Crystal combinatorics from Lusztig's PBW bases

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Representation Theory, Mathematical Physics and Integrable Systems
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[1]Includes work with John Claxton, Jackson Criswell, Ben Salisbury and Adam Schultze

Outline

1. What (for this talk) are canonical bases?
2. ‘Elementary’ construction of canonical and crystal bases from PBW bases
3. Extracting explicit combinatorics!
4. Relation to more standard combinatorics

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- All is basically due to Lusztig.

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- For each reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$, Lusztig defines an order

$$\alpha_{i_1} = \beta_1 < \beta_2 < \ldots < \beta_N$$

on positive roots of $\mathfrak{g}$, and elements F_{β_j} in $U_q^-(\mathbf{g})_{\beta_j}$:

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- Fact: $\{F_{\beta_1}^{(a_1)} \cdots F_{\beta_N}^{(a_N)}\}$ is a basis of $U_q^-(\mathbf{g})$.
- This can be thought of as a first attempt at defining a canonical basis, but it isn't canonical, since you get a different basis $B_{\mathbf{i}}$ for each expression $\mathbf{i}$ of w_0.

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- So we have a canonical $\mathbb{Z}[q]$ lattice $\mathcal{L}$ and basis for $\mathcal{L}/q\mathcal{L}$! These coincide with Kashiwara's crystal lattice/basis (see Grojnowski-Lusztig, Saito). So for combinatorial goals we are already good!

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- For this audience I will also explain how to get the actual canonical basis.

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Furthermore, the change of basis from any $\mathbf{B_i}$ *to* $\mathbf{B}$ *is unit-triangular.*

- You get the same basis starting with any PBW basis! So now we have a single chosen basis $\mathbf{B}$! This is Lusztig's canonical basis.

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- Wow, that was easy!
- In some sense that is the biggest punch line of this talk...but I'm kind of a combinatorist, so let's think about how to do combinatorics from this perspective.

Crystal operators for $\mathfrak{sl}_3$

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- The interesting calculations are in relating the two PBW bases.

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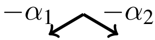

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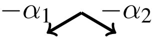
Relating the two PBW bases for $\mathfrak{sl}_3$



- Recall $\quad F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \quad \mod q.$

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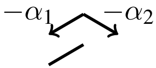

- Recall $\quad F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \quad \bmod q.$

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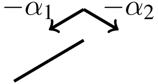

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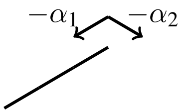

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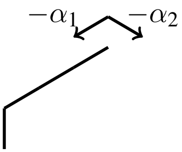

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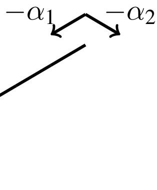

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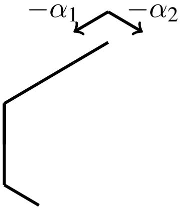
Relating the two PBW bases for $\mathfrak{sl}_3$



- Recall $\quad F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \quad \mod q.$

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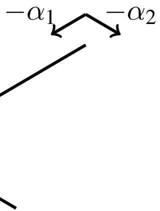

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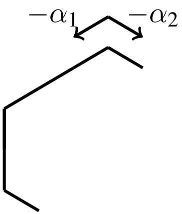

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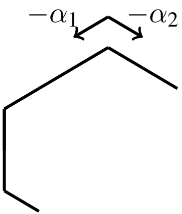

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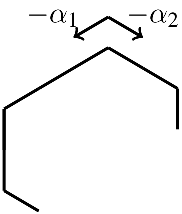

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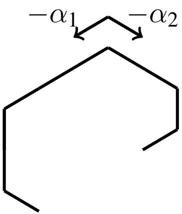

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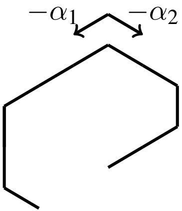

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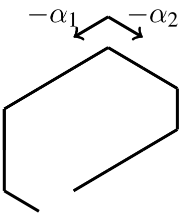

- Recall $\quad F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \quad \mod q.$

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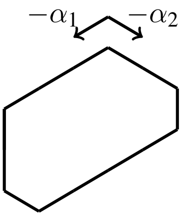

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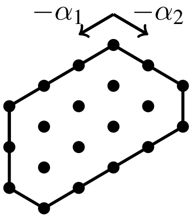

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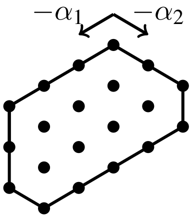
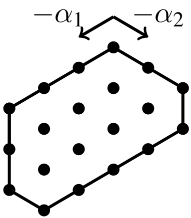


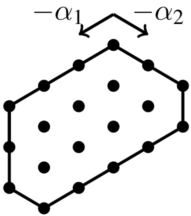
- Recall $\quad F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \quad \mod q$.
- The polygons that show up this way are exactly those where the horizontal width is the max of the two diagonal widths.

