Capelli eigenvalue problem for Lie superalgebras and supersymmetric polynomials

joint with S. Sahi, H. Salmasian

Lumini, June 7, 2018

Let $\mathfrak{g}$ be a reductive Lie (super)algebra and V be a spherical representation of $\mathfrak{g}$.

$$\mathbb{C}[V] \simeq \mathrm{Sym}(V^*) = \bigoplus_{\lambda\in\Lambda} V_\lambda.$$

Let $\mathcal{D}(V)$ denote the algebra of polynomial differential operators on V and $\mathcal{D}(V)^{\mathfrak{g}}$ the algebra of invariant operators.

$$\mathcal{D}(V) \simeq \mathrm{Sym}(V^*) \otimes \mathrm{Sym}(V) = \bigoplus_{\lambda,\mu\in\Lambda} V_\lambda \otimes V_\mu^*,$$

$$\mathcal{D}(V)^{\mathfrak{g}} = \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda) = \bigoplus_{\lambda\in\Lambda} \mathbb{C}\iota_\lambda.$$

The Capelli basis: $\{\iota_\lambda \mid \lambda \in \Lambda\}$.

Let $\mathfrak{g}$ be a reductive Lie (super)algebra and V be a spherical representation of $\mathfrak{g}$.

$$\mathbb{C}[V] \simeq \mathrm{Sym}(V^*) = \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

Let $\mathcal{D}(V)$ denote the algebra of polynomial differential operators on V and $\mathcal{D}(V)^\mathfrak{g}$ the algebra of invariant operators.

$$\mathcal{D}(V) \simeq \mathrm{Sym}(V^*) \otimes \mathrm{Sym}(V) = \bigoplus_{\lambda, \mu \in \Lambda} V_\lambda \otimes V_\mu^*,$$

$$\mathcal{D}(V)^\mathfrak{g} = \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_\mathfrak{g}(V_\lambda, V_\lambda) = \bigoplus_{\lambda \in \Lambda} \mathbb{C}\iota_\lambda.$$

The Capelli basis: $\{\iota_\lambda \mid \lambda \in \Lambda\}$.

Let $\mathfrak{g}$ be a reductive Lie (super)algebra and V be a spherical representation of $\mathfrak{g}$.

$$\mathbb{C}[V] \simeq \mathrm{Sym}(V^*) = \bigoplus_{\lambda\in\Lambda} V_\lambda.$$

Let $\mathcal{D}(V)$ denote the algebra of polynomial differential operators on V and $\mathcal{D}(V)^{\mathfrak{g}}$ the algebra of invariant operators.

$$\mathcal{D}(V) \simeq \mathrm{Sym}(V^*) \otimes \mathrm{Sym}(V) = \bigoplus_{\lambda,\mu\in\Lambda} V_\lambda \otimes V_\mu^*,$$

$$\mathcal{D}(V)^{\mathfrak{g}} = \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda) = \bigoplus_{\lambda\in\Lambda} \mathbb{C}\iota_\lambda.$$

The Capelli basis: $\{\iota_\lambda \mid \lambda \in \Lambda\}$.

By Schur's lemma

$$\iota_\lambda|_{V_\mu} = c_\lambda(\mu)\,\mathrm{Id},$$

one can show that $P_\lambda := c_\lambda(\mu)$ is a polynomial function on Λ.

- Kostant, Sahi: $P_\lambda = \varphi_\lambda(x+\rho)$ for some symmetric polynomial φ_λ.
- Okounkov, Olshansky: Defined uniquely (up to rescaling) by symmetry and vanishing property:

$$\varphi_\lambda(\mu) = 0 \quad \text{unless} \quad \lambda \subseteq \mu,$$

  λ, μ are partitions.
- Knop–Sahi, Okounkov–Olshansky: Shifted Jack polynomials $P_\lambda(x,\theta)$

$$P_\lambda(\mu,\theta) = \begin{cases} h(\lambda) \text{ if } \mu = \lambda \\ 0 \text{ if } |\mu| \leq |\lambda|, \lambda \neq \mu. \end{cases}$$

- Jack polynomials are eigenfunctions of CMS operators.

By Schur's lemma

$$\iota_\lambda|_{V_\mu} = c_\lambda(\mu)\,\mathrm{Id},$$

one can show that $P_\lambda := c_\lambda(\mu)$ is a polynomial function on Λ.

- Kostant, Sahi: $P_\lambda = \varphi_\lambda(x+\rho)$ for some symmetric polynomial φ_λ.
- Okounkov, Olshansky: Defined uniquely (up to rescaling) by symmetry and vanishing property:

$$\varphi_\lambda(\mu) = 0 \quad \text{unless} \quad \lambda \subseteq \mu,$$

  λ, μ are partitions.
- Knop–Sahi, Okounkov–Olshansky: Shifted Jack polynomials $P_\lambda(x,\theta)$

$$P_\lambda(\mu,\theta) = \begin{cases} h(\lambda) \text{ if } \mu = \lambda \\ 0 \text{ if } |\mu| \leq |\lambda|, \lambda \neq \mu. \end{cases}$$

- Jack polynomials are eigenfunctions of CMS operators.

By Schur's lemma

$$\iota_\lambda|_{V_\mu} = c_\lambda(\mu)\,\mathrm{Id},$$

one can show that $P_\lambda := c_\lambda(\mu)$ is a polynomial function on Λ.

- Kostant, Sahi: $P_\lambda = \varphi_\lambda(x+\rho)$ for some symmetric polynomial φ_λ.
- Okounkov, Olshansky: Defined uniquely (up to rescaling) by symmetry and vanishing property:

$$\varphi_\lambda(\mu) = 0 \quad \text{unless} \quad \lambda \subseteq \mu,$$

  λ, μ are partitions.
- Knop–Sahi, Okounkov–Olshansky: Shifted Jack polynomials $P_\lambda(x,\theta)$

$$P_\lambda(\mu,\theta) = \begin{cases} h(\lambda) \text{ if } \mu = \lambda \\ 0 \text{ if } |\mu| \leq |\lambda|, \lambda \neq \mu. \end{cases}$$

- Jack polynomials are eigenfunctions of CMS operators.

