Scattering theory, Riemann hypothesis and Harmonic analysis on symmetric spaces

To the memory of professor Faddeev

Michel Semenov-Tian-Shansky
Institut Mathématique de Bourgogne, Dijon, France


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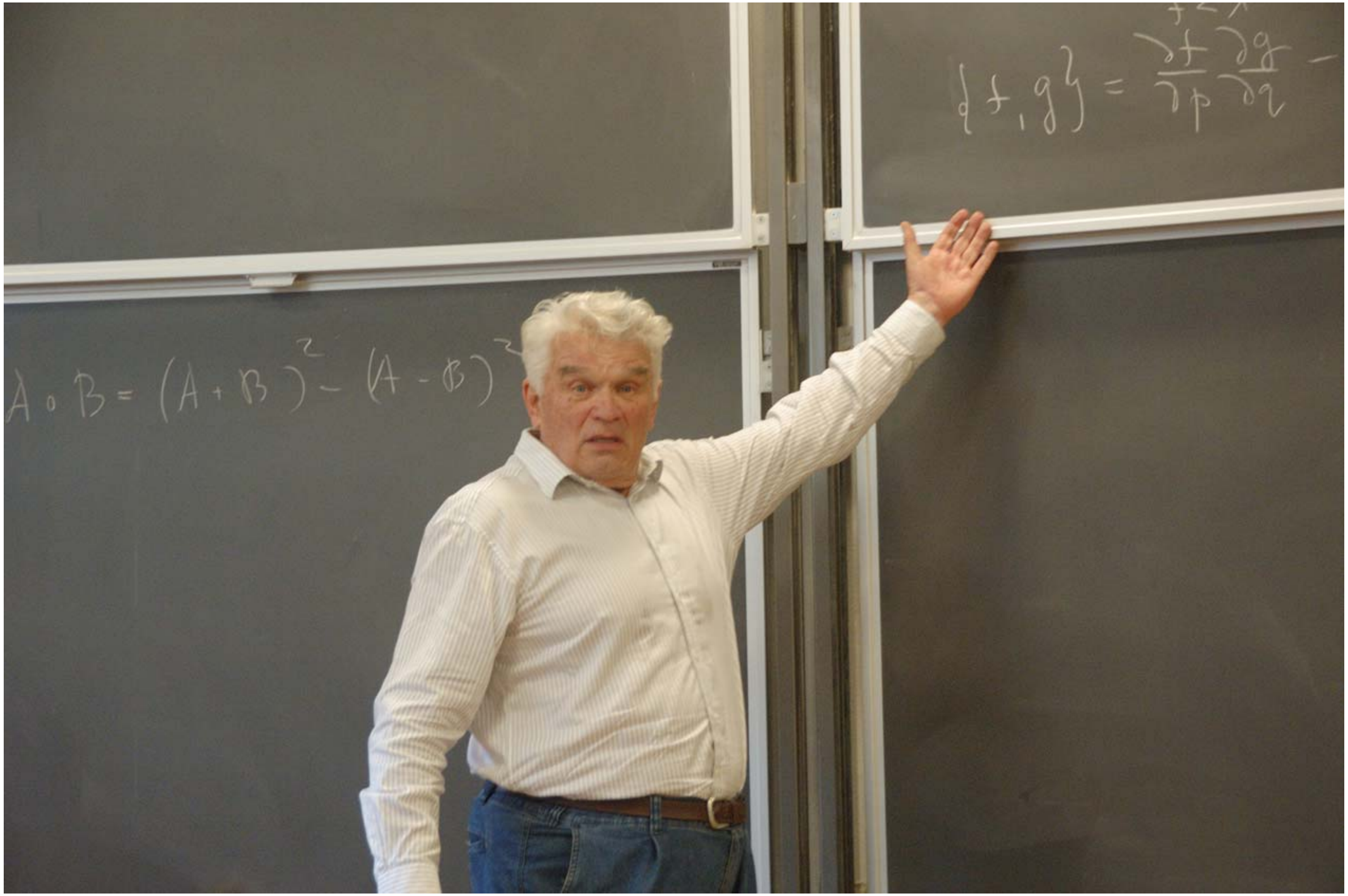
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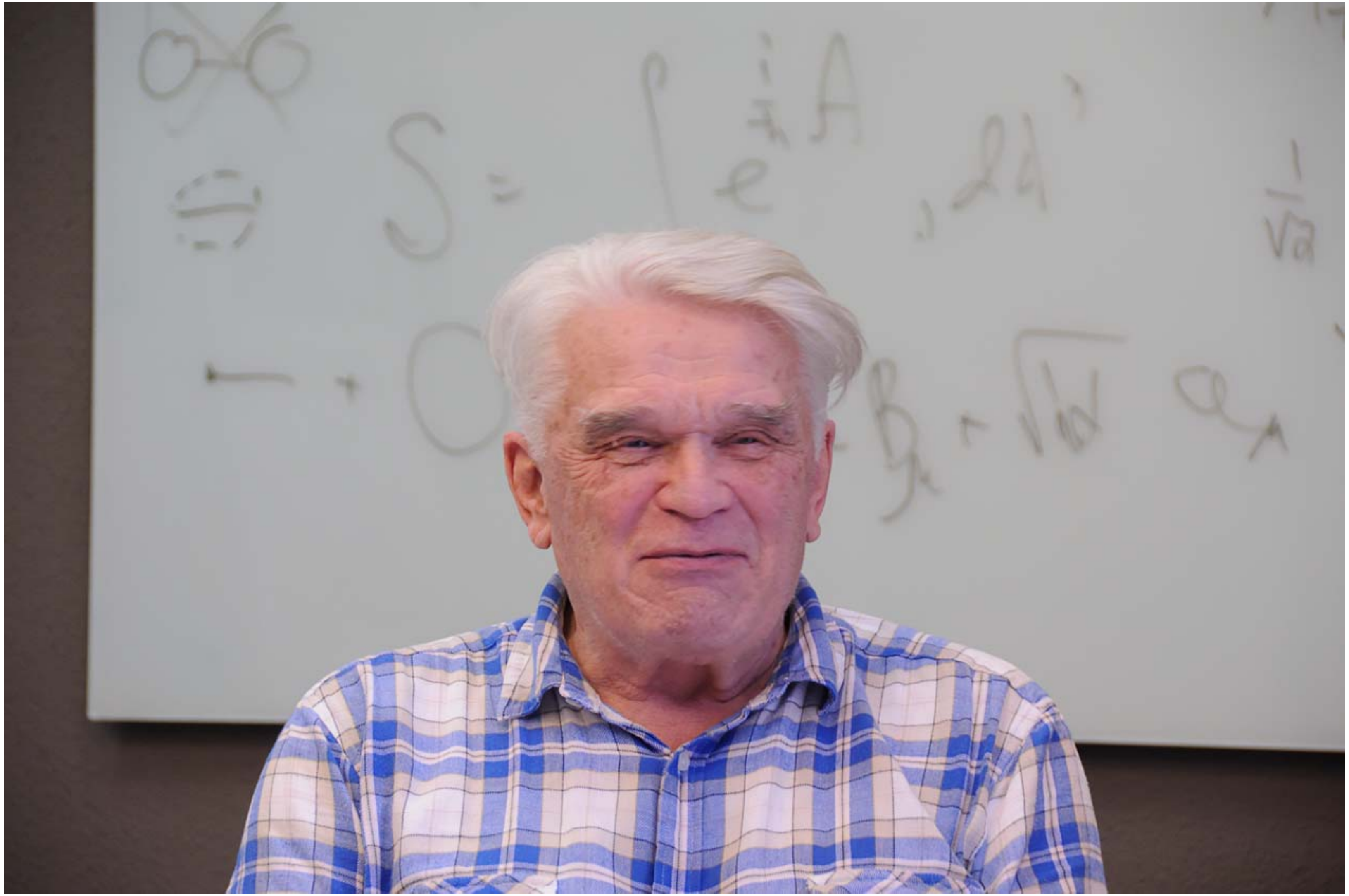

