Non-projective K3 surfaces containing Levi-flat hypersurfaces

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Goal of this talk:

Gluing construction of non-projective K3 surfaces.

We will construct a K3 surface X by holomorphically patching two open complex surfaces, say M and M'.

- M (M') is the complement of a (appropriate) tubular neighborhood of an elliptic curve in the blow-up S (S') of the projective plane $\mathbb{P}^2$ at (appropriate) nine points.
- Neither S nor S' admit elliptic fibration structure (nine points are "general")
- In order to patch M and M' holomorphically, we need to take "nice neighborhood". For this purpose, we need to choose nine points carefully.
- Use a technique from Complex Dynamics for taking "nice neighborhood" (Arnol'd's theorem).

Remarks, Known results

- For the case where S and S' are elliptic, a similar construction of K3 surfaces is known. The resulting K3 surfaces are also elliptic.
- M. Doi showed that a similar construction of K3 surfaces if one admit (slight) deformations of the complex structures of M and M'.

  (Doi, Mamoru, Gluing construction of compact complex surfaces with trivial canonical bundle. J. Math. Soc. Japan 61 (2009), no. 3, 853–884)
- The idea to use Arnol'd-type theorem for patching two open manifolds is also used by H. Tsuji in order to study complex structures on $S^3 \times S^3$.

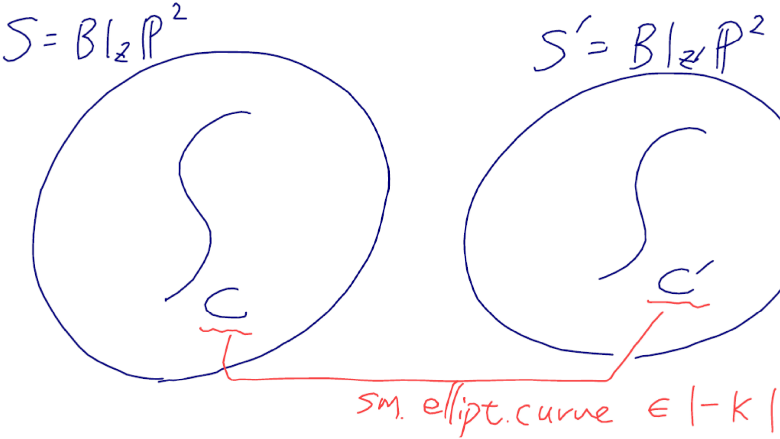
  (H. Tsuji, Complex structures on $S^3 \times S^3$, Tohoku Math. J. (2) Volume 36, Number 3 (1984), 351–376)

Take a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and nine points $Z := \{p_1, p_2, \ldots, p_9\} \subset C_0$.

- $S := \mathrm{Bl}_Z\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$: blow-up at Z
- $C := \pi_*^{-1}C_0$: the strict transform of C_0

Note that $N_{C/S} \cong \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9)$. When Z is special, S is an elliptic surface ($N_{C/S} \in \mathrm{Pic}^0(C)$ is torsion in this case). We are interested in the case where $\underline{Z \text{ is general}}$.

Let (S', C') be another model which is constructed by another choice of an elliptic curve C_0' and another nine points configuration $Z' := \{p_1', p_2', \ldots, p_9'\} \subset C_0'$.


S = Bl_Z P^2
S' = Bl_Z' P^2
C
C'
sm. ellipt. curve ∈ |−K|

Assumptions

In what follows, we always assume the following:

Assumption

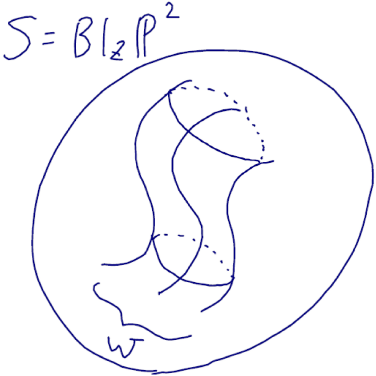
- $\exists g \colon C \cong_{\text{bihol.}} C'$
- $N_{C/S} = g^* N_{C'/S'}^{-1}$
- $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine

$N_{C/S} \in \mathrm{Pic}^0(C)$ is said to be *Diophantine* if $\exists A, \alpha > 0$ such that $\mathrm{dist}(\mathbb{I}_C, N_{C/S}^{\otimes n}) \geq A \cdot n^{-\alpha}$ for $\forall n > 0$.

- $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine for almost every choice of Z in the sense of Lebesgue measure.
- We will explain why do we need this condition latter.

