ON THE EXISTENCE OF KCSC METRICS ON RESOLUTIONS OF ISOLATED SINGULARITIES

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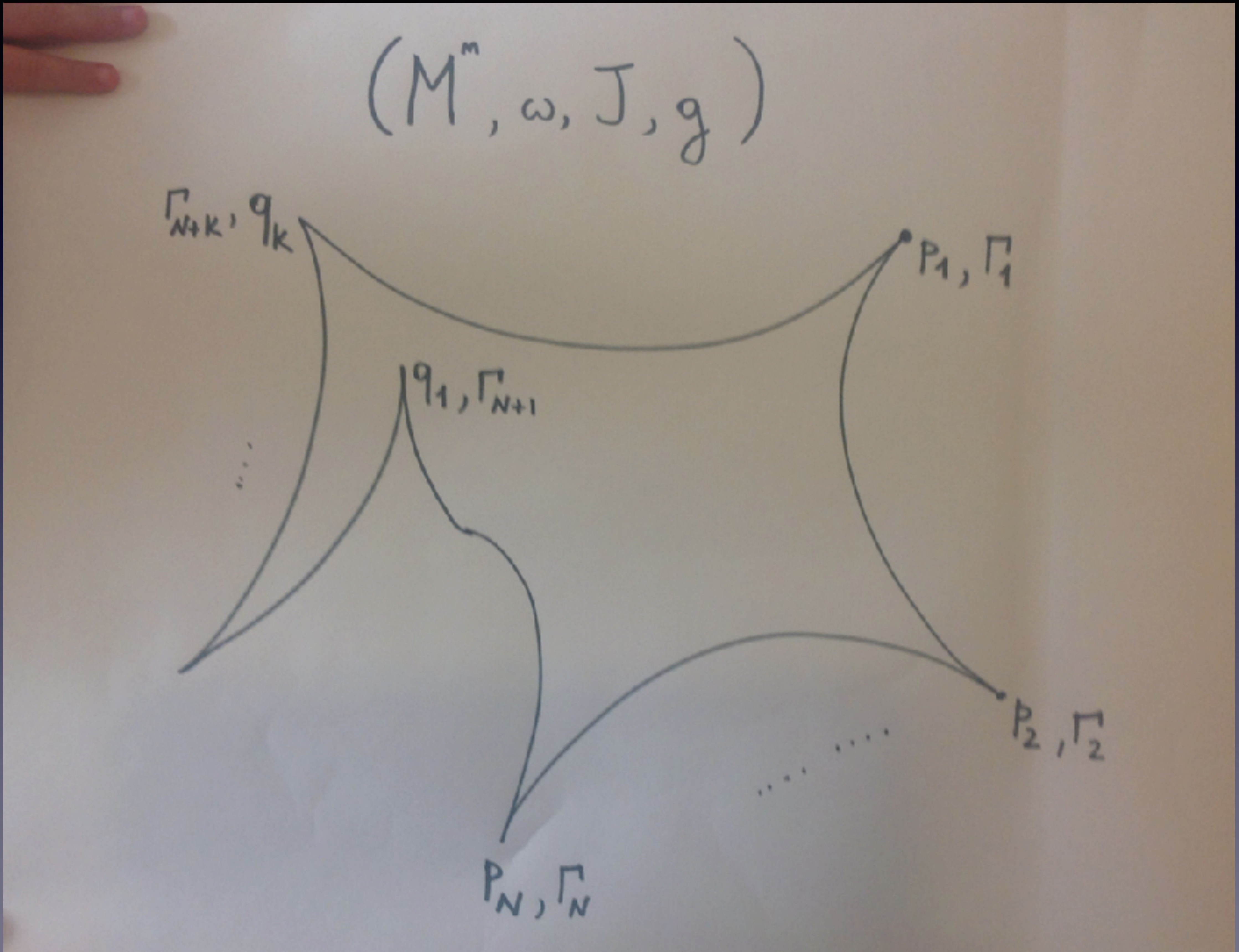
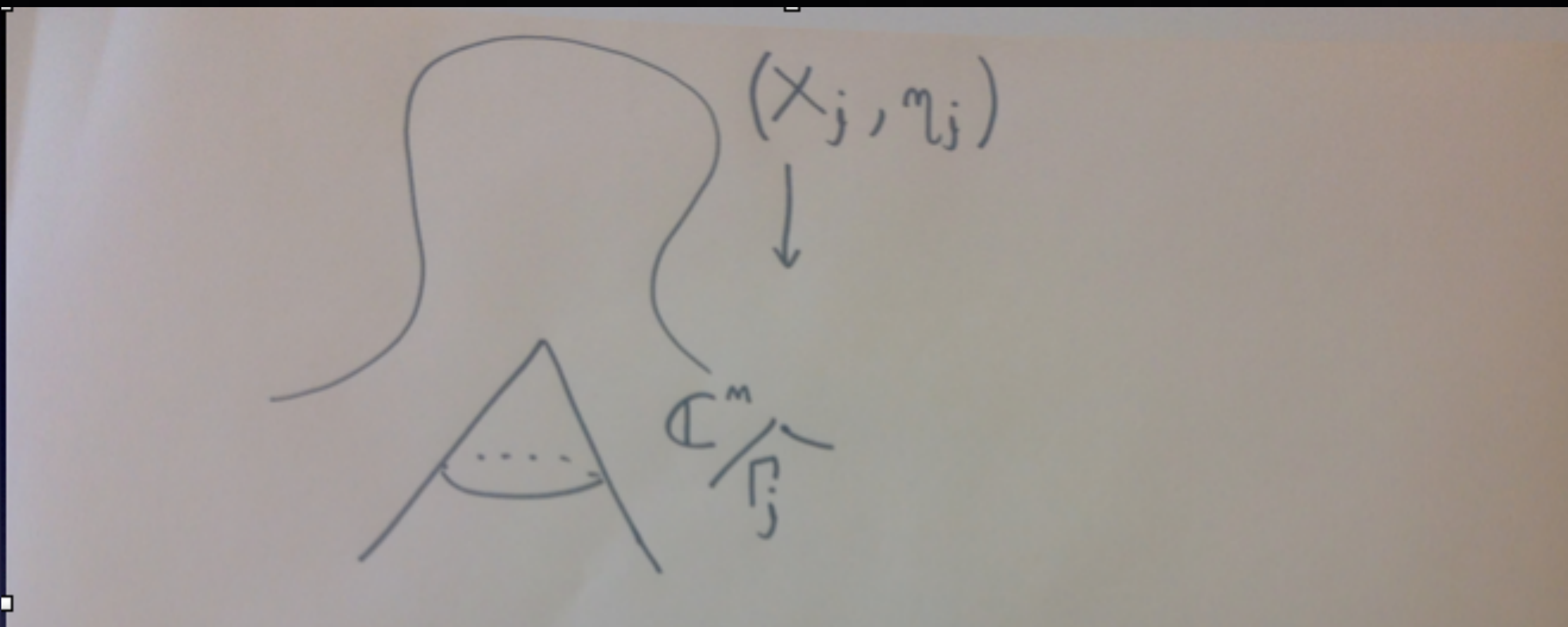
Building blocks: 1) Base Kcsc or extremal Orbifold