By Schur's lemma

$$\iota_\lambda|_{V_\mu} = c_\lambda(\mu)\,\mathrm{Id},$$

one can show that $P_\lambda := c_\lambda(\mu)$ is a polynomial function on Λ.

- Kostant, Sahi: $P_\lambda = \varphi_\lambda(x+\rho)$ for some symmetric polynomial φ_λ.
- Okounkov, Olshansky: Defined uniquely (up to rescaling) by symmetry and vanishing property:

$$\varphi_\lambda(\mu) = 0 \quad \text{unless} \quad \lambda \subseteq \mu,$$

  λ, μ are partitions.
- Knop–Sahi, Okounkov–Olshansky: Shifted Jack polynomials $P_\lambda(x,\theta)$

$$P_\lambda(\mu,\theta) = \begin{cases} h(\lambda) \text{ if } \mu = \lambda \\ 0 \text{ if } |\mu| \leq |\lambda|, \lambda \neq \mu. \end{cases}$$

- Jack polynomials are eigenfunctions of CMS operators.

By Schur's lemma

$$\iota_\lambda|_{V_\mu} = c_\lambda(\mu)\,\mathrm{Id},$$

one can show that $P_\lambda := c_\lambda(\mu)$ is a polynomial function on Λ.

- Kostant, Sahi: $P_\lambda = \varphi_\lambda(x+\rho)$ for some symmetric polynomial φ_λ.
- Okounkov, Olshansky: Defined uniquely (up to rescaling) by symmetry and vanishing property:

$$\varphi_\lambda(\mu) = 0 \quad \text{unless} \quad \lambda \subseteq \mu,$$

  λ, μ are partitions.
- Knop–Sahi, Okounkov–Olshansky: Shifted Jack polynomials $P_\lambda(x,\theta)$

$$P_\lambda(\mu,\theta) = \begin{cases} h(\lambda) \text{ if } \mu = \lambda \\ 0 \text{ if } |\mu| \leq |\lambda|, \lambda \neq \mu. \end{cases}$$

- Jack polynomials are eigenfunctions of CMS operators.

Sergeev and Veselov developed theory of super Jack polynomials, which are polynomials of two independent vector variables $(x_1, \dots, x_m | y_1, \dots, y_n)$ which can be considered as Frobenius coordinates on the set of $H(m, n)$ of (m, n)-hook partitions, those are partitions which can be covered by an infinite (m, n)-hook, or equivalently do not contain a $(m+1) \times (n+1)$-rectangle as a subdiagram.

$H(m, n)$ enumerate polynomial representations of $\mathfrak{gl}(m|n)$.

With Sahi and Salmasian we study Cappeli problem for superalgebras with reductive even part. We show that the polynomials $P_\lambda(\mu)$ in this case are either shifted super Jack polynomials for different values of parameter θ or so called factorial Schur polynomials studied previously by Olshansky, Okounkov and Ivanov. Those are related to representation theory of the Lie superalgebra $Q(n)$ and Sergeev duality for projective representations of symmetric group.

Sergeev and Veselov developed theory of super Jack polynomials, which are polynomials of two independent vector variables $(x_1, \ldots, x_m | y_1, \ldots, y_n)$ which can be considered as Frobenius coordinates on the set of $H(m, n)$ of (m, n)-hook partitions, those are partitions which can be covered by an infinite (m, n)-hook, or equivalently do not contain a $(m+1) \times (n+1)$-rectangle as a subdiagram.
$H(m, n)$ enumerate polynomial representations of $\mathfrak{gl}(m|n)$.
With Sahi and Salmasian we study Cappeli problem for superalgebras with reductive even part. We show that the polynomials $P_\lambda(\mu)$ in this case are either shifted super Jack polynomials for different values of parameter θ or so called factorial Schur polynomials studied previously by Olshansky, Okounkov and Ivanov. Those are related to representation theory of the Lie superalgebra $Q(n)$ and Sergeev duality for projective representations of symmetric group.

Sergeev and Veselov developed theory of super Jack polynomials, which are polynomials of two independent vector variables $(x_1, \dots, x_m | y_1, \dots, y_n)$ which can be considered as Frobenius coordinates on the set of $H(m, n)$ of (m, n)-hook partitions, those are partitions which can be covered by an infinite (m, n)-hook, or equivalently do not contain a $(m+1) \times (n+1)$-rectangle as a subdiagram.
$H(m, n)$ enumerate polynomial representations of $\mathfrak{gl}(m|n)$.
With Sahi and Salmasian we study Cappeli problem for superalgebras with reductive even part. We show that the polynomials $P_\lambda(\mu)$ in this case are either shifted super Jack polynomials for different values of parameter θ or so called factorial Schur polynomials studied previously by Olshansky, Okounkov and Ivanov. Those are related to representation theory of the Lie superalgebra $Q(n)$ and Sergeev duality for projective representations of symmetric group.

Four classical series of Lie superalgebras

$E = E_0 \oplus E_1$ be a vector superspace of dimension $(m|n)$.

- The general linear Lie superalgebra $\mathfrak{gl}(m,n) = \mathrm{End}_{\mathbb{C}}(E)$ with the supercommutator $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX$.
- Let $\Pi \colon E \to E$ be odd and $\Pi^2 = 1$ The Lie superalgebra $Q(n)$:

$$Q(n) = \{X \in \mathrm{End}(E) \mid X\Pi = \Pi X\}$$

.

- The ortho-symplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ preserves a non-degenerate even symmetric form b on E.
- The Lie superalgebra $\mathfrak{p}(n)$ preserves a non-degenerate odd symmetric form β on E, $\dim E = (n|n)$.

Four classical series of Lie superalgebras

$E = E_0 \oplus E_1$ be a vector superspace of dimension $(m|n)$.

- The general linear Lie superalgebra $\mathfrak{gl}(m,n) = \mathrm{End}_{\mathbb{C}}(E)$ with the supercommutator $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX$.
- Let $\Pi\colon E \to E$ be odd and $\Pi^2 = 1$ The Lie superalgebra $Q(n)$:

$$Q(n) = \{X \in \mathrm{End}(E) \mid X\Pi = \Pi X\}$$

.

