $\mathbb{C}H^{n_r}_{\lambda_r} \longrightarrow \ell^2(\mathbb{C})$, $r = 1, \ldots, l$, are Calabi's immersions.

Homogeneous Kähler manifolds into $\mathbb{C}H^{N\leq\infty}$

Theorem 2 (-, A. J. Di Scala, H Hishi, 2012) *Let* (M, g) *be a* n*-dimensional h.K.m. which can be Kähler immersed into* $\mathbb{C}H^{N\leq\infty}$*. Then, up to a unitary transformation of* $\mathbb{C}H^{N}$*,*

$$(M, g) = \mathbb{C}H^n \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}H^N.$$

Two theorems on h.K.m. into $\mathbb{C}P^{N\leq\infty}$

Theorem 3 (-, A. J. Di Scala, H Hishi, 2012) *Let* (M, g) *be a* n*-dimensional h.K.m. which can be Kähler immersed into* $\mathbb{C}P^{N\leq\infty}$*.* Then ω is integral, $\pi_1(M) = 1$ and the immersion is injective.

Theorem 4 (-, R. Mossa, 2014) *Let* (M, g) *be a simply-connected h.K.m. such that its associated Kähler form* ω *is integral. Then there exists* $m_0 \in \mathbb{Z}$ *such that*

$$(M, m_0 g) \to \mathbb{C}P^{N\leq\infty}.$$

Remarks on the compact case

When M is compact Theorem 3 and Theorem 4 were proved by M. Takeuchi (1978) using the theory of semisimple Lie groups and Dynkin diagrams (one can take $m_0 = 1$ in Theorem 4).

Notice that if a h.K.m. can be Kähler immersed into $\mathbb{C}P^{N<\infty}$ then M is a C-space, i.e. is a compact (simply-connected) Kähler manifold.

Viceversa if M is any compact (not necessarily homogeneous) Kähler manifold which can be Kähler immersed into $\mathbb{C}P^{N\leq\infty}$ one can assume $N < \infty$.

The necessity of taking m_0 in Theorem 4

Let Ω be an irreducible bounded symmetric domain. The Wallach set[†] $W(\Omega) \subset \mathbb{R}$ is a subset of $\mathbb{R}$ which "looks like":

$$\underbrace{0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot}\ \overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$\uparrow$ *discrete part of* $W(\Omega)$ $\qquad$ $\uparrow$ *continuous part of* $W(\Omega)$

Important property of the Wallach set: $W(\Omega) = \mathbb{R}$ (and hence the discrete part of $W(\Omega)$ is empty) if and only if $\Omega = \mathbb{C}H^n$.

[†] $W(\Omega)$ consists of all $\lambda \in \mathbb{R}$ such that there exists a Hilbert space $\mathcal{H}_\lambda$ whose reproducing kernel is $K^{\frac{\lambda}{\gamma}}$, γ the genus of Ω, where K is the reproducing kernel for the Hilbert space of holomorphic L^2-functions on Ω.

The Wallach set and Kähler immersions into $\mathbb{C}P^\infty$

Theorem W (–, M. Zedda, 2010) *Let* (Ω, g_B) *be a irreducible bounded symmetric domain. Then* $(\Omega, \lambda g_B)$ *can be Kähler immersed into* $\mathbb{C}P^\infty$ *if and only if* $\lambda\gamma \in W(\Omega) \setminus \{0\}$*, where* γ *denotes the genus of* Ω*.*

Two consequences of Theorem W

First consequence: Let $(\Omega, g_B) \neq \mathbb{C}H^n$ be a irreducible bounded symmetric domain. One can find $\lambda > 0$ such that $\lambda\gamma \notin W(\Omega)$:

$$0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad * \quad ——————————$$

$$\uparrow$$

$$\lambda\gamma \notin W(\Omega)$$

By Theorem W, λg_B is not projectively induced and $\lambda\omega_B$ is integral (this shows the necessity of taking $m_0 > 1$ in Theorem 4).