Relating the two PBW bases for $\mathfrak{sl}_3$



- Recall $\quad F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \quad \text{mod } q.$
- The polygons that show up this way are exactly those where the horizontal width is the max of the two diagonal widths.
- Given one side can easily figure out the other.

Relating the two PBW bases for $\mathfrak{sl}_3$



- Recall $F_2^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}F_1^{(3)} = F_1^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}F_2^{(2)} \mod q.$
- The polygons that show up this way are exactly those where the horizontal width is the max of the two diagonal widths.
- Given one side can easily figure out the other.
- So we can explicitly relate the two PBW bases, and hence we can apply crytsal operators as in the previous slide.

Crystal operators for $\mathfrak{sl}_4$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

$$s_1 \qquad s_2 \qquad s_3 \qquad s_1 \qquad s_2 \qquad s_1$$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

s_1	s_2	s_3	s_1	s_2	s_1
α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

	s_1	s_2	s_3	s_1	s_2	s_1
	α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3
Take $b =$	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

	s_1	s_2	s_3	s_1	s_2	s_1
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Take $b =$	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

$$
\begin{array}{ccccccc}
 & s_1 & s_2 & s_3 & s_1 & s_2 & s_1 \\
 & \alpha_1 & (\alpha_1{+}\alpha_2) & (\alpha_1{+}\alpha_2{+}\alpha_3) & \alpha_2 & (\alpha_2{+}\alpha_3) & \alpha_3 \\
\text{Take } b = & F_1^{(2)} & F_{12}^{(3)} & F_{123}^{(1)} & F_2^{(3)} & F_{23}^{(3)} & F_3^{(2)} \\
 & F_1^{(2)} & F_{12}{}^{(3)} & F_{123}^{(1)} & \color{blue}{F_3^{(3)}} & \color{blue}{F_{32}^{(2)}} & \color{blue}{F_2^{(4)}}
\end{array}
$$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

	s_1	s_2	s_3	s_1	s_2	s_1
	α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3
Take $b =$	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(3)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

	s_1	s_2	s_3	s_1	s_2	s_1
	α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3
Take $b =$	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$
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	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
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$$
\begin{array}{rcccccc}
 & s_1 & s_2 & s_3 & s_1 & s_2 & s_1 \\
 & \alpha_1 & (\alpha_1+\alpha_2) & (\alpha_1+\alpha_2+\alpha_3) & \alpha_2 & (\alpha_2+\alpha_3) & \alpha_3 \\
\text{Take } b = & F_1^{(2)} & F_{12}^{(3)} & F_{123}^{(1)} & F_2^{(3)} & F_{23}^{(3)} & F_3^{(2)} \\
 & F_1^{(2)} & F_{12}{}^{(3)} & F_{123}^{(1)} & F_3^{(3)} & F_{32}^{(2)} & F_2^{(4)} \\
 & F_1^{(2)} & F_3{}^{(1)} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)} \\
 & F_3^{(1)} & F_1{}^{(2)} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)} \\
f_3 b = & F_3^{(2)} & F_1{}^{(2)} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)}
\end{array}
$$

Crystal operators for $\mathfrak{sl}_4$

- We apply f_3 to a $b \in B(\infty)$, using PBW basis for $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$

$$
\begin{array}{rcccccc}
 & s_1 & s_2 & s_3 & s_1 & s_2 & s_1 \\
 & \alpha_1 & (\alpha_1+\alpha_2) & (\alpha_1+\alpha_2+\alpha_3) & \alpha_2 & (\alpha_2+\alpha_3) & \alpha_3 \\
\text{Take } b = & F_1^{(2)} & F_{12}^{(3)} & F_{123}^{(1)} & F_2^{(3)} & F_{23}^{(3)} & F_3^{(2)} \\
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\end{array}
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$$
\begin{array}{rcccccc}
 & s_1 & s_2 & s_3 & s_1 & s_2 & s_1 \\
 & \alpha_1 & (\alpha_1+\alpha_2) & (\alpha_1+\alpha_2+\alpha_3) & \alpha_2 & (\alpha_2+\alpha_3) & \alpha_3 \\
\text{Take } b = & F_1^{(2)} & F_{12}^{(3)} & F_{123}^{(1)} & F_2^{(3)} & F_{23}^{(3)} & F_3^{(2)} \\
 & F_1^{(2)} & F_{12}^{(3)} & F_{123}^{(1)} & F_3^{(3)} & F_{32}^{(2)} & F_2^{(4)} \\
 & F_1^{(2)} & F_3^{(1)} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)} \\
 & F_3^{(1)} & F_1^{(2)} & F_{312}^{(3)} & F_{12}^{(1)} & F_{32}^{(2)} & F_2^{(4)} \\
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	s_1	s_2	s_3	s_1	s_2	s_1
	α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3
Take $b =$	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(3)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_3^{(1)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
$f_3 b =$	$F_3^{(2)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(4)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(2)}$	$F_{23}^{(4)}$	$F_3^{(2)}$