- The ortho-symplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ preserves a non-degenerate even symmetric form b on E.
- The Lie superalgebra $\mathfrak{p}(n)$ preserves a non-degenerate odd symmetric form β on E, $\dim E = (n|n)$.

Four classical series of Lie superalgebras

$E = E_0 \oplus E_1$ be a vector superspace of dimension $(m|n)$.

- The general linear Lie superalgebra $\mathfrak{gl}(m,n) = \mathrm{End}_{\mathbb{C}}(E)$ with the supercommutator $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX$.
- Let $\Pi \colon E \to E$ be odd and $\Pi^2 = 1$ The Lie superalgebra $Q(n)$:

$$Q(n) = \{X \in \mathrm{End}(E) \mid X\Pi = \Pi X\}$$

.

- The ortho-symplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ preserves a non-degenerate even symmetric form b on E.
- The Lie superalgebra $\mathfrak{p}(n)$ preserves a non-degenerate odd symmetric form β on E, $\dim E = (n|n)$.

Four classical series of Lie superalgebras

$E = E_0 \oplus E_1$ be a vector superspace of dimension $(m|n)$.

- The general linear Lie superalgebra $\mathfrak{gl}(m,n) = \mathrm{End}_{\mathbb{C}}(E)$ with the supercommutator $[X,Y] = XY - (-1)^{\bar{X}\bar{Y}}YX$.
- Let $\Pi\colon E \to E$ be odd and $\Pi^2 = 1$ The Lie superalgebra $Q(n)$:

$$Q(n) = \{X \in \mathrm{End}(E) \mid X\Pi = \Pi X\}$$

.

- The ortho-symplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ preserves a non-degenerate even symmetric form b on E.
- The Lie superalgebra $\mathfrak{p}(n)$ preserves a non-degenerate odd symmetric form β on E, $\dim E = (n|n)$.

Four classical series of Lie superalgebras

$E = E_0 \oplus E_1$ be a vector superspace of dimension $(m|n)$.

- The general linear Lie superalgebra $\mathfrak{gl}(m,n) = \mathrm{End}_{\mathbb{C}}(E)$ with the supercommutator $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX$.
- Let $\Pi\colon E \to E$ be odd and $\Pi^2 = 1$ The Lie superalgebra $Q(n)$:

$$Q(n) = \{X \in \mathrm{End}(E) \mid X\Pi = \Pi X\}$$

.

- The ortho-symplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ preserves a non-degenerate even symmetric form b on E.
- The Lie superalgebra $\mathfrak{p}(n)$ preserves a non-degenerate odd symmetric form β on E, $\dim E = (n|n)$.

Schur–Weyl–Sergeev duality.

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, define $S_\lambda(E)$ by

$$E^{\otimes d} = \bigoplus_{\lambda \in H(m,n), |\lambda| = d} S_\lambda(E) \boxtimes Y_\lambda.$$

Y_λ is the irreducible representation of $\mathbb{S}_d$. $H(m,n)$ is the set of partitions fitting in the fat (m,n)-hook.

$\mathfrak{g} = Q(n)$, define $SQ_\lambda(E)$ by

$$E^{\otimes d} = \bigoplus_{\lambda \in DP(n), |\lambda| = d} SQ_\lambda(E) \boxtimes Z_\lambda.$$

Here $DP(n)$ is the set of strict partitions $\lambda_1 > \lambda_2 > \dots \lambda_k > 0$ with $k \leq n$.
Z_λ irreducible representation of $U_d := \mathbb{C}[\mathbb{S}_d] \ltimes \mathrm{Cliff}_d$. U_d is Morita equivalent to $\mathbb{C}[\hat{\mathbb{S}}_d]/(z+1)$.
$\hat{\mathbb{S}}_d$ is the double cover of the symmetric group.

Back to Capelli problem

How to get a good source of examples of $(\mathfrak{g}, V)$ where V is a spherical representation of $\mathfrak{g}$?

Kantor–Koeher–Tits construction of Jordan superalgebras.

$\mathfrak{G}$: Lie superalgebra,
$\{e, h, f\}$: $\mathfrak{sl}(2)$-triple such that $\frac{1}{2}\,\mathrm{ad}_h$ has eigenvalues $\pm 1, 0$.
$\frac{1}{2}\,\mathrm{ad}_h$ defines a $\mathbb{Z}$-grading

$$\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1.$$

Set $\mathfrak{g} = \mathfrak{G}_0$. Then $V := \mathfrak{G}_1$ is a Jordan superalgebra with operation

$$\{x, y\} := [[x, f], y], \quad x, y \in \mathfrak{G}_1.$$

Classification of simple Jordan superalgebras (V. Kac).

Almost always V is a spherical representation of $\mathfrak{g}$. The orbit of e is open, isomorphic to a symmetric superspace G/K.

Back to Capelli problem

How to get a good source of examples of $(\mathfrak{g}, V)$ where V is a spherical representation of $\mathfrak{g}$?

Kantor–Koeher–Tits construction of Jordan superalgebras.

$\mathfrak{G}$: Lie superalgebra,

$\{e, h, f\}$: $\mathfrak{sl}(2)$-triple such that $\frac{1}{2}\operatorname{ad}_h$ has eigenvalues $\pm 1, 0$.

$\frac{1}{2}\operatorname{ad}_h$ defines a $\mathbb{Z}$-grading

$$\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1.$$

Set $\mathfrak{g} = \mathfrak{G}_0$. Then $V := \mathfrak{G}_1$ is a Jordan superalgebra with operation

$$\{x, y\} := [[x, f], y], \quad x, y \in \mathfrak{G}_1.$$

Classification of simple Jordan superalgebras (V. Kac).

Almost always V is a spherical representation of $\mathfrak{g}$. The orbit of e is open, isomorphic to a symmetric superspace G/K.

Back to Capelli problem

How to get a good source of examples of $(\mathfrak{g}, V)$ where V is a spherical representation of $\mathfrak{g}$?