2) Local ALE scalar flat resolutions of singularities



We assume the existence of local resolutions:

1) ALE
2) Kähler
3) Scalar flat

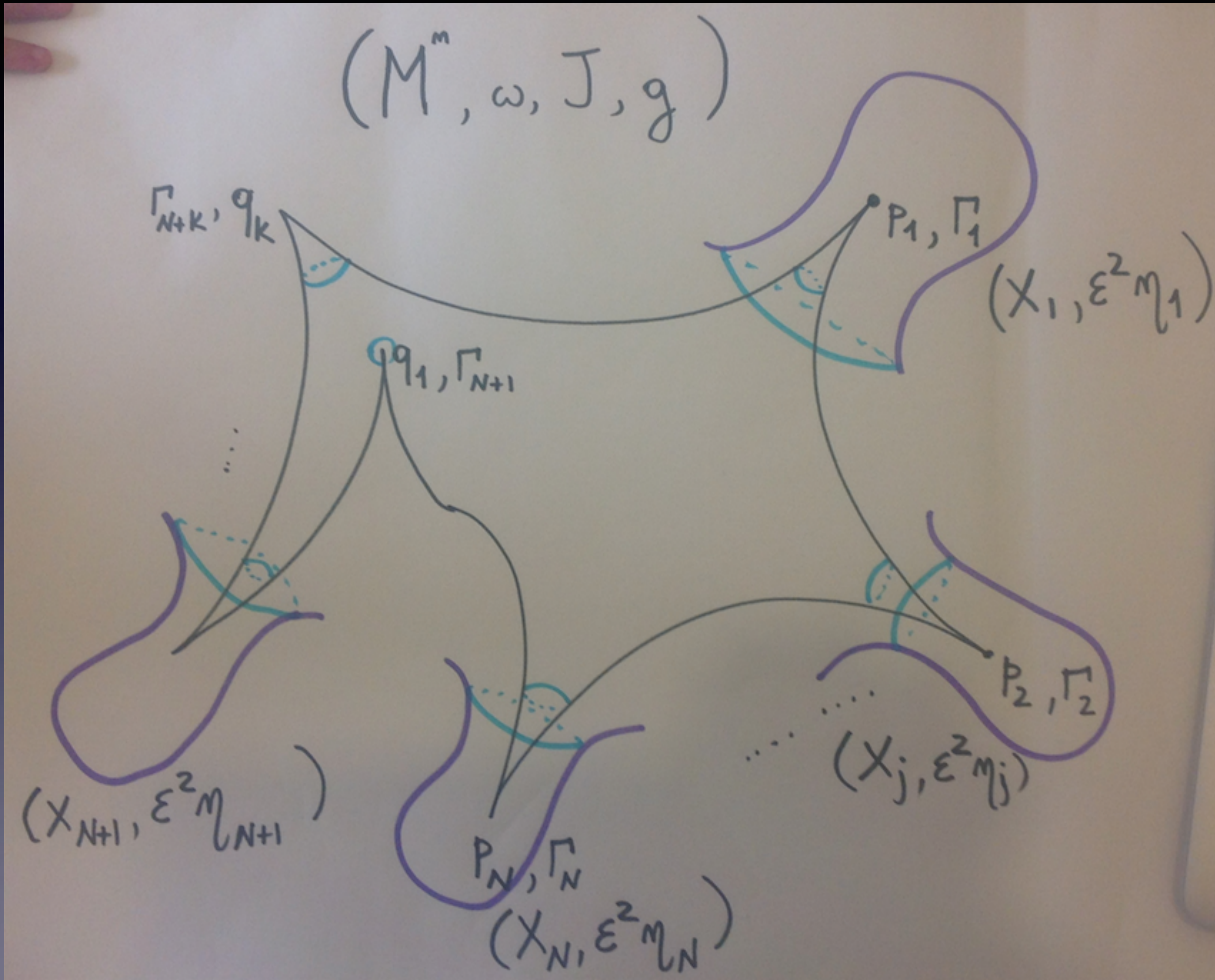
Such models are known to exist for any finite subgroup of U(2) [Kronheimer-Calderbank-Singer-Lock-Viaclovsky] and SU(3) [Nakajima] plus some sporadic examples.