Crystal operators 3: using segments/Kostant partitions

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$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

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$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

$$\begin{array}{cccccccccccccc}
 & & & & & \boxed{3} & & & & & & & & \\
 & & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & & & & \boxed{3} & \boxed{3} & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}$$

Crystal operators 3: using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

$$\boxed{1}\ \boxed{1}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ \boxed{2}\ \boxed{2}\ \boxed{2}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \boxed{3}\ \boxed{3}$$

$$\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \boxed{1}\ \boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{2}\ \boxed{3}\ \boxed{3}$$

Crystal operators 3: using segments/Kostant partitions

$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$

					3								
		2	2	2	2				3	3	3		
1	1	1	1	1	1	2	2	2	2	2	2	3	3

f_3

3													
2	2	2	2	3	3	3							
1	1	1	1	2	2	2	1	1	2	2	2	3	3

Crystal operators 3: using segments/Kostant partitions

	$F_1^{(2)}$		$F_{12}{}^{(3)}$			$F_{123}^{(1)}$	$F_2^{(3)}$			$F_{23}^{(3)}$			$F_3^{(2)}$	
	$\boxed{1}$	$\boxed{1}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

Crystal operators 3: using segments/Kostant partitions

	$F_1^{(2)}$		$F_{12}{}^{(3)}$			$F_{123}^{(1)}$	$F_2^{(3)}$			$F_{23}^{(3)}$			$F_3^{(2)}$	
	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{3}$ $\boxed{2}$ $\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\boxed{3}$ $\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

Crystal operators 3: using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

	1	1	2 1	2 1	2 1	3 2 1	2	2	2	3 2	3 2	3 2	3	3
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	3 2 1	2 1	2 1	2 1	3 2	3 2	3 2	1	1	3 2	2	2	3	3

Crystal operators 3: using segments/Kostant partitions

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$$

	$\boxed{1}$	$\boxed{1}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\\\boxed{1}\end{array}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\begin{array}{c}\boxed{3}\\\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\boxed{1}$	$\boxed{1}$	$\begin{array}{c}\boxed{3}\\\boxed{2}\end{array}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(2)} \quad F_{23}^{(4)} \quad F_3^{(2)}$$

Crystal operators 3: using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

						$\boxed{3}$								
			$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$				$\boxed{3}$	$\boxed{3}$	$\boxed{3}$		
	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\boxed{3}$													
	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$			$\boxed{3}$				
	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(2)} \qquad F_{23}^{(4)} \qquad F_3^{(2)}$$

Crystal operators 3: using segments/Kostant partitions

$F_1^{(2)}$ $\quad F_{12}(3)$ $\quad F_{123}^{(1)}$ $\quad F_2^{(3)}$ $\quad F_{23}^{(3)}$ $\quad F_3^{(2)}$

	$\boxed{1}$	$\boxed{1}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			$\color{red}{(}$	(	(	)	)
	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\boxed{1}$	$\boxed{1}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

$F_1^{(2)}$ $\quad F_{12}(3)$ $\quad F_{123}^{(1)}$ $\quad \color{blue}{F_2^{(2)}}$ $\quad \color{blue}{F_{23}^{(4)}}$ $\quad F_3^{(2)}$

- Get a bracketing rule as long as each α_i can be moved to left with all 3-term moves involving α_i.

Crystal operators 3: using segments/Kostant partitions

$$F_1^{(2)} \quad F_{12}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$$

	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$
f_3	)	(	(	(	)	)	)			$\color{red}{(}$	(	(	)	)
	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$

$$F_1^{(2)} \quad F_{12}^{(3)} \quad F_{123}^{(1)} \quad \color{blue}{F_2^{(2)}} \quad \color{blue}{F_{23}^{(4)}} \quad F_3^{(2)}$$

- Get a bracketing rule as long as each α_i can be moved to left with all 3-term moves involving α_i. A reduced expression with this property exists in all types except E_8 and F_4 (see Littelmann "Cones, crystals and patterns").

Crystal operators 3: using segments/Kostant partitions

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$$

	1	1	$\begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}3\\2\\1\end{smallmatrix}$	2	2	2	$\begin{smallmatrix}3\\2\end{smallmatrix}$	$\begin{smallmatrix}3\\2\end{smallmatrix}$	$\begin{smallmatrix}3\\2\end{smallmatrix}$	3	3
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\begin{smallmatrix}3\\2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}3\\2\end{smallmatrix}$	$\begin{smallmatrix}3\\2\end{smallmatrix}$	$\begin{smallmatrix}3\\2\end{smallmatrix}$	1	1	$\begin{smallmatrix}3\\2\end{smallmatrix}$	2	2	3	3

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(2)} \quad F_{23}^{(4)} \quad F_3^{(2)}$$

- Get a bracketing rule as long as each α_i can be moved to left with all 3-term moves involving α_i. A reduced expression with this property exists in all types except E_8 and F_4 (see Littelmann “Cones, crystals and patterns"). There are non-trivially different such expressions.