Kantor–Koeher–Tits construction of Jordan superalgebras.

$\mathfrak{G}$: Lie superalgebra,

$\{e, h, f\}$: $\mathfrak{sl}(2)$-triple such that $\frac{1}{2}\,\mathrm{ad}_h$ has eigenvalues $\pm 1, 0$.

$\frac{1}{2}\,\mathrm{ad}_h$ defines a $\mathbb{Z}$-grading

$$\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1.$$

Set $\mathfrak{g} = \mathfrak{G}_0$. Then $V := \mathfrak{G}_1$ is a Jordan superalgebra with operation

$$\{x, y\} := [[x, f], y], \quad x, y \in \mathfrak{G}_1.$$

Classification of simple Jordan superalgebras (V. Kac).

Almost always V is a spherical representation of $\mathfrak{g}$. The orbit of e is open, isomorphic to a symmetric superspace G/K.

Back to Capelli problem

How to get a good source of examples of $(\mathfrak{g}, V)$ where V is a spherical representation of $\mathfrak{g}$?

Kantor–Koeher–Tits construction of Jordan superalgebras.

$\mathfrak{G}$: Lie superalgebra,

$\{e, h, f\}$: $\mathfrak{sl}(2)$-triple such that $\frac{1}{2}\,\mathrm{ad}_h$ has eigenvalues $\pm 1, 0$.

$\frac{1}{2}\,\mathrm{ad}_h$ defines a $\mathbb{Z}$-grading

$$\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1.$$

Set $\mathfrak{g} = \mathfrak{G}_0$. Then $V := \mathfrak{G}_1$ is a Jordan superalgebra with operation

$$\{x, y\} := [[x, f], y], \quad x, y \in \mathfrak{G}_1.$$

Classification of simple Jordan superalgebras (V. Kac).

Almost always V is a spherical representation of $\mathfrak{g}$. The orbit of e is open, isomorphic to a symmetric superspace G/K.

Back to Capelli problem

How to get a good source of examples of $(\mathfrak{g}, V)$ where V is a spherical representation of $\mathfrak{g}$?

Kantor–Koeher–Tits construction of Jordan superalgebras.

$\mathfrak{G}$: Lie superalgebra,

$\{e, h, f\}$: $\mathfrak{sl}(2)$-triple such that $\frac{1}{2}\,\mathrm{ad}_h$ has eigenvalues $\pm 1, 0$.

$\frac{1}{2}\,\mathrm{ad}_h$ defines a $\mathbb{Z}$-grading

$$\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1.$$

Set $\mathfrak{g} = \mathfrak{G}_0$. Then $V := \mathfrak{G}_1$ is a Jordan superalgebra with operation

$$\{x, y\} := [[x, f], y], \quad x, y \in \mathfrak{G}_1.$$

Classification of simple Jordan superalgebras (V. Kac).

Almost always V is a spherical representation of $\mathfrak{g}$. The orbit of e is open, isomorphic to a symmetric superspace G/K.

Back to Capelli problem

How to get a good source of examples of $(\mathfrak{g}, V)$ where V is a spherical representation of $\mathfrak{g}$?

Kantor–Koeher–Tits construction of Jordan superalgebras.

$\mathfrak{G}$: Lie superalgebra,

$\{e, h, f\}$: $\mathfrak{sl}(2)$-triple such that $\frac{1}{2}\operatorname{ad}_h$ has eigenvalues $\pm 1, 0$.

$\frac{1}{2}\operatorname{ad}_h$ defines a $\mathbb{Z}$-grading

$$\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1.$$

Set $\mathfrak{g} = \mathfrak{G}_0$. Then $V := \mathfrak{G}_1$ is a Jordan superalgebra with operation

$$\{x, y\} := [[x, f], y], \quad x, y \in \mathfrak{G}_1.$$

Classification of simple Jordan superalgebras (V. Kac).

Almost always V is a spherical representation of $\mathfrak{g}$. The orbit of e is open, isomorphic to a symmetric superspace G/K.

List. (Kac)

I. Classical:

1. $\mathfrak{G} = \mathfrak{gl}(2m|2n)$, $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, $V = E \otimes E$ where E is the natural representation of $\mathfrak{g}$.
2. $\mathfrak{G} = \mathfrak{spo}(2m|4n)$, $\mathfrak{g} = \mathfrak{gl}(m|2n)$, $V = \mathrm{Sym}^2(E)$.
3. $\mathfrak{G} = \mathfrak{osp}(m+2|2n)$, $\mathfrak{g} = \mathfrak{gosp}(m|2n)$, $V = E$.

Lemma

In the first two cases V is a spherical $\mathfrak{g}$-module, and in the third case V is spherical iff $m - 2n > 0$ or odd.

List. (Kac)

I. Classical:

1. $\mathfrak{G} = \mathfrak{gl}(2m|2n)$, $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, $V = E \otimes E$ where E is the natural representation of $\mathfrak{g}$.
2. $\mathfrak{G} = \mathfrak{spo}(2m|4n)$, $\mathfrak{g} = \mathfrak{gl}(m|2n)$, $V = \mathrm{Sym}^2(E)$.
3. $\mathfrak{G} = \mathfrak{osp}(m+2|2n)$, $\mathfrak{g} = \mathfrak{gosp}(m|2n)$, $V = E$.

Lemma

In the first two cases V is a spherical $\mathfrak{g}$-module, and in the third case V is spherical iff $m - 2n > 0$ or odd.

II. Exceptional.

1. $\mathfrak{G} = D(2,1,t)$, $\mathfrak{g} = \mathfrak{gl}(1|2)$, $V = K(t,0,0)$ a $(2|2)$-dimensional module. Deformation of $I(3)$ for $\mathfrak{G} = \mathfrak{osp}(4|2)$.
2. $\mathfrak{G} = F(1,3)$, $\mathfrak{g} = \mathfrak{gosp}(2|4)$, $\dim V = (6|4)$.

Lemma

In the first case V is a spherical $\mathfrak{g}$-module iff $t \notin \mathbb{Q}^+$, in the second case V is spherical.