It is well known that such a metric has the following shape

$$\eta \;=\; i\partial\overline{\partial}\left(\frac{|x|^2}{2} + e_{X_\Gamma}\,|x|^{4-2m} - c_{X_\Gamma}\,|x|^{2-2m} + \psi_\eta(x)\right), \qquad \text{with} \qquad \psi_\eta = \mathcal{O}(|x|^{-2m}),$$

for some real constants e_{X_Γ} and c_{X_Γ}. In particular the number e_{X_Γ} is called the ADM mass of the model.

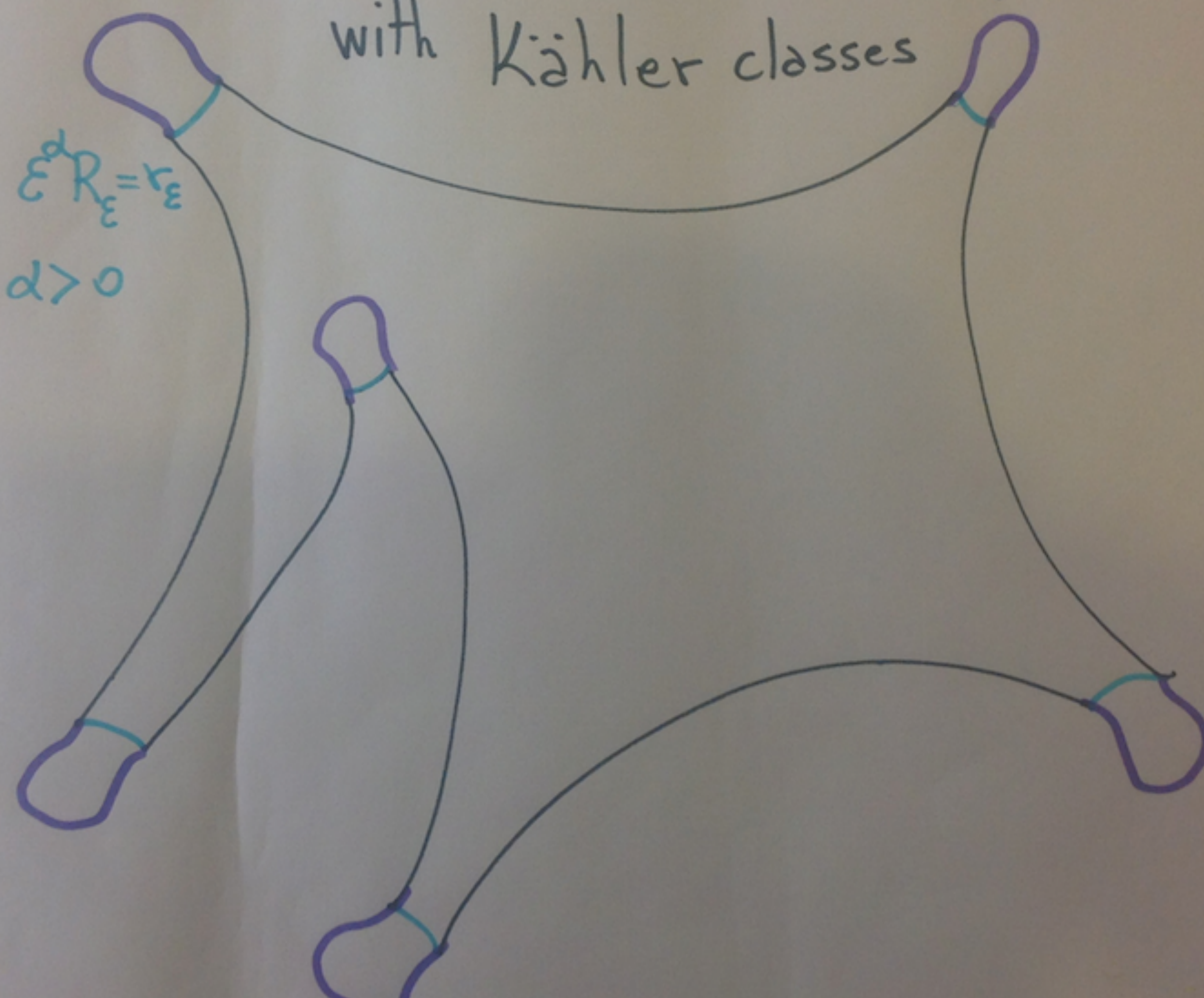
Generalized connected sums



Final output

$$\tilde{M} = M \coprod_{p_{i,\varepsilon}} X_{\Gamma_i} \coprod_{q_{j,\varepsilon}} X_{\Gamma_j}$$