Second consequence: The complex hyperbolic space is the only irreducible bounded symmetric domain (Ω, g_B) where λg_B is projectively induced, for all $\lambda > 0$.

A Lemma for homogeneous bounded domains

Lemma H (-, A. J. Di Scala, H Hishi, 2012) *Let* (Ω, g_Ω) *be a h.b.d. If* $(\Omega, \lambda g_\Omega)$ *can be Kähler immersed into* $\mathbb{C}P^\infty$ *for all* $\lambda > 0$, *then* $(\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

Ingredients for the proof. Unitary representation of semisimple Lie groups; reproducing kernels of weighted Bergman spaces.

5. Sketch of the proofs of Theorem 1, 2, 3, 4

Sketch of the proof of Theorem 1

$(M,g) \stackrel{f}{\to} \mathbb{C}^{N\leq\infty}$ we want to prove that:

$(M,g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$ and $f = f_0 \times f_1 \times \cdots \times f_l$.

1. Theorem FC + Calabi's rigidity theorem+ max principle $\Rightarrow$

$$\mathcal{F} = \mathbb{C}^k \times \not{F} \times \not{\mathbb{C}} \stackrel{\text{Kähler}}{\hookrightarrow} \begin{array}{c} (M,g) \\ \pi\downarrow \\ (\Omega, g_\Omega) \end{array} \to \mathbb{C}^{N\leq\infty}$$

2. Riemannian geometry + homogeneity $\Rightarrow$

$$(M,g) \stackrel{\text{Kähler}}{=} \mathbb{C}^k \times (\Omega, g_\Omega) \Rightarrow (\Omega, \lambda g_\Omega) \to \mathbb{C}^{N\leq\infty},\ \forall\lambda > 0.$$

3. S. Bochner (1947) $\Rightarrow (\Omega, \lambda g_\Omega) \to \mathbb{C}P^\infty, \ \forall \lambda > 0$.

4. Lemma H $\Rightarrow (\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow$

$\Rightarrow (M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

5. The fact that the immersion f is, up to a unitary transformation of $\mathbb{C}^N$, of the form $f = f_0 \times f_1 \times \cdots \times f_l$ follows by the reducibility of a Kähler product into $\mathbb{C}^{N \leq \infty}$ and by Calabi's rigidity theorem. □

Sketch of the proof of Theorem 2 (based on Theorem 1)

If $(M, g) \to \mathbb{C}H^{N \leq \infty}$ we want to prove that

$$(M, g) = \mathbb{C}H^n \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}H^N.$$

1. $(M, g) \to \mathbb{C}H^{N \leq \infty} \Rightarrow (M, g) \to \ell^2(\mathbb{C})$.

2. Theorem 1 $\Rightarrow (M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow M = \mathbb{C}H^n$. $\square$

Sketch of the proof of Theorem 3

Let $f : (M, g) \to \mathbb{C}P^{N \leq \infty}$ be a Kähler immersion.

The integrality of $\omega = f^*\omega_{FS}$ is immediate since ω_{FS} is integral.

Th. FC $\Rightarrow$ $\mathcal{F} = \mathbb{C}^k \times \not{F} \times C \overset{\text{Kähler}}{\hookrightarrow} \begin{matrix} (M, g) \\ \pi\downarrow \\ (\Omega, g_\Omega) \end{matrix} \to \mathbb{C}P^{N \leq \infty}$ $\Rightarrow M \overset{top}{=}$ $\Omega \times \mathbb{C}^n \times C$ is simply-connected.

Calabi's rigidity $\Rightarrow$ $f \circ g = \mathcal{U}_g \circ f$, $\forall g \in G =$ Aut$(M) \cap$ Isom(M, g) $\Rightarrow$ $f(M)$ is a h.K.m. $\Rightarrow$ $f(M) \subset \mathbb{C}P^N$ is simply-connected.