III. Strange.

1. $\mathfrak{G} = Q(2n)$, $\mathfrak{g} = Q(n) \oplus Q(n)$, $V = E \boxtimes E$.
2. $\mathfrak{G} = P(2n)$, $\mathfrak{g} = \mathfrak{gl}(n|n)$, $V = \Pi\,\mathrm{Sym}^2(E)$ where Π is the switch of parity.

Lemma

Both cases are spherical.

II. Exceptional.

1. $\mathfrak{G} = D(2,1,t)$, $\mathfrak{g} = \mathfrak{gl}(1|2)$, $V = K(t,0,0)$ a $(2|2)$-dimensional module. Deformation of $\mathrm{I}(3)$ for $\mathfrak{G} = \mathfrak{osp}(4|2)$.
2. $\mathfrak{G} = F(1,3)$, $\mathfrak{g} = \mathfrak{gosp}(2|4)$, $\dim V = (6|4)$.

Lemma

In the first case V is a spherical $\mathfrak{g}$-module iff $t \notin \mathbb{Q}^+$, in the second case V is spherical.

III. Strange.

1. $\mathfrak{G} = Q(2n)$, $\mathfrak{g} = Q(n) \oplus Q(n)$, $V = E \boxtimes E$.
2. $\mathfrak{G} = P(2n)$, $\mathfrak{g} = \mathfrak{gl}(n|n)$, $V = \Pi\,\mathrm{Sym}^2(E)$ where Π is the switch of parity.

Lemma

Both cases are spherical.

II. Exceptional.

1. $\mathfrak{G} = D(2,1,t)$, $\mathfrak{g} = \mathfrak{gl}(1|2)$, $V = K(t,0,0)$ a $(2|2)$-dimensional module. Deformation of $\mathrm{I}(3)$ for $\mathfrak{G} = \mathfrak{osp}(4|2)$.
2. $\mathfrak{G} = F(1,3)$, $\mathfrak{g} = \mathfrak{gosp}(2|4)$, $\dim V = (6|4)$.

Lemma

In the first case V is a spherical $\mathfrak{g}$-module iff $t \notin \mathbb{Q}^+$, in the second case V is spherical.

III. Strange.

1. $\mathfrak{G} = Q(2n)$, $\mathfrak{g} = Q(n) \oplus Q(n)$, $V = E \boxtimes E$.
2. $\mathfrak{G} = P(2n)$, $\mathfrak{g} = \mathfrak{gl}(n|n)$, $V = \Pi\,\mathrm{Sym}^2(E)$ where Π is the switch of parity.

Lemma

Both cases are spherical.

II. Exceptional.

1. $\mathfrak{G} = D(2,1,t)$, $\mathfrak{g} = \mathfrak{gl}(1|2)$, $V = K(t,0,0)$ a $(2|2)$-dimensional module. Deformation of $\mathrm{I}(3)$ for $\mathfrak{G} = \mathfrak{osp}(4|2)$.
2. $\mathfrak{G} = F(1,3)$, $\mathfrak{g} = \mathfrak{gosp}(2|4)$, $\dim V = (6|4)$.

Lemma

In the first case V is a spherical $\mathfrak{g}$-module iff $t \notin \mathbb{Q}^+$, in the second case V is spherical.

III. Strange.

1. $\mathfrak{G} = Q(2n)$, $\mathfrak{g} = Q(n) \oplus Q(n)$, $V = E \boxtimes E$.
2. $\mathfrak{G} = P(2n)$, $\mathfrak{g} = \mathfrak{gl}(n|n)$, $V = \Pi \operatorname{Sym}^2(E)$ where Π is the switch of parity.

Lemma

Both cases are spherical.

Super Jack polynomials.

Definition

Let $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\theta \in \mathbb{C}$.
$\mathcal{SJ}_{m,n,\theta} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ a subring given by relations:

- $f(\sigma(\mathbf{x}), \tau(\mathbf{y})) = f(\mathbf{x}, \mathbf{y})$ for all $(\sigma, \tau) \in \mathbb{S}_m \times \mathbb{S}_n$.
- If $x_i + \theta y_j = 0$ then

$$f(\mathbf{x} + \frac{1}{2}e_i, \mathbf{y} - \frac{1}{2}e_j) = f(\mathbf{x} - \frac{1}{2}e_i, \mathbf{y} + \frac{1}{2}e_j).$$

If $\theta = 1$ we get supersymmetric polynomials:

- $\mathcal{SJ}_{m,n,1} \simeq \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$.
- $\mathcal{SJ}_{m,n,1}$ has basis of super Schur polynomials $\{s_\lambda(\mathbf{x}, \mathbf{y}) \mid \lambda \in H(m,n)\}$, which are the characters of $S_\lambda(E)$.

Super Jack polynomials.

Definition

Let $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\theta \in \mathbb{C}$.
$\mathcal{SJ}_{m,n,\theta} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ a subring given by relations:

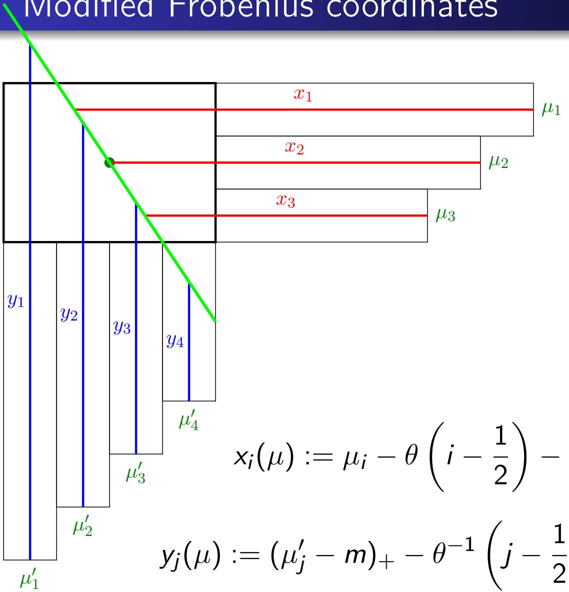
- $f(\sigma(\mathbf{x}), \tau(\mathbf{y})) = f(\mathbf{x}, \mathbf{y})$ for all $(\sigma, \tau) \in \mathbb{S}_m \times \mathbb{S}_n$.
- If $x_i + \theta y_j = 0$ then

$$f(\mathbf{x} + \frac{1}{2}e_i, \mathbf{y} - \frac{1}{2}e_j) = f(\mathbf{x} - \frac{1}{2}e_i, \mathbf{y} + \frac{1}{2}e_j).$$

If $\theta = 1$ we get supersymmetric polynomials:

- $\mathcal{SJ}_{m,n,1} \simeq \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$.
- $\mathcal{SJ}_{m,n,1}$ has basis of super Schur polynomials $\{s_\lambda(\mathbf{x}, \mathbf{y}) \mid \lambda \in H(m,n)\}$, which are the characters of $S_\lambda(E)$.