with Kähler classes



The above strategy has been described for the first time in the 70's by the Cambridge physicists applied to the Calabi-Yau equation (KE, Ricci flat), in the case when the base orbifold is a flat torus quotient by an involution.

By this procedure one gets some feeling of how a Ricci-flat metric actually looks like at least on some special K3 surfaces, and it has then been called "Kummer construction".

A rigorous proof in this case was given by Topinwala-LeBrun-Singer.

It was then used in 1997 by Joyce to construct, starting again from quotients of flat 7-tori, the first compact manifolds with holonomy G2 and Spin(7).

The Extremal case A.-Lena-Mazzieri

The extremal case

If g is an extremal metric and X_s its extremal vector field, we denote with $G := Iso_0(M,g) \cap Ham(M,\omega)$ the identity component of the group of Hamiltonian isometries and with $\mathfrak{g}$ its Lie algebra. Moreover we denote with $T \subset G$ the maximal torus whose Lie algebra $\mathfrak{t}$ contains the extremal vector field X_s and $\tilde{T}$ its lift to the resolution.
The key observation is the following:

Theorem

If $X \in \mathfrak{t}$ and denoted with $\tilde{X}$ its lift to X_Γ, we can always find a Hamiltonian potential $\left\langle \mu_\eta, \tilde{X} \right\rangle$ such that $\bar{\partial} \left\langle \mu_\eta, \tilde{X} \right\rangle = \tilde{X} \lrcorner \eta$.

Theorem

Let (M, g, ω) be a compact extremal orbifold with T-invariant metric g and singular points $\{x_1, \ldots, x_S\}$. Then there exists $\bar{\varepsilon}$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ the resolution

$$\tilde{M} := M \sqcup_{x_1,\varepsilon} X_{\Gamma_1} \sqcup_{x_2,\varepsilon} \cdots \sqcup_{x_S,\varepsilon} X_{\Gamma_S}$$

has a $\tilde{T}$-invariant extremal Kähler metric.

This fits well with the deformation theory of LeBrun-Simanca for extremal metrics on smooth manifolds when keeping the complex structure fixed and moving the Kähler class.

- (M, ω, g) a compact m-dimensional Kcsc orbifold with isolated singularities,
- $S = \{p_1, \ldots, p_N\} \subseteq M$ the set of points with neighborhoods biholomorphic to a ball of $\mathbb{C}^m/\Gamma_j$ where, for $j = 1, \ldots, N$, the Γ_j's are nontrivial subgroups of $U(m)$,
- $\mathbb{C}^m/\Gamma_j$ admits an ALE Kahler scalar-flat resolution (X_{Γ_j}, η_j),
- $\pi \colon \tilde{M} \to M$ the resolution of singularities from the generalised connected sum of M minus small balls replaced by copies of (X_p, η_p).

Theorem 1.1. *If* $(\tilde{M}, \pi^* [\omega] + \sum_{j=1}^{N} \varepsilon^{2m} b_j^{2m} [\tilde{\eta}_j])$ *is K-stable, then $\tilde{M}$ has a Kcsc metric in the class* $\pi^* [\omega] + \sum_{j=1}^{N} \varepsilon^{2m} b_j^{2m} [\tilde{\eta}_j]$.

The converse implication (restricting K-stability to smooth test configurations) was proved by Szekelyhidi

The Kcsc case: the PDE approach

The linearized equation

For a smooth real function $f \in C^\infty(M)$ such that $\omega + i\partial\overline{\partial} f > 0$, we set

$$\omega_f \; = \omega + i\partial\overline{\partial} f \, ,$$

Since we want to understand the behavior of the scalar curvature under deformations of this type, it is convenient to consider the following differential operator

$$\mathbf{S}_\omega(\cdot) \, : \, C^\infty(M) \longrightarrow C^\infty(M) \, , \qquad\qquad f \longmapsto \mathbf{S}_\omega(f) := s_{\omega + i\partial\overline{\partial} f} \, ,$$

$$\mathbf{S}_\omega(f) \; = \; s_\omega \, - \, \frac{1}{2} \mathbb{L}_\omega f \, + \, \frac{1}{2} \mathbb{N}_\omega(f) \, ,$$

where the linearized scalar curvature operator $\mathbb{L}_\omega$ is given by

$$\mathbb{L}_\omega f \; = \; \Delta^2_\omega f \, + \, 4 \, \langle \rho_\omega \, | \, i\partial\overline{\partial} f \, \rangle \, .$$