Crystal operators using type D Kostant partition

Crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

Crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

Crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \begin{smallmatrix}2\\1\end{smallmatrix} \quad \begin{smallmatrix}3\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}3\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}3\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}3\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}4\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}4\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}3\,4\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}2\\3\,4\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}2\\3\,4\\2\\1\end{smallmatrix} \quad \begin{smallmatrix}2\\3\,4\\2\\1\end{smallmatrix} \quad 2 \quad 2 \quad 2 \quad \begin{smallmatrix}4\\2\end{smallmatrix} \quad \begin{smallmatrix}3\\2\end{smallmatrix} \quad \begin{smallmatrix}3\\2\end{smallmatrix} \quad \begin{smallmatrix}3\,4\\2\end{smallmatrix} \quad 3 \quad 3$$

f_3

Crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \;\; 1 \;\; \substack{2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{4\\2\\1} \;\; \substack{4\\2\\1} \;\; \substack{3\,4\\2\\1} \;\; \substack{2\\3\,4\\2\\1} \;\; \substack{2\\3\,4\\2\\1} \;\; \substack{2\\3\,4\\2\\1} \;\; 2 \;\; 2 \;\; 2 \;\; \substack{4\\2} \;\; \substack{3\\2} \;\; \substack{3\\2} \;\; \substack{3\,4\\2} \;\; 3 \;\; 3$$

f_3

Crystal operators using type D Kostant partition

$$F_1^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_2^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_3^{(2)} \quad F_4^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{3\,4\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

Crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{34\\2\\1}}^{(1)} \quad F_{\substack{2\\34\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{34\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{34\\2\\1} \quad \substack{2\\34\\2\\1} \quad \substack{2\\34\\2\\1} \quad \substack{2\\34\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{34\\2} \quad 3 \quad 3$$

$$) \quad) \quad) \quad) \quad (\quad) \quad (\quad (\quad) \quad) \quad (\quad (\quad (\quad) \quad (\quad) \quad) \quad (\ldots$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{34\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{34\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

Crystal operators using type D Kostant partition

$$F_1^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_2^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_3^{(2)} \quad F_4^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$) \quad) \quad) \quad) \quad (\quad) \quad (\quad (\quad) \quad) \quad (\quad (\quad (\quad) \quad (\quad) \quad) \quad (\ldots$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{3\,4\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

Crystal operators using type D Kostant partition

$$F_1^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_2^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_3^{(2)} \quad F_4^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$) \quad) \quad) \quad) \quad (\quad) \quad (\quad (\quad) \quad) \quad (\quad (\quad (\quad) \quad (\quad) \quad) \quad (\ldots$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{3\,4\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

Crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{34\\2\\1}}^{(1)} \quad F_{\substack{2\\34\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{34\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \;\; 1 \;\; \substack{2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{4\\2\\1} \;\; \substack{4\\2\\1} \;\; \substack{34\\2\\1} \;\; \substack{2\\34\\2\\1} \;\; \substack{2\\34\\2\\1} \;\; \substack{2\\34\\2\\1} \;\; 2 \;\; 2 \;\; 2 \;\; \substack{4\\2} \;\; \substack{3\\2} \;\; \substack{3\\2} \;\; \substack{34\\2} \;\; 3 \;\; 3$$

$$) \;\;) \;\;) \;\;) \;\; (\;\;) \;\; (\;\; (\;\;) \;\;) \;\; \color{red}{(} \;\; (\;\; (\;\;) \;\; (\;\;) \;\;) \;\; (\ldots$$

$$f_3 \quad \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{3\\2\\1} \;\; \substack{2\\1} \;\; \substack{34\\2\\1} \;\; \substack{4\\2\\1} \;\; \substack{4\\2\\1} \;\; \substack{3\\2} \;\; \substack{3\\2} \;\; 2 \;\; 2 \;\; 2 \;\; \substack{34\\2} \;\; \substack{4\\2} \;\; 3 \;\; 3$$

Crystal operators using type D Kostant partition

$$F_1^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_2^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_3^{(2)} \quad F_4^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$\begin{array}{lcccccccccccccccccc} &) &) &) &) & (&) & (& (&) &) & (& (& (&) & (&) &) & (\ldots \\ f_3 & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{2\\1} & \substack{3\,4\\2\\1} & \substack{4\\2\\1} & \substack{4\\2\\1} & \substack{3\\2} & \substack{3\\2} & 2 & 2 & 2 & \substack{3\,4\\2} & \substack{4\\2} & 3 & 3 & \end{array}$$

$$F_1^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_2^{(2)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(3)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_3^{(2)} \quad F_4^{(0)}$$