Modified Frobenius coordinates



$$x_i(\mu) := \mu_i - \theta\left(i - \frac{1}{2}\right) - \frac{1}{2}(n - \theta m),$$

$$y_j(\mu) := (\mu'_j - m)_+ - \theta^{-1}\left(j - \frac{1}{2}\right) + \frac{1}{2}\left(\theta^{-1} n + m\right).$$

$$h_\theta(\lambda) := \prod_{1\leq i\leq \ell(\lambda)} \prod_{1\leq j\leq \lambda_i} (\lambda_i - j + \theta(\lambda'_j - i) + 1).$$

Theorem (Sergeev–Veselov, Knop–Sahi)

Let $\theta \in \mathbb{C}$ be a complex number such that $\theta \notin \mathbb{Q}_{\leq 0}$. Then for each $\lambda \in H(m,n)$, there exists a unique polynomial $P_\lambda(\mathbf{x},\mathbf{y},\theta) \in \mathcal{SJ}_{m,n,\theta}$ which satisfies the following properties.

- $\deg(P_\lambda) \leq |\lambda|$,
- $P_\lambda(\mathbf{x}_\mu, \mathbf{y}_\mu, \theta) = 0$ *for all* $\mu \in H(m,n)$ *such that* $|\mu| \leq |\lambda|$ *and* $\mu \neq \lambda$.
- $P_\lambda(\mathbf{x}_\mu, \mathbf{y}_\mu, \theta) = h_\theta(\lambda)$.

Furthermore, the family of polynomials $P_\lambda(\mathbf{x},\mathbf{y},\theta)_{\lambda\in H(m,n)}$ form a basis of $\mathcal{SJ}_{m,n,\theta}$. They are called shifted super Jack polynomials.

Theorem (Sahi, Salmasian, S.)

The solutions to the Capelli eigenvalues problem are given by shifted super Jack polynomials P_λ for the following value of the parameter θ.

1. $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, $V = E \otimes E$, $\theta = 1$, $P_\lambda \in \mathcal{SJ}_{m,n,\theta}$.
2. $\mathfrak{g} = \mathfrak{gl}(m|2n)$, $V = \mathrm{Sym}^2(V)$, $\theta = \frac{1}{2}$, $P_\lambda \in \mathcal{SJ}_{m,n,\theta}$.
3. $\mathfrak{g} = \mathfrak{gosp}(m|2n)$, $V = E$, $\theta = \frac{m-2n}{2}$, $P_\lambda \in \mathcal{SJ}_{2,0,\theta}$.
4. $\mathfrak{g} = \mathfrak{gl}(1|2)$, $\theta = -\frac{1}{t}$, $P_\lambda \in \mathcal{SJ}_{1,1,\theta}$.
5. $\mathfrak{g} = \mathfrak{gosp}(2|4)$, $\theta = \frac{3}{2}$, $P_\lambda \in \mathcal{SJ}_{2,1,\theta}$.

Symmetric Q-polynomials.

Definition

$\mathcal{SQ}_n$ is a subring of the ring of symmetric polynomials $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ satisfying the additional symmetry property

$$f(t, -t, x_3, \ldots, x_n) = 0 \quad \text{for all} \quad t, x_3, \ldots, x_n \in \mathbb{C}.$$

- $\mathcal{SQ}_n$ is generated by $\sum_{i=1}^{n} x_i^{2k+1}$.
- $\mathcal{SQ}_n$ is isomorphic to the ring of invariant polynomials in the coadjoint representation of $Q(n)$.
- $\mathcal{SQ}_n$ has a basis enumerated by strict partitions Q_λ given by the characters of $SQ_\lambda(E)$.
- $\mathcal{SQ}_n$ is a particular case of Macdonald polynomials.

Symmetric Q-polynomials.

Definition

$\mathcal{SQ}_n$ is a subring of the ring of symmetric polynomials $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ satisfying the additional symmetry property

$$f(t, -t, x_3, \dots, x_n) = 0 \quad \text{for all} \quad t, x_3, \dots, x_n \in \mathbb{C}.$$

- $\mathcal{SQ}_n$ is generated by $\sum_{i=1}^n x_i^{2k+1}$.
- $\mathcal{SQ}_n$ is isomorphic to the ring of invariant polynomials in the coadjoint representation of $Q(n)$.
- $\mathcal{SQ}_n$ has a basis enumerated by strict partitions Q_λ given by the characters of $SQ_\lambda(E)$.
- $\mathcal{SQ}_n$ is a particular case of Macdonald polynomials.

Factorial Schur Q functions

Theorem (Ivanov)

For every $\lambda \in DP(n)$, there exists a unique polynomial $Q^*_\lambda \in \mathcal{SQ}_n$ which satisfies the following properties.

- $\deg(Q^*_\lambda) \leq |\lambda|$.
- $Q^*_\lambda(\mu) = 0$ *for all* $\mu \in DP(n)$ *such that* $|\mu| \leq |\lambda|$ *and* $\mu \neq \lambda$.
- $Q^*_\lambda(\lambda) = h(\lambda)$, *where* $h(\lambda) := \lambda! \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}$.

Furthermore, the family of polynomials $\left(Q^*_\lambda\right)_{\lambda \in DP(n)}$ is a basis of $\mathcal{SQ}_n$. Q^*_λ are called factorial Schur Q-polynimials.