$f : M \to f(M)$ is a local isometry $\Rightarrow$ f is a covering map $\Rightarrow$ f is injective. $\square$

Sketch of the proof of Theorem 4

Let (M, g) be a simply-connected h.K.m. with ω integral we want to show that $(M, m_0 g) \rightarrow \mathbb{C}P^{N \leq \infty}$, for some $m_0 \in \mathbb{Z}$.

1. Let L be a holomorphic line bundle with $c_1(L) = [\omega]$ and consider the Hilbert space

$$\mathcal{H}_m = \{s \in H^0(L) \mid \int_M h_m(s, s) \frac{\omega^n}{n!} < \infty\}$$

where h_m is an Hermitian metric on L^m such that Ric$(h_m)^{\ddagger} = m\omega$.

2. There exists $m_0 \in \mathbb{Z}$ such that $\mathcal{H}_{m_0} \neq \{0\}$ (J. Rosenberg, M. Vergne, 1984);

‡Ric$(h_m) = -\frac{i}{2}\partial\bar{\partial} \log h_m(\sigma(x), \sigma(x))$, where $\sigma : U \rightarrow L^m$ is a trivialising holomorphic section of L^m.

3. Consider the smooth function on M given by:

$$\epsilon_{m_0}(x) = \sum_{j=0}^{d_{m_0}} h_{m_0}(s_j(x), s_j(x)),$$

where $\{s_0, \ldots, s_{d_{m_0}}\}$ is an orthonormal basis of $\mathcal{H}_{m_0}$.

Homogeneity + $\pi_1(M) = 1$ $\Rightarrow$ $\epsilon_{m_0}(x)$ is a positive constant.

4. Therefore the "Kodaira map"

$$\varphi_{m_0} : M \to \mathbb{C}P^{d_{m_0}}, x \mapsto [s_0(x), \ldots, s_{d_{m_0}}(x)]$$

is well-defined and it satisfies

$$\varphi_{m_0}^* \omega_{FS} = m_0\omega + \frac{i}{2}\partial\bar{\partial} \log \epsilon_{m_0} = m_0\omega.$$

$\square$

5. The Kähler-Einstein case

The Kähler-Einstein case

Theorem (M. Umehara, 1987) *Let* (M, g) *be a complete KE manifold of complex dimension* n *which admits a Kähler immersion into* $\mathbb{C}^N$ *(resp.* $\mathbb{C}H^N$*). Then* $(M, g) = \mathbb{C}^n$ *(resp.*$(M, g) = \mathbb{C}H^n$*).*

Conjecture A: *A compact KE manifold which admits a Kähler immersion into a complex projective space is homogeneous (Chern (1967), Tsukada (1986), Hulin (2000)).*

Remark: *The conjecture cannot be weakened to the noncompact case. There exist examples (even continuous family) of noncompact and nonhomogeneous KE submanifolds of* $\mathbb{C}P^\infty$ *(–, M. Zedda, 2010)*.

The Ricci flat case

Conjecture B: *A Ricci flat projectively induced Kähler metric is flat.*

Theorem (-, F. Salis, F. Zuddas, 2018) *A projectively induced Ricci flat and radial Kähler metric is flat.*

Conjecture B cannot be weakened to scalar flat metrics

Let S be the blow-up of $\mathbb{C}^2$ at the origin and denote by E the exceptional divisor. Simanca constructs a scalar flat Kähler complete (not Ricci-flat) metric g_S on S whose Kähler potential on $S \setminus E = \mathbb{C}^2 \setminus \{0\}$ can be written as

$$\Phi_S(|z|^2) = |z|^2 + \log|z|^2, |z|^2 = |z_1|^2 + |z_2|^2.$$

The holomorphic map

$$\varphi : S \setminus E \to \mathbb{C}P^\infty : (z_1, z_2) \mapsto (z_1, z_2, \dots, \sqrt{\frac{j+k}{j!k!}} z_1^j z_2^k, \dots), \ j+k \neq 0,$$

is a Kähler immersion. By Calabi's extension theorem it extends to a Kähler immersion of (S, g_S) into $\mathbb{C}P^\infty$.

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Thank you for your attention!