Relation to other combinatorics

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

1	1	2	3	3	4
2	3	3	4		
3	4	4			

$$\longrightarrow$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3}\end{array} \qquad \longrightarrow \qquad \boxed{1}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3} \end{array} \qquad \longrightarrow \qquad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \; \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline 2&3&3&4\\ \cline{1-4} 3&4&4\\ \cline{1-3}\end{array} \qquad \longrightarrow \qquad \begin{array}{c} \\ \boxed{1} \end{array} \; \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} \; \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{cccccc} 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 & & \\ 3 & 4 & 4 & & & \end{array} \longrightarrow \begin{array}{cccc} & & & 3 \\ & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline 2&3&3&4 & \multicolumn{2}{c}{} \\ \cline{1-4} 3&4&4 & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array} \quad \longrightarrow \quad \begin{array}{ccccc} &&&\boxed{3}&\\ &\boxed{2}&\boxed{2}&\boxed{2}&\\ \boxed{1}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

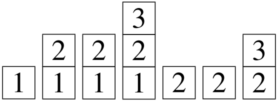
$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline 2&3&3&4&&\\ \cline{1-4} 3&4&4&&&\\ \cline{1-3}\end{array} \qquad \longrightarrow \qquad \begin{array}{cccccc} & & & \boxed{3} & & \\ & \boxed{2} & \boxed{2} & \boxed{2} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