- The subspace of $\ker(\mathbb{L})$ given by the elements with zero mean is in one to one correspondence with the space of holomorphic vector fields which vanish somewhere in M.
- From now on we set

$$\ker\left(\mathbb{L}_\omega\right) = span_{\mathbb{R}} \left\{ 1, \varphi_1, \ldots, \varphi_d \right\} \, .$$

The extremal case is unsensitive of the geometry of the local models. But the Kcsc is not. We need to distinguish two very different cases:

Let $\mathbf{q} := \{q_1, \dots, q_K\} \subseteq M$ *is the set of points with neighborhoods biholomorphic to a ball of* $\mathbb{C}^m/\Gamma_{N+l}$ *such that* $\mathbb{C}^m/\Gamma_{N+l}$ *admits a scalar flat ALE resolution* $(Y_{\Gamma_{N+l}}, k_l, \theta_l)$ *with* $e(Y_{\Gamma_{N+l}}) \neq 0$.

Let $\mathbf{p} = \{p_1, \dots, p_N\} \subseteq M$ *the set of points with neighborhoods biholomorphic to a ball of* $\mathbb{C}^m/\Gamma_j$ *such that* $\mathbb{C}^m/\Gamma_j$ *admits a scalar flat ALE resolution* $\left(X_{\Gamma_j}, h_j, \eta_j\right)$ *with* $e\left(X_{\Gamma_j}\right) = 0$.

If there exist $\mathbf{a} := (a_1, \dots, a_K) \in (\mathbb{R}^+)^K$ *such that*

$$\begin{cases} \sum_{l=1}^{K} a_l e\left(\Gamma_{N+l}\right) \varphi_i\left(q_l\right) = 0 & i = 1, \dots, d \\ \\ \left(a_l e(\Gamma_{N+l}) \varphi_i\left(q_l\right)\right)_{\substack{1 \leq i \leq d \\ 1 \leq l \leq K}} & \textit{has rank } d \end{cases} \tag{1.1}$$

then there exists $\varepsilon_0 > 0$ *such that, for any* $\varepsilon < \varepsilon_0$ *and any* $\mathbf{b} = (b_1, \dots, b_n) \in (\mathbb{R}^+)^N$, *the manifold*

$$\tilde{M} := M \sqcup_{p_1, \varepsilon} X_{\Gamma_1} \sqcup_{p_2, \varepsilon} \cdots \sqcup_{p_N, \varepsilon} X_{\Gamma_N} \sqcup_{q_1, \varepsilon} X_{\Gamma_{N+1}} \sqcup_{q_2, \varepsilon} \cdots \sqcup_{q_{N+K}, \varepsilon} X_{\Gamma_{N+K}},$$

admits a Kcsc metric.

The following improvement of the classical estimates for ALE Ricci-flat is essential to solve the zero mass case:

Asymptotics of scalar flat ALE

Theorem

Let (X_Γ, h, η) be a scalar flat ALE Kähler resolution of an isolated quotient singularitiy. Moreover assume $\Gamma \lhd U(m)$ be nontrivial and $e(\Gamma) = 0$. Then for $R > 0$ large enough, we have that on $X_\Gamma \setminus \pi^{-1}(B_R)$ the Kähler form can be written as

$$\eta \;=\; i\partial\overline{\partial}\left(\frac{|x|^2}{2} - c(\Gamma)\,|x|^{2-2m} + \psi_\eta(x)\right), \qquad \text{with} \qquad \psi_\eta \;=\; \mathcal{O}(|x|^{-2m}),$$

for some positive real constant $c(\Gamma) > 0$. Moreover, the radial component $\psi_\eta^{(0)}$ in the Fourier decomposition of ψ_η is such that

$$\psi_\eta^{(0)}(|x|) = \mathcal{O}\left(|x|^{2-4m}\right).$$

“Zero Mass” Singularities
A.-Della Vedova-Lena-Mazzieri

Theorem 1.2. *Suppose that each (X_p, η_p) has vanishing ADM mass and let $\mathrm{span}_{\mathbb{R}}\{1, \varphi_1, \dots, \varphi_d\}$ be the space of Hamiltonian potentials of Killing fields with zeros. Suppose moreover that for all $p \in S$ there exists $b_p > 0$ such that*

$$\begin{cases} \sum_{p\in S} b_p\,(\Delta_\omega \varphi_j + s_\omega \varphi_j)\,(p) = 0 & j = 1, \dots, d \\ \left((\Delta_\omega \varphi_j + s_\omega \varphi_j)\,(p)\right)_{1\le j\le d, p\in S} & \text{has rank } d. \end{cases}$$

Then there exists $\bar\varepsilon$ such that for every $\varepsilon \in (0, \bar\varepsilon)$ $\tilde M$ has a Kcsc metric in the class

$$\pi^*[\omega] + \sum_{p\in S} \varepsilon^2 \tilde b_p^{2m}\,[\tilde\eta_p] \qquad \text{with} \qquad \mathfrak{i}_j^*\,[\tilde\eta_j] = [\eta_j]$$

where $\mathfrak{i}_j$ the natural embedding of $X_{\Gamma_j, R_\varepsilon}$ into $\tilde M$. Moreover

$$\left| \tilde b_p^{2m} - \frac{|\Gamma_p| b_p}{2\,(m-1)} \right| \le \mathsf{C}\varepsilon^\gamma \qquad \text{for some} \qquad \gamma > 0\,,$$

where $|\Gamma_p|$ denotes the order of the group.