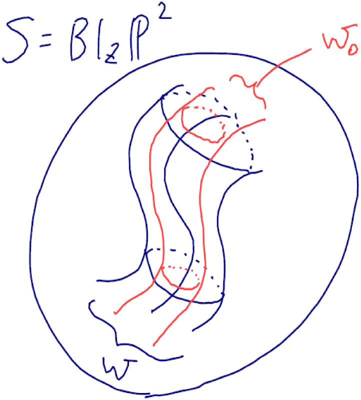
Outline of the construction –Step 1

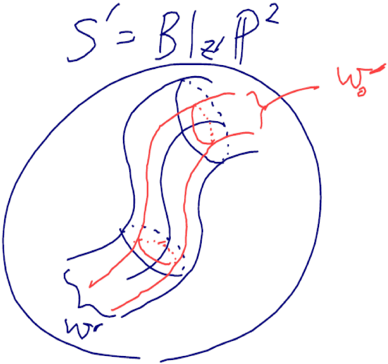
First, we take "nice" neighborhoods W of C in S and W' of C' in S':



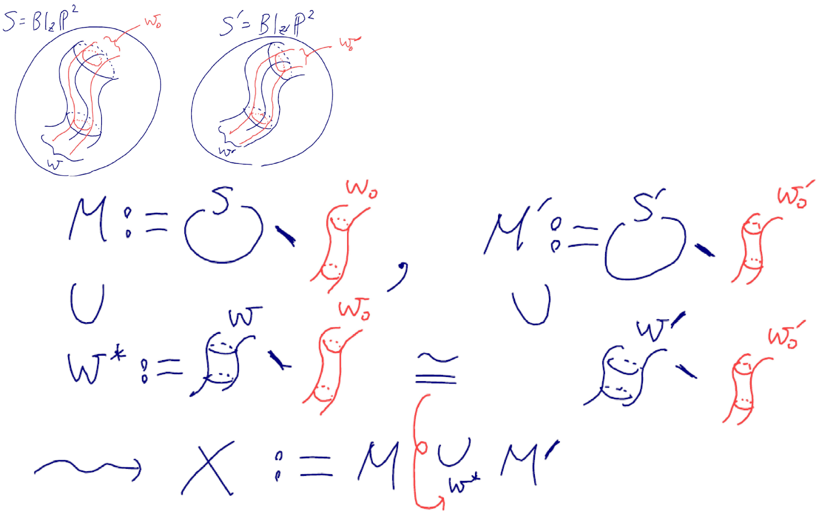
Outline of the construction –Step 2

Next, we take “nice” neighborhoods $W_0 \Subset W$ of C and $W_0' \Subset W'$ of C' appropriately:





Outline of the construction –Step 3



Question

How should we choose "nice" neighborhoods W, W_0, W', and W_0' (in order to patch M and M' holomorphically)?

Here we use the following:

Theorem (Arnol'd (1976))

Assume C is a smooth elliptic curve and $N_{C/S} \in \mathrm{Pic}^0(C)$ is Diophantine.
Then C admits a **holomorphic tubular neighborhood** W *(i.e. W can be chosen so that W is biholomorphic to a neighborhood of the zero-section in $N_{C/S}$).*

Arnol'd's theorem is shown by using complex dynamical technique as in the proof of **Siegel's linearization theorem**, which is the reason why Diophantine condition is needed in our assumption.

What follows from Arnol'd's theorem and our assumptions

- "$N_{C/S}$: Diophantine" + Arnol'd's thm
  $\Rightarrow W$, W_0: holomorphic tubular neighborhoods of C
  $\Rightarrow W \backslash W_0 \cong_{\text{bihol.}}$ (an annulus bundle over C)

- "$N_{C/S}$: Diophantine" + "$N_{C/S} = g^* N_{C'/S'}^{-1}$" + Arnol'd's thm
  $\Rightarrow W'$, W_0': holomorphic tubular neighborhoods of C'
  $\Rightarrow W' \backslash W_0' \cong_{\text{bihol.}}$ (an annulus bundle over C')

- "$g\colon C \cong C'$" + "$N_{C/S} = g^* N_{C'/S'}^{-1}$" + observations above
  $\Rightarrow (W^* :=)\ W \backslash W_0 \cong_{\text{bihol.}} W' \backslash W_0'$
  $\Rightarrow$ One can glue M and M' holomorphically by using W^* as a "tab for gluing".

Observation

W^* admits a foliation $\mathcal{F}$ which is naturally defined by considering the flat connection on $N_{C/S}$. Each leaf is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

It is easily observed that $\pi_1(X) = 0$. Therefore, for proving that X is a K3 surface, it is sufficient to show the following:

Proposition

There exists a nowhere vanishing holomorphic 2-form σ on X.