1	1	2	3	3	4
2	3	3	4		
3	4	4			

$\longrightarrow$



Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

1	1	2	3	3	4
2	3	3	4		
3	4	4			

$\longrightarrow$

			3				
	2	2	2			3	
1	1	1	1	2	2	2	3

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

1	1	2	3	3	4
2	3	3	4		
3	4	4			

$\longrightarrow$

			3					
	2	2	2			3		
1	1	1	1	2	2	2	3	3

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\ \cline{1-4} 3&4&4\\ \cline{1-3}\end{array} \qquad \longrightarrow \qquad \begin{array}{ccccccccc} &&&\boxed{3}&&&&&\\ &\boxed{2}&\boxed{2}&\boxed{2}&&&\boxed{3}&&\\ \boxed{1}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}&\boxed{2}&\boxed{2}&\boxed{3}&\boxed{3}\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\ \hline
2 & 3 & 3 & 4 \\ \cline{1-4}
3 & 4 & 4 \\ \cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccccccc}
 & & & \boxed{3} & & & & & \\
 & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\ \hline
2 & 2 & 2 & \overline{4} & \overline{2} \\ \cline{1-5}
3 & 4 \\ \cline{1-2}
\end{array}
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
\\ \\ \boxed{1}
\end{array}
\begin{array}{c}
\\ \boxed{2} \\ \boxed{1}
\end{array}
\begin{array}{c}
\\ \boxed{2} \\ \boxed{1}
\end{array}
\begin{array}{c}
\boxed{3} \\ \boxed{2} \\ \boxed{1}
\end{array}
\begin{array}{c}
\\ \\ \boxed{2}
\end{array}
\begin{array}{c}
\\ \\ \boxed{2}
\end{array}
\begin{array}{c}
\\ \boxed{3} \\ \boxed{2}
\end{array}
\begin{array}{c}
\\ \\ \boxed{3}
\end{array}
\begin{array}{c}
\\ \\ \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \bar{3} & \bar{1} \\
\hline
2 & 2 & 2 & \bar{4} & \bar{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\quad \longrightarrow
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline 2&3&3&4\\ \cline{1-4} 3&4&4\\ \cline{1-3}\end{array} \quad\longrightarrow\quad \begin{array}{|c|}\hline 1\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}\ \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ \hline\end{array}\ \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array}\ \begin{array}{|c|}\hline 3\\ \hline\end{array}\ \begin{array}{|c|}\hline 3\\ \hline\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&\color{red}{3}&\color{red}{\bar{3}}&\bar{1}\\ \hline 2&2&2&\bar{4}&\bar{2}\\ \cline{1-5} 3&4\\ \cline{1-2}\end{array} \quad\longrightarrow$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3}\end{array} \quad\longrightarrow\quad \begin{array}{c}\\ \\ \boxed{1}\end{array}\ \begin{array}{c}\\ \boxed{2}\\ \boxed{1}\end{array}\ \begin{array}{c}\\ \boxed{2}\\ \boxed{1}\end{array}\ \begin{array}{c}\boxed{3}\\ \boxed{2}\\ \boxed{1}\end{array}\ \begin{array}{c}\\ \\ \boxed{2}\end{array}\ \begin{array}{c}\\ \\ \boxed{2}\end{array}\ \begin{array}{c}\\ \boxed{3}\\ \boxed{2}\end{array}\ \begin{array}{c}\\ \\ \boxed{3}\end{array}\ \begin{array}{c}\\ \\ \boxed{3}\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&\color{red}{3}&\color{red}{\bar{3}}&\bar{1}\\\hline 2&2&2&\bar{4}&\bar{2}\\\cline{1-5} 3&4\\\cline{1-2}\end{array} \quad\longrightarrow\quad \begin{array}{cc}3&4\\2&2\\1&1\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccccccc}
 & & & \boxed{3} & & & & & \\
 & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\
\hline
2 & 2 & 2 & \overline{4} & \overline{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc}
3 & 4 \\
2 & 2 \\
1 & 1
\end{array}
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\longrightarrow
\begin{array}{c}
\\ \\
\boxed{1}
\end{array}
\begin{array}{c}
\\
\boxed{2} \\
\boxed{1}
\end{array}
\begin{array}{c}
\\
\boxed{2} \\
\boxed{1}
\end{array}
\begin{array}{c}
\boxed{3} \\
\boxed{2} \\
\boxed{1}
\end{array}
\begin{array}{c}
\\ \\
\boxed{2}
\end{array}
\begin{array}{c}
\\ \\
\boxed{2}
\end{array}
\begin{array}{c}
\\
\boxed{3} \\
\boxed{2}
\end{array}
\begin{array}{c}
\\ \\
\boxed{3}
\end{array}
\begin{array}{c}
\\ \\
\boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \bar{3} & \color{red}{\bar{1}} \\
\hline
2 & 2 & 2 & \bar{4} & \bar{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\longrightarrow
\begin{array}{cc}
3 & 4 \\
2 & 2 \\
1 & 1
\end{array}
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline 2&3&3&4\\ \cline{1-4} 3&4&4\\ \cline{1-3}\end{array} \longrightarrow \begin{array}{ccccccccc} &&&\boxed{3}&&&&&\\ &\boxed{2}&\boxed{2}&\boxed{2}&&&\boxed{3}&&\\ \boxed{1}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}&\boxed{2}&\boxed{2}&\boxed{3}&\boxed{3}\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&3&\overline{3}&\color{red}{\overline{1}}\\ \hline 2&2&2&\overline{4}&\overline{2}\\ \cline{1-5} 3&4\\ \cline{1-2}\end{array} \longrightarrow \begin{array}{cccccc} &&&2&&\\ 3&4&3&&4&\\ 2&2&&2&&\\ 1&1&&1&&1\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3} \end{array} \longrightarrow \begin{array}{ccccccccc} &&&3&&&&& \\ &2&2&2&&&3&& \\ 1&1&1&1&2&2&2&3&3 \end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\ \hline 2 & 2 & 2 & \overline{4} & \overline{2} \\ \cline{1-5} 3 & 4 \\ \cline{1-2} \end{array} \longrightarrow \begin{array}{ccccccc} &&&&2&& \\ 3&&4&3&&4& \\ 2&&2&&2&& \\ 1&&1&&1&&1 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3}\end{array} \quad \longrightarrow \quad \begin{array}{ccccccccc} & & & \boxed{3} & & & & & \\ & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} \end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\ \hline 2 & 2 & 2 & \overline{4} & \overline{2} \\ \cline{1-5} 3 & 4 \\ \cline{1-2}\end{array} \quad \longrightarrow \quad \begin{array}{cccccc} & & & 2 & & \\ 3 & 4 & 3 & & 4 & \\ 2 & 2 & & 2 & & \\ 1 & 1 & & 1 & & 1 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3}\end{array} \quad \longrightarrow \quad \begin{array}{ccccccccc} & & & \boxed{3} & & & & & \\ & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} \end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&\color{red}{2}&3&3&\bar{3}&\bar{1}\\\hline 2&2&2&\bar{4}&\bar{2}\\\cline{1-5} 3&4\\\cline{1-2}\end{array} \quad \longrightarrow \quad \begin{array}{ccccc} & & 2 & & \\ 3 & 4 & 3\;4 & & \\ 2 & 2 & 2 & & \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3}\end{array} \longrightarrow \begin{array}{ccccccccc} & & & \boxed{3} & & & & & \\ & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} \end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \bar{3} & \bar{1} \\ \hline 2 & 2 & 2 & \bar{4} & \bar{2} \\ \cline{1-5} 3 & 4 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{ccccc} & & 2 & & \\ 3 & 4 & 3\;4 & & \\ 2 & 2 & 2 & & \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