On the Futaki invariant for generalized connected sums A.-Della Vedova-Mazzieri

For any singular p, let η_p be an ALE Kähler metric on the resolution X_p. We will assume that it has the form

$$\eta_p = \xi_p + dd^c\psi_p,$$

where ξ_p is a $(1,1)$-form supported in $\pi_p^{-1}(B(r)/\Gamma_p)$, and ψ_p is a smooth function. Since we constructed $\tilde{M}$ by replacing each singular ball U_p with the resolved ball $\pi_p^{-1}(B(r)/\Gamma_p)$, we can think of each ξ_p as a $(1,1)$-form on $\tilde{M}$. Thus, for all real ε, we can consider the following $(1,1)$-form on $\tilde{M}$

$$\omega_\varepsilon = \pi^*\omega + \varepsilon \sum_{p\in S} \xi_p.$$

Lemma 9.5. *For $\varepsilon > 0$ sufficiently small, ω_ε defines a Kähler metric on $\tilde{M}$.*

Lemma 9.6. *Any holomorphic vector field V on $\tilde{M}$ descends to an holomorphic vector field on $\pi_* V$ on M wich vanishes at all points of S.*

Proof. Since π is a biholomorphism on the complement of $\pi^{-1}(S)$, pushing down the restriction to that set of V defines a vector field V' on $M\setminus S$. Given $p\in S$, the restriction of V' to $U_p\setminus\{p\}$ lifts to a Γ_p-invariant vector field on the punctured ball $B'(r)$ of $\mathbf{C}^n$. By Hartog's theorem such a vector field extends to a holomorphic vector field on the whole ball $B(r)$. Of course such a vector field is Γ_p-invariant, and so it gives a holomorphic vector field on U_p which is equal to V' on $U_p\setminus\{p\}$. Therefore one ends up with a holomorphic vector field $\pi_* V$ on M.

Lemma 9.7. *If V is a holomorphic vector field on $\tilde{M}$, which is Hamiltonian with respect to ω_ε, then $\pi_* V$ is Hamiltonian with respect to ω on M. Moreover, if ϕ_ε and ϕ are Hamiltonian potentials for V and $\pi_* V$ respectively, then one has*

$$\phi_\varepsilon = \pi^*\phi + \varepsilon \sum_{p \in S} \phi_p + c(\varepsilon), \tag{9.14}$$

where ϕ_p is a smooth function supported in $\pi^{-1}(U_p)$ satisfying $d\phi_p = i_V \xi_p$, and $c(\varepsilon)$ is a constant.

Given an holomorphic vector field V on the resolution $\tilde{M}$, and supposing that V is Hamiltonian with respect to ω_ε with potential ϕ_ε, one can form the Futaki invariant

$$\mathrm{Fut}(V, \omega_\varepsilon) = \int_{\tilde{M}} (\phi_\varepsilon - \underline{\phi_\varepsilon}) \frac{\rho_\varepsilon \wedge \omega_\varepsilon^{n-1}}{(n-1)!},$$

where ρ_ε is the Ricci form of ω_ε, and $\underline{\phi_\varepsilon} = \int \phi_\varepsilon \omega_\varepsilon^n / \int \omega_\varepsilon^n$ is the mean value of ϕ_ε with respect to ω_ε.

On the other hand, thanks to Lemmata 9.6 and 9.7, V descends to a holomorphic vector field $\pi_* V$ on M wich is Hamiltonian with respect to ω with potential, say, ϕ. Thus one can also consider the Futaki invariant

$$\mathrm{Fut}(\pi_* V, \omega) = \int_M (\phi - \underline{\phi}) \frac{\rho \wedge \omega^{n-1}}{(n-1)!},$$

where ρ is the Ricci form of ω, and $\underline{\phi} = \int \phi \omega^n / \int \omega^n$. The Futaki invariants $\mathrm{Fut}(V, \omega_\varepsilon)$ and $\mathrm{Fut}(\pi_* V, \omega)$ are related by the following

Theorem 3.1. *As $\varepsilon \to 0$ one has*

$$\mathrm{Fut}(V, \omega_\varepsilon) = \mathrm{Fut}(\pi_* V, \omega) + \varepsilon^{n-1} \sum_{p \in S} (\phi(p) - \underline{\phi}) \int_{X_p} \frac{\rho_p \wedge \xi_p^{n-1}}{(n-1)!}$$
$$- \varepsilon^n \sum_{p \in S} \left(\underline{s}(\phi(p) - \underline{\phi}) + \Delta\phi(p) \right) \int_{X_p} \frac{\xi_p^n}{n!} + O(\varepsilon^{n+1}). \quad (3.1)$$

where ρ_p is the Ricci form of the chosen ALE Kähler metric η_p on the model resolution X_p, and $\underline{s} = n \int \rho \wedge \omega^{n-1} / \int \omega^n$ is the mean scalar curvature of ω.