Theorem (Sahi, Salmasian, S.)

In the cases III(1) and III(2) factorial Schur Q-polynomials Q^*_λ give the solution to the Capelli eigenvalue problem.

Factorial Schur Q functions

Theorem (Ivanov)

For every $\lambda \in DP(n)$, there exists a unique polynomial $Q^*_\lambda \in \mathcal{SQ}_n$ which satisfies the following properties.

- $\deg(Q^*_\lambda) \leq |\lambda|$.
- $Q^*_\lambda(\mu) = 0$ *for all* $\mu \in DP(n)$ *such that* $|\mu| \leq |\lambda|$ *and* $\mu \neq \lambda$.
- $Q^*_\lambda(\lambda) = h(\lambda)$, *where* $h(\lambda) := \lambda! \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}$.

Furthermore, the family of polynomials $\left(Q^*_\lambda\right)_{\lambda \in DP(n)}$ is a basis of $\mathcal{SQ}_n$. Q^*_λ are called factorial Schur Q-polynimials.

Theorem (Sahi, Salmasian, S.)

In the cases III(1) and III(2) factorial Schur Q-polynomials Q^*_λ give the solution to the Capelli eigenvalue problem.

Example.

$\mathfrak{g} = \mathfrak{gl}(n|n)$ and $V = \Pi\,\mathrm{Sym}^2(E)$.

- $\mathrm{Sym}^d(\Pi X) \simeq \Pi^d \Lambda^d(X)$.
- R. Howe's result:
$$\mathbb{C}[V] = \bigoplus_{\lambda \in QH(n,n)} S_\lambda(E^*),$$
- $QH(n,n)$ the set of quasisymmetric hook partitions. Bijection $QH(n,n) \to DP(n)$.
- Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$.
- $\Phi : Z(\mathfrak{g}) \to D(V)^{\mathfrak{g}}$ is surjective.
- $Z(\mathfrak{g}) \simeq \mathcal{S}\mathcal{J}_{n,n,1}$ via Harish-Chandra map.
- $\Phi(\mathcal{S}\mathcal{J}_{n,n,1}) = \mathcal{S}\mathcal{Q}_n$.
- Vanishing conditions of Ivanov are automatic for the Capelli problem. Hence the result.

The map $\Phi : Z(\mathfrak{g}) \to D(V)^{\mathfrak{g}}$ in all cases but the exceptional case II(2).

About proof:

1. Identify Λ with certain set of partitions.
2. Check symmetry conditions. In almost all cases we can use Harish-Chandra homomorphism.
3. Vanishing conditions go automatically.

The map $\Phi : Z(\mathfrak{g}) \to D(V)^{\mathfrak{g}}$ in all cases but the exceptional case II(2).
About proof:

1. Identify Λ with certain set of partitions.
2. Check symmetry conditions. In almost all cases we can use Harish-Chandra homomorphism.
3. Vanishing conditions go automatically.

Generalized roots system (S)

X vector space over $\mathbb{R}$,
$(\cdot,\cdot)$ non-degenerate scalar product, maybe not positive definite,
$p : \Delta \to \mathbb{Z}_2$.

Definition

$R \subset X$ is a generalized root system if

- If $\alpha, \beta \in R$ and $(\alpha, \alpha) \neq 0$, then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $s_\alpha(\beta) \in R$.
- If $\alpha, \beta \in R$ and $(\alpha, \alpha) = 0$, then at least one of $\beta \pm \alpha$ belongs to R.
- $R = -R$.
- $(\alpha, \alpha) = 0$ implies $p(\alpha) = 1$.

Theorem (S)

Any irreducible generalized root system is the root system of a basic simple Lie superalgebra.

Generalized roots system (S)

X vector space over $\mathbb{R}$,
$(\cdot, \cdot)$ non-degenerate scalar product, maybe not positive definite,
$p : \Delta \to \mathbb{Z}_2$.

Definition

$R \subset X$ is a generalized root system if

- If $\alpha, \beta \in R$ and $(\alpha, \alpha) \neq 0$, then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $s_\alpha(\beta) \in R$.
- If $\alpha, \beta \in R$ and $(\alpha, \alpha) = 0$, then at least one of $\beta \pm \alpha$ belongs to R.
- $R = -R$.
- $(\alpha, \alpha) = 0$ implies $p(\alpha) = 1$.

Theorem (S)

Any irreducible generalized root system is the root system of a basic simple Lie superalgebra.

Supersymmetric polynomials

- Weyl group $W_0 := \langle s_\alpha \rangle$ invariance.
- Odd reflection invariance

$$f(x+\alpha) = f(x), \quad (\alpha,\alpha) = 0, (x,\alpha) = 0.$$

- Weyl groupoid.

Harish-Chandra restriction theorem:
The restriction $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ establish an isomorphism between $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ and the ring of supersymmetric polynomials.

Supersymmetric polynomials

- Weyl group $W_0 := \langle s_\alpha \rangle$ invariance.
- Odd reflection invariance

$$f(x+\alpha) = f(x), \quad (\alpha,\alpha) = 0, (x,\alpha) = 0.$$

- Weyl groupoid.

Harish-Chandra restriction theorem:
The restriction $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ establish an isomorphism between $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ and the ring of supersymmetric polynomials.

Deformed generalized root systems (Sergeev–Veselov)

- The same set of roots R.
- New function $m : \Delta \to \mathbb{C}$, multiplicity.
- New bilinear form $B(\cdot,\cdot)$.

Compatibility condition

- B is s_α-invariant for all non-isotropic α;
- If $(\alpha,\alpha) = 0$ then $m(\alpha) = 2$;
- The function $\psi(x) := \prod_{\alpha\in R^+} sin^{-m(\alpha)} B(\alpha, x)$ is an eigenfunction of the generalized Schroedinger operator

$$-\Delta_B + \sum_{\alpha\in R^+} \frac{m(\alpha)(m(\alpha)+2m(2\alpha)+1)B(\alpha,\alpha)}{sin^2(B(\alpha,x))}.$$

Classification (Sergeev, Veselov) involves θ.