1	1	2	3	3	4
2	3	3	4		
3	4	4			

$\longrightarrow$

			3					
	2	2	2			3		
1	1	1	1	2	2	2	3	3

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

1	1	1	1	1	1	2	3	3	$\bar{3}$	$\bar{1}$
2	2	2	$\bar{4}$	$\bar{2}$						
3	4									

$\longrightarrow$

		2		
3	4	3 4		
2	2	2		
1	1	1	1	1

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\ \hline 2&3&3&4\\ \cline{1-4} 3&4&4\\ \cline{1-3}\end{array} \quad\longrightarrow\quad \begin{array}{ccccccccc} &&&\boxed{3}&&&&&\\ &\boxed{2}&\boxed{2}&\boxed{2}&&&\boxed{3}&&\\ \boxed{1}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}&\boxed{2}&\boxed{2}&\boxed{3}&\boxed{3}\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&\color{red}{3}&3&\overline{3}&\overline{1}\\ \hline 2&2&2&\overline{4}&\overline{2}\\ \cline{1-5} 3&4\\ \cline{1-2}\end{array} \quad\longrightarrow\quad \begin{array}{cccccccc} &&&2&&&&\\ 3&4&3&&4&&&\\ 2&2&&2&&&&2\\ 1&1&&1&&1&1&1\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\longrightarrow
\begin{array}{ccccccccc}
 & & & \boxed{3} & & & & & \\
 & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \bar{3} & \bar{1} \\
\hline
2 & 2 & 2 & \bar{4} & \bar{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\longrightarrow
\begin{array}{cccccccc}
 & & & 2 & & & & \\
3 & 4 & 3 & & 4 & & & \\
2 & 2 & & 2 & & & & 2 \\
1 & 1 & & 1 & & 1 & 1 & 1
\end{array}
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3}\end{array} \longrightarrow \begin{array}{ccccccccc} &&&3&&&&&\\ &2&2&2&&&3&&\\ 1&1&1&1&2&2&2&3&3\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&3&\overline{3}&\overline{1}\\\hline 2&2&2&\overline{4}&\color{red}{\overline{2}}\\\cline{1-5} 3&4\\\cline{1-2}\end{array} \longrightarrow \begin{array}{cccccc} &&2&&&\\ 3&4&3\;\,4&&&\\ 2&2&2&&&2\\ 1&1&1&1&1&1\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

1	1	2	3	3	4
2	3	3	4		
3	4	4			

$\longrightarrow$

$$\begin{array}{ccccccccc} & & & \boxed{3} & & & & & \\ & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} \end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

1	1	1	1	1	1	2	3	3	$\bar{3}$	$\bar{1}$
2	2	2	$\bar{4}$	$\bar{2}$						
3	4									

$\longrightarrow$

$$\begin{array}{cccccccccccc} & & & 2 & & & & & & & & \\ 3 & 4 & 3 & & 4 & & & & & & & \\ 2 & 2 & & 2 & & & & 2 & 3 & & 4 & \\ 1 & 1 & & 1 & & 1 & 1 & 1 & & 2 & & 2 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccccccc}
 & & & \boxed{3} & & & & & \\
 & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\
\hline
2 & 2 & 2 & \overline{4} & \overline{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\quad \longrightarrow \quad
\begin{array}{cccccccccccc}
 & & & 2 & & & & & & & & \\
3 & 4 & 3 & & 4 & & & & & & & \\
2 & 2 & & 2 & & & & 2 & 3 & & 4 & \\
1 & 1 & & 1 & & 1 & 1 & 1 & & 2 & & 2
\end{array}
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccccccc}
 & & & \boxed{3} & & & & & \\
 & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \bar{3} & \bar{1} \\
\hline
2 & 2 & 2 & \color{red}{\bar{4}} & \bar{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\quad \longrightarrow \quad
\begin{array}{cccccccccccc}
 & & & 2 & & & & & & & & \\
3 & 4 & 3 & & 4 & & & & & & & \\
2 & 2 & & 2 & & & & 2 & 3 & & 4 & \\
1 & 1 & & 1 & & 1 & 1 & 1 & & 2 & & 2
\end{array}
$$

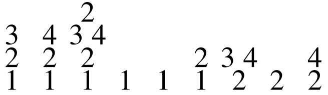
Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3}\end{array} \longrightarrow \boxed{1}\ \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ \boxed{2}\ \boxed{2}\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \boxed{3}\ \boxed{3}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&3&\bar{3}&\bar{1}\\\hline 2&2&2&\color{red}{\bar{4}}&\bar{2}\\\cline{1-5} 3&4\\\cline{1-2}\end{array} \longrightarrow$$



Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3}\end{array} \longrightarrow \begin{array}{ccccccccc} &&&\boxed{3}&&&&&\\ &\boxed{2}&\boxed{2}&\boxed{2}&&&\boxed{3}&&\\ \boxed{1}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}&\boxed{2}&\boxed{2}&\boxed{3}&\boxed{3}\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&3&\bar{3}&\bar{1}\\\hline 2&2&2&\bar{4}&\bar{2}\\\cline{1-5} 3&4\\\cline{1-2}\end{array} \longrightarrow \begin{array}{ccccccccc} &&2&&&&&&\\ 3&4&3\;\;4&&&&&&\\ 2&2&2&&&2&3\;4&&4\\ 1&1&1&1&1&1&2&2&2\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3}\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline\end{array}\ \begin{array}{|c|}\hline 2\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline\end{array}\ \begin{array}{|c|}\hline 3\\\hline\end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&3&3&\bar{3}&\bar{1}\\\hline 2&2&2&\bar{4}&\bar{2}\\\cline{1-5} 3&\color{red}{4}\\\cline{1-2}\end{array} \longrightarrow \begin{array}{c}3\\2\\1\end{array}\ \begin{array}{c}4\\2\\1\end{array}\ \begin{array}{c}2\\3\ \ 4\\2\\1\end{array}\ \begin{array}{c}1\end{array}\ \begin{array}{c}1\end{array}\ \begin{array}{c}2\\1\end{array}\ \begin{array}{c}3\ \ 4\\2\end{array}\ \begin{array}{c}2\end{array}\ \begin{array}{c}4\\2\end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 \\ \cline{1-3}\end{array} \quad \longrightarrow \quad \begin{array}{ccccccccc} & & & \boxed{3} & & & & & \\ & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} \end{array}$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\ \hline 2 & 2 & 2 & \overline{4} & \overline{2} \\ \cline{1-5} 3 & {\color{red}4} \\ \cline{1-2}\end{array} \quad \longrightarrow \quad \begin{array}{cccccccccccccc} & & & 2 & & & & & & & & & & \\ 3 & 4 & 3 & & 4 & & & & & & & & & \\ 2 & 2 & & 2 & & & & 2 & 3 & & 4 & & 4 & \\ 1 & 1 & & 1 & & 1 & 1 & 1 & & 2 & & 2 & 2 & 3 \end{array}$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\
\hline
2 & 3 & 3 & 4 \\
\cline{1-4}
3 & 4 & 4 \\
\cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccccccc}
 & & & \boxed{3} & & & & & \\
 & \boxed{2} & \boxed{2} & \boxed{2} & & & \boxed{3} & & \\
\boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{1} \\
\hline
2 & 2 & 2 & \overline{4} & \overline{2} \\
\cline{1-5}
3 & 4 \\
\cline{1-2}
\end{array}
\quad \longrightarrow \quad
\begin{array}{cccccccccccccc}
 & & & 2 & & & & & & & & & & \\
3 & 4 & 3 & & 4 & & & & & & & & & \\
2 & 2 & & 2 & & & & 2 & 3 & & 4 & & 4 & \\
1 & 1 & & 1 & & 1 & 1 & 1 & & 2 & & 2 & 2 & 3
\end{array}
$$

Relation to other combinatorics

- Type A (see Claxton-T...although previously known):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 3 & 4 \\ \hline
2 & 3 & 3 & 4 \\ \cline{1-4}
3 & 4 & 4 \\ \cline{1-3}
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
\\ \\ \boxed{1}
\end{array}
\begin{array}{c}
\\ \boxed{2} \\ \boxed{1}
\end{array}
\begin{array}{c}
\\ \boxed{2} \\ \boxed{1}
\end{array}
\begin{array}{c}
\boxed{3} \\ \boxed{2} \\ \boxed{1}
\end{array}
\begin{array}{c}
\\ \\ \boxed{2}
\end{array}
\begin{array}{c}
\\ \\ \boxed{2}
\end{array}
\begin{array}{c}
\\ \boxed{3} \\ \boxed{2}
\end{array}
\begin{array}{c}
\\ \\ \boxed{3}
\end{array}
\begin{array}{c}
\\ \\ \boxed{3}
\end{array}
$$

- Type D: See Salisbury-Schultze-T.
  Must use large tableau for b (i.e. B_λ for λ large).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \bar{3} & \bar{1} \\ \hline
2 & 2 & 2 & \bar{4} & \bar{2} \\ \cline{1-5}
3 & 4 \\ \cline{1-2}
\end{array}
\quad \longrightarrow \quad
\begin{array}{cccccccccc}
 & & 2 & & & & & & & \\
3 & 4 & 3\;4 & & & & & & & \\
2 & 2 & 2 & & & 2\;3\;4 & & & 4 & \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3
\end{array}
$$

- Type B and C: Fairly similar to type D. See Criswell-Salisbury-T.

Some citation information

Some citation information

1. Lusztig.

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2. My notes “An elementary construction of Lusztig’s canonical bases," published in a Contemporary Mathematics conference proceedings.

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1. Lusztig.
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3. "Combinatorial descriptions of the crystal structure on certain PBW bases" with Ben Salisbury and Adam Schultze. For general results on combinatorics/bracketing rules from PBW bases.
4. Explicit descriptions of combinatorics and relations to tableaux are in a series of three papers: “Young Tableaux, Multisegments, and PBW Bases" with Claxton for type A, “PBW bases and marginally large tableaux in type D" with Salisbury an Schultze, and “PBW bases and marginally large tableaux in types B and C" with Criswell and Salisbury. Sorry, there are some good reasons it got split up...

Thanks!!!

Thanks!!!
And happy birthday Kolya!!!!!