$$\int_{X_p} \frac{\xi_p^n}{n!} = \lim_{R \to +\infty} \int_{\pi_p^{-1}(B(R)/\Gamma_p)} \frac{\eta_p^n}{n!} - \frac{\pi^n}{n! |\Gamma_p|} \left(R^{2n} - n(n-2) e_p R^2 + n(n-1) c_p \right).$$

$$\int_{X_p} \frac{\rho_p \wedge \xi_p^{n-1}}{(n-1)!} = \int_{X_p} s_p \frac{\eta_p^n}{n!} - \frac{\pi^n e_p}{(n-3)! |\Gamma_p|},$$

Moral: our “balancing conditions” are the first order terms in the expansion of the Futaki invariants of the resolved orbifolds.

So Fut = 0 at first order iff =0 at any order for small perturbation of cohomology classes.

Outputs:

1) New Proofs of the existence Theorems via the extremal construction.

2) Non existence results:

Theorem 1.4. *Under the assumptions of Theorem 1.2, given* $\mathbf{b} \in (\mathbb{R}^+)^N$ *and* $\mathbf{c} \in \mathbb{R}^N$ *such that*

$$\left(b_j \Delta_\omega \varphi_i\left(p_j\right) + c_j \varphi_i\left(p_j\right)\right)_{\substack{1 \le i \le d \\ 1 \le j \le N}} \text{ has rank } d.$$

and

$$\sum_{j=1}^{N} b_j \Delta_\omega \varphi_i\left(p_j\right) + c_j \varphi_i\left(p_j\right) \neq 0 \text{ for some } i = 1, \ldots, d$$

then there exists $\bar{\varepsilon}$ *such that for every* $\varepsilon \in (0, \bar{\varepsilon})$ $\tilde{M}$ *has no Kcsc metric in the class*

$$\pi^*[\omega] + \sum_{j=1}^{N} \varepsilon^{2m} b_j^{2m} \left[\tilde{\eta}_j\right]$$

Applications:

1) Examples:

Example

Consider $(\mathbb{P}^1 \times \mathbb{P}^1, \pi_1^* \omega_{FS} + \pi_2^* \omega_{FS})$ and let $\mathbb{Z}_2$ act in the following way

$$([x_0 : x_1], [y_0 : y_1]) \longrightarrow ([x_0 : -x_1], [y_0 : -y_1])$$

It's immediate to check that this action is in $SU(2)$ with four fixed points

$$\begin{aligned} p_1 &= ([1:0],[1:0]) \\ p_2 &= ([1:0],[0:1]) \\ p_3 &= ([0:1],[1:0]) \\ p_4 &= ([0:1],[0:1]) \end{aligned}$$

The quotient space $X_2 := \mathbb{P}^1 \times \mathbb{P}^1/\mathbb{Z}_2$ is a Kähler-Einstein, Fano orbifold and thanks to the embedding into $\mathbb{P}^4$

$$([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0^2 y_0^2 : x_0^2 y_1^2 : x_1^2 y_0^2 : x_1^2 y_1^2 : x_0 x_1 y_0 y_1]$$

it is isomorphic to the intersection of singular quadrics

$$\left\{z_0 z_3 - z_4^2 = 0\right\} \cap \left\{z_1 z_2 - z_4^2 = 0\right\}$$

Example

Consider $(\mathbb{P}^2, \omega_{FS})$ and let $\mathbb{Z}_3$ act in the following way

$$[z_0 : z_1 : z_2] \longrightarrow [x_0 : \zeta_3 x_1 : \zeta_3^2 x_2] \qquad \zeta_3 \neq 1, \zeta_3^3 = 1$$

It's immediate to check that this action is in $SU(2)$ with three fixed points

$$\begin{aligned} p_1 &= [1:0:0] \\ p_2 &= [0:1:0] \\ p_3 &= [0:0:1] \end{aligned}$$

noindent The quotient space $X_3 := \mathbb{P}^2/\mathbb{Z}_3$ is a Kähler-Einstein, Fano orbifold and it is isomorphic, via the embedding

$$[x_0 : x_1 : x_2] \mapsto [x_0^3 : x_1^3 : x_2^3 : x_0 x_1 x_2],$$

to the singular cubic surface in $\mathbb{P}^3$

$$\left\{z_0 z_1 z_2 - z_3^3 = 0\right\}.$$

Example 8.5. Let Y be the toric Kähler-Einstein threefold whose 1-dimensional fan Σ_1 is generated by points

$$\Sigma_1 = \{(2, -1, 0), (1, 3, 1), (0, 0, 1), (-3, -2, -2)\}$$

and its 3-dimensional fan Σ_3 is generated by 6 cones

$$\begin{aligned} C_1 &:= \langle (1, 3, 1), (0, 0, 1), (-3, -2, -2), \rangle \\ C_2 &:= \langle (2, -1, 0), (0, 0, 1), (-3, -2, -2) \rangle \\ C_3 &:= \langle (2, -1, 0), (1, 3, 1), (-3, -2, -2) \rangle \\ C_4 &:= \langle (2, -1, 0), (1, 3, 1), (0, 0, 1) \rangle \end{aligned}$$

The cone C_1 is relative to affine open subsets of Y containing a $SU(3)$ singularity and the other cones are relative to affine open subsets of Y containing a $U(3)$ singularity.