Deformed generalized root systems (Sergeev–Veselov)

- The same set of roots R.
- New function $m : \Delta \to \mathbb{C}$, multiplicity.
- New bilinear form $B(\cdot, \cdot)$.

Compatibility condition

- B is s_α-invariant for all non-isotropic α;
- If $(\alpha, \alpha) = 0$ then $m(\alpha) = 2$;
- The function $\psi(x) := \prod_{\alpha \in R^+} sin^{-m(\alpha)} B(\alpha, x)$ is an eigenfunction of the generalized Schroedinger operator

$$-\Delta_B + \sum_{\alpha \in R^+} \frac{m(\alpha)(m(\alpha) + 2m(2\alpha) + 1)B(\alpha, \alpha)}{sin^2(B(\alpha, x))}.$$

Classification (Sergeev, Veselov) involves θ.

Deformed generalized root systems (Sergeev–Veselov)

- The same set of roots R.
- New function $m : \Delta \to \mathbb{C}$, multiplicity.
- New bilinear form $B(\cdot,\cdot)$.

Compatibility condition

- B is s_α-invariant for all non-isotropic α;
- If $(\alpha,\alpha) = 0$ then $m(\alpha) = 2$;
- The function $\psi(x) := \prod_{\alpha \in R^+} sin^{-m(\alpha)} B(\alpha, x)$ is an eigenfunction of the generalized Schroedinger operator

$$-\Delta_B + \sum_{\alpha \in R^+} \frac{m(\alpha)(m(\alpha) + 2m(2\alpha) + 1)B(\alpha,\alpha)}{sin^2(B(\alpha,x))}.$$

Classification (Sergeev, Veselov) involves θ.

Supersymmetry condition:

- $f(s_\alpha(x)) = f(x)$ Weyl group invariance;
- Let $(\alpha,\alpha) = 0$. On the hyperplane $B(x,\alpha) = 0$ we have $f(x + \frac{1}{2}\alpha) = f(x - \frac{1}{2}\alpha)$.

Super Jack polynomials $\mathcal{SJ}_{m,n,\theta}$ correspond to the deformed root system $A(m-1,n-1)_\theta$. Shifted super Jack polynomials are used to prove integrability of CMS:

$$L_{m,n,\theta} = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} - k\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} +$$

$$\sum_{1\le i<j\le m} \frac{2k(k+1)}{sin^2(x_i - x_j)} + \sum_{1\le i<j\le n} \frac{2(k^{-1}+1)}{sin^2(y_i - y_j)} + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{2(k+1)}{sin^2(x_i - y_j)}$$

Supersymmetry condition:

- $f(s_\alpha(x)) = f(x)$ Weyl group invariance;
- Let $(\alpha, \alpha) = 0$. On the hyperplane $B(x, \alpha) = 0$ we have $f(x + \frac{1}{2}\alpha) = f(x - \frac{1}{2}\alpha)$.

Super Jack polynomials $\mathcal{SJ}_{m,n,\theta}$ correspond to the deformed root system $A(m-1, n-1)_\theta$. Shifted super Jack polynomials are used to prove integrability of CMS:

$$L_{m,n,\theta} = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} - k \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} +$$

$$\sum_{1 \le i < j \le m} \frac{2k(k+1)}{sin^2(x_i - x_j)} + \sum_{1 \le i < j \le n} \frac{2(k^{-1}+1)}{sin^2(y_i - y_j)} + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{2(k+1)}{sin^2(x_i - y_j)}$$

Restricted root system

We assume that $\mathfrak{g}$ has an invariant form. Recall: open orbit G/K in V.
$\mathfrak{k} = \mathfrak{g}^w$, w the element in the Weyl group of $\mathfrak{sl}(2)$, $(\mathfrak{g}, \mathfrak{k})$ is a symmetric superpair.
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.
$\mathfrak{a}$ maximal toral subalgebra in $\mathfrak{m}$, $\gamma : R \to \mathfrak{a}^*$.

Theorem

The restricted root system $\Gamma = \gamma(R) \setminus \{0\}$ is a deformed root system of type $A(m-1|, n-1)_\theta$.

In this way Calogero–Moser operator is the radial part of Laplacian.
Remark. Theorem should hold for all symmetric superspaces.

Restricted root system

We assume that $\mathfrak{g}$ has an invariant form. Recall: open orbit G/K in V.
$\mathfrak{k} = \mathfrak{g}^w$, w the element in the Weyl group of $\mathfrak{sl}(2)$, $(\mathfrak{g}, \mathfrak{k})$ is a symmetric superpair.
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.
$\mathfrak{a}$ maximal toral subalgebra in $\mathfrak{m}$, $\gamma : R \to \mathfrak{a}^*$.

Theorem

The restricted root system $\Gamma = \gamma(R) \setminus \{0\}$ *is a deformed root system of type* $A(m-1|, n-1)_\theta$.

In this way Calogero–Moser operator is the radial part of Laplacian.
Remark. Theorem should hold for all symmetric superspaces.

Restricted root system

We assume that $\mathfrak{g}$ has an invariant form. Recall: open orbit G/K in V.
$\mathfrak{k} = \mathfrak{g}^w$, w the element in the Weyl group of $\mathfrak{sl}(2)$, $(\mathfrak{g}, \mathfrak{k})$ is a symmetric superpair.
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.
$\mathfrak{a}$ maximal toral subalgebra in $\mathfrak{m}$, $\gamma : R \to \mathfrak{a}^*$.

Theorem

The restricted root system $\Gamma = \gamma(R) \setminus \{0\}$ is a deformed root system of type $A(m-1|, n-1)_\theta$.

In this way Calogero–Moser operator is the radial part of Laplacian.
Remark. Theorem should hold for all symmetric superspaces.

Questions.

- Can we obtain all deformed roots system with rational multiplicity functions as restricted root system of symmetric spaces?
- When $\mathbb{C}[G/K]$ has a nice structure (multiplicity free).
- Consider integrable system on $\mathfrak{a}^*$ coming from the radial part of the Casimir element.