Outline of the proof: As $K_S = -C$, there exists a meromorphic 2-form η on S with $\mathrm{div}(\eta) = -C$. We can also take a meromorphic 2-form η' on S' with $\mathrm{div}(\eta') = -C'$. σ is obtained by patching $\eta|_M$ and $-\eta'|_{M'}$ after appropriate normalizations. $\square$

Some remark on the construction of the 2-form σ on X

For patching $\eta|_M$ and $-\eta'|_{M'}$ on W^*, we use the following:

Key Lemma

$H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

Key Lemma is shown by considering the restriction of a given holomorphic function on W^* to a leaf of $\mathcal{F}$ and considering the Maximum principle.

By using this Key Lemma, one can describe the 2-form $\sigma|_{W^*}$ very explicitly.
$\Rightarrow$ We could explicitly compute the integrations $\int \sigma$ along 20 2-cycles of 22 appropriately chosen 2-cycles ("marking" of X).

Main results

We constructed K3 surfaces in such a manner with independent (at least) 19 parameters. As an conclusion of the construction, we have the following:

Theorem (K-, T. Uehara. improved version of the main result in arXiv:1703.03663)

There exists a deformation $\pi\colon \mathcal{X} \to B$ of K3 surfaces over a (at least) 19 dimensional complex manifold B with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic map $F_b\colon \mathbb{C} \to X_b$ with the following property: The Euclidean closure of $F_b(\mathbb{C})$ is a real analytic compact hypersurface of X_b. Especially, $F_b(\mathbb{C})$ is Zariski dense whereas it is not Euclidean dense. X_b is non-Kummer and non-projective for general $b \in B$.

"Degrees of freedom" in our construction

- Choice of C_0, C_0', and a Diophantine line bundle L on C_0 (dimension=1 because of $C_0 \cong C_0'$ and Dioph. condition).
- Choice of points $p_1, p_2, \dots, p_8 \in C_0$ (dimension=8).
- Choice of points $p_1', p_2', \dots, p_8{}' \in C_0'$ (dimension=8).
- Points $p_9 \in C_0$ and $p_9' \in C_0'$ are automatically decided by the condition $N_{C/S} = g^* N_{C'/S'}^{-1} = L$ (dimension=0).
- Choice of an isomorphism $g \colon C \cong C'$ (dimension=1).
- Choice of the "size" of the tab for gluing W^* (dimension=1)

lattice	cycle	$\frac{1}{2\pi\sqrt{-1}}\int\sigma$	corresponding parameter
U	$A_{\beta,\gamma}$	τ	choice of C_0 (and C_0')
	B_α	???	choice of w_j's (R, R',...)
U	$A_{\gamma,\alpha}$	1	—
	B_β	???	choice of w_j's (R, R',...)
$E_8(-1)$	$C_{1,2}$	"$p_2 - p_1$" in C	choice of $p_2 - p_1$
	$C_{2,3}$	"$p_3 - p_2$" in C	choice of $p_3 - p_2$
	$\vdots$	$\vdots$	$\vdots$
	$C_{7,8}$	"$p_8 - p_7$" in C	choice of $p_8 - p_7$
	$C_{6,7,8}$	"$p_6 + p_7 + p_8$" in C	choice of $p_6 + p_7 + p_8$
$E_8(-1)$	$C'_{1,2}$	"$p'_2 - p'_1$" in C'	choice of $p'_2 - p'_1$
	$C'_{2,3}$	"$p'_3 - p'_2$" in C'	choice of $p'_3 - p'_2$
	$\vdots$	$\vdots$	$\vdots$
	$C'_{7,8}$	"$p'_8 - p'_7$" in C'	choice of $p'_8 - p'_7$
	$C'_{6,7,8}$	"$p'_6 + p'_7 + p'_8$" in C	choice of $p'_6 + p'_7 + p'_8$
U	$A_{\alpha,\beta}$	$a_\beta - \tau \cdot a_\alpha$	choice of p_9 and p'_9 (i.e. $N_{C/S}$ and $N_{C'/S'}$)
	B_γ	"$p'_9 - g(p_9)$"	choice of $g\colon C \cong C'$

Question

For the previous example $(C_0, Z = \{p_1, p_2, \ldots, p_9\}, C, S)$*, does* C *admit a holomorphic tubular neighborhood when* $N_{C/S} \in \mathrm{Pic}^0(C)$ *is* ***not*** *Diophantine?*