The 7-anticanonical polytope $P_{-7K_{X^{(6)}}}$ is the convex hull of vertices

$$P_{-7K_{X^{(6)}}} := \langle (1, 9, -7), (-3, 1, -7), (9, -3, -7), (-7, -7, 21) \rangle$$

With 2-faces

$$\begin{aligned} F_1 &:= \langle (-3, 1, -7), (9, -3, -7), (-7, -7, 21) \rangle \\ F_2 &:= \langle (1, 9, -7), (9, -3, -7), (-7, -7, 21) \rangle \\ F_3 &:= \langle (1, 9, -7), (-3, 1, -7), (-7, -7, 21) \rangle \\ F_4 &:= \langle (1, 9, -7), (-3, 1, -7), (9, -3, -7) \rangle \end{aligned}$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-7K_{X^{(6)}}}$

$$C_1 \longleftrightarrow F_1 \cap F_2 \cap F_4 = \{(9, -3, -7)\}$$

It is now clear that this example does not satisfy either the balancing condition on the $SU(3)$ point, nor the one found in [2] on the remaining 3 $U(3)$ singularities. So even if a local model exists (and we do not know if this is indeed the case) Theorem 1.2 shows that Kcsc metrics on its resolution do not exist in the adiabatic classes.

Extensions to non-compact spaces

(work in progress with C. Spotti)

Theorem 0.1. *Let $\Gamma \lhd U(m)$ finite acting freely on $\mathbb{S}^{2m-1}$, let $(X_\Gamma, \omega_\Gamma)$ be a scalar-flat ALE Kähler manifold such that there is a compact $K \subset X_\Gamma$ such that*

$$X_\Gamma \setminus K \simeq (\mathbb{C}^m \setminus B_R)/\Gamma .$$

where $\simeq$ stands for biholomorphic. Let $q \in X_\Gamma$, then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ $Bl_q X_\Gamma$, the blow up at q of X_Γ, carries a scalar-flat ALE Kähler metric $\tilde{\omega}_a$ in the class

$$[\tilde{\omega}_a] := [\omega_\Gamma] - \varepsilon^{2m-2} a \left[c_1\left(\mathcal{O}\left(E_q\right)\right)\right] \qquad \forall a > 0$$

with E_q the exceptional divisor. Moreover $m\left(Bl_q X_\Gamma, \tilde{\omega}_a\right)$, the mass of $Bl_q X_\Gamma$, is a small perturbation of $m\left(X_\Gamma, \omega_\Gamma\right)$ i.e.

$$\lim_{\varepsilon \to 0} m\left(Bl_q X_\Gamma, \tilde{\omega}_a\right) = m\left(X_\Gamma, \omega_\Gamma\right) .$$

Theorem 0.2. *Let $\Gamma \lhd U(m)$ finite acting freely on $\mathbb{S}^{2m-1}$, let $(X_\Gamma, \omega_\Gamma)$ be a scalar-flat ALE Kähler orbifold with isolated singular points such that there is a compact $K \subset X_\Gamma$ such that*

$$X_\Gamma \setminus K \simeq (\mathbb{C}^m \setminus B_R)/\Gamma .$$

Let $q \in K$ be a singular point with nontirvial (finite) local orbifold group $G \lhd U(m)$. Suppose there is a scalar-flat ALE Kähler manifold (X_G, η) that is a resolution of $\mathbb{C}^m/G$. Then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ $\tilde{X}_\Gamma$, the orbifold obtained by gluing topologically $X_\Gamma \setminus q$ and X_G , carries a scalar-flat ALE Kähler metric $\tilde{\omega}_a$ in the class

$$[\tilde{\omega}_a] := [\omega_\Gamma] + \varepsilon^{2m-2\alpha} a [\tilde{\eta}] \qquad \forall a > 0$$

with $\alpha = 1$ if $m(X_G, \eta) \neq 0$ and $\alpha = 0$ if $m(X_G, \eta) = 0$ and $[\tilde{\eta}] \in H^2\left(\tilde{X}_\Gamma, \mathbb{R}\right)$ induced by $[\eta]$ via the natural embedding in $\tilde{X}_\Gamma$ of a neighborhood of the exceptional locus of X_G. Moreover $m\left(\tilde{X}_\Gamma, \tilde{\omega}_a\right)$, the mass of $\tilde{X}_\Gamma$, is a small perturbation of $m\left(X_\Gamma, \omega_\Gamma\right)$ i.e.

$$\lim_{\varepsilon \to 0} m\left(\tilde{X}_\Gamma, \tilde{\omega}_a\right) = m\left(X_\Gamma, \omega_\Gamma\right) .$$

Interesting problem:
how does this construction
depend on the distance
from the exceptional divisor?