Introduction

Three subject which play a very distinguished rôle in Faddeev heritage are :

- Scattering theory ;
- Quantum and Classical Integrable systems.
- Lie groups and Lie algebras

His work brought to light non-trivial and unexpected links between these subjects
I shall talk about some of the less known aspects of these links :

- Scattering theory and Riemann hypothesis.
- Scattering theory and representation theory of semi-simple Lie groupes.

Starting point :

Perturbation theory for operators with continuous spectrum.

- Model exemple : $H_0 = -\Delta, H = -\Delta + V$
- Hilbert Identity for the resolvent :
  $R(\lambda) = R_0(\lambda) - R_0(\lambda) V R(\lambda)$
- Typical difficulty : because of the presence of continuous spectrum this integral equation is not of Fredholm type.

This difficulty has been successfully resolved by Faddeev (in particular, for operators with a complicated structure of continuous spectrum (three-body problem).

- Gelfand's suggestion : apply similar methods for the harmonic analysis on $SL(2,\mathbb{R})/\Gamma$; here the continuous spectrum is present when the discrete subgroup Γ is not co-compact.

The trick :

(suggeszted by the Quantum Scattering theory)
The regularization of the integral equation for the resolvent is suggested by Quantum mechanics :

- Put $T(\lambda) = VR(\lambda)V$; T is closely related to the scattering amplitude.
- Then $R(\lambda) = R_0(\lambda) - R_0(\lambda)T(\lambda)R_0(\lambda)$, and

$$T(\lambda) = V - VR_0(\lambda)T(\lambda).$$

This latter equation is already of Fredholm type (to prove this assertion one has to make a very clever and nontrivial choice of the pertinent functional spaces).

Note that the scattering operator is close to the operator $I + T(\lambda)$

Case of automorphic Laplace operator

Here $H = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{4}$ acting in $L_2(\mathbb{H}/\Gamma)$. The treatment of H by means of the Perturbation theory is not obvious! For simplicity we assume that $\Gamma = SL(2, \mathbb{Z})$; its fundamental domain F decomposes intp the union of a compact domain F_0 and of the infinite band

$$F_1 = \{x + iy\,;\, -1/2 \leq x \leq 1/2,\ y \geq a\}\,.$$

Denote by P_0, P_1 the associated projection operators (which act as multiplication operators by the characteristic functions of F_0, F_1). Put $R(k) = (H - k^2 I)^{-1}$,
$R_{ij}(s) = P_i R(k) P_j$, $i, j = 0, 1$.
Then all R_{ij} are compact, except R_{11}.

Elimination of continuous spectrum

Let us intrduce the integration operator over the horocycles,

$$Pf(y) = \int_{-1/2}^{-1/2} f(x+iy)\, dy.$$

It is easy to check that $R_{11}(\varkappa) = PR_{11}(\varkappa)P + R'_{11}(\varkappa)$, where $R_{11}(\varkappa)'$ is compact for $\varkappa > 3/2$ and $PR_{11}(\varkappa)P := T(\varkappa)$ coincides with the resolvent $R_0(k) := B - k^2 I)^{-1}$ of a simple differential operator

$$B = -y^2 \frac{d^2}{dy^2} - \frac{1}{4}$$

(Whittaker operator) with the boundary condition $\phi(a) = \varkappa\phi'(a)$; its spectral resolution is known explicitely.

Elimination of continuous spectrum–2

we conclude that the resolvent $R(\varkappa) = T(\varkappa) + V$, where V is compact.
From the Hilbert identity

$$R(k) - R(\varkappa) = (k^2 - \varkappa^2)R(\varkappa)R(k) := \omega(k)RR(k)$$

and the identity $(I - \omega(k)T(\varkappa))^{-1} = I + \omega(k)R_0(k)$ one can deduce, with a little of algebra, that

$$R(k) = R_0(k) + (I + \omega(k)R_0(k)V + \omega(k)(I + \omega(k)R_0(k)V)R(k).$$

Put $R(k) = R_0(k) + (I + \omega(k)R_0(k))B(k)(I + \omega(k)R_0(k))$;
then we get for $B(s)$:

$$B(k) = V + H(k)B(k), \quad \text{where} \quad H(k) := V(I + \omega(k)R_0(k)).$$

Elimination of continuous spectrum–3

The gain reached by these clever transformations : operator $H(k)$ is already of Fredholm type (in an ppropriate Banach space) ; it is holomorphic in k in the band $-1/2 < \mathrm{Re}\, k < 3/2$. This immediately implies that the kernel of the resolvent $R(k)$ admits an analytic continuation into the band ${}_1/2 < \mathrm{Re}\, k \leq 1/2$ with eventually some poles of finite multiplicity ; moreover, all poles in the band $0 \leq \mathrm{Re}\, k < 3/2$ are purely imaginary.
The analytic continuation of the resolvent implies also that of the continuous spectrum eigenfunctions of H ; this yields an operator-theory proof of all basic analytic properties of the Eisenstein series,

$$E(z,k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^{\frac{1}{2}+k}.$$

Eisenstein series and the zeta function

- Notice that the continuous spectrum of H corresponds to the purely imaginary values of k.

It is well known from Quantum mechanics that the asymptotic behavior of the continuous spectrum eigenfunctions is closely related to the scattering. For the Eisenstein series we have for $y = \operatorname{Im} z \to \infty$:

$$E(z,k) = y^{\frac{1}{2}+k} + S_R(k) y^{\frac{1}{2}-k} + o(1), \quad \operatorname{Re} k = 0,$$

where $S_R(k)$ is the "reflection coefficient"
$S_R(k) = \frac{B(1/2,k)\zeta(2k)}{\zeta(2k+1)}$.

- Besides the continuous spectrum, operator H has got an eigenvalue $-1/4$ which corresponds to the constants and an infinite number of positive eigenvalues $k_n^2 > 0, k_n \to \infty$.

Non-stationary point of view

- In “elementary” scattering theory one defines first the wave operators,

$$W_\pm = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0},$$

The scattering operator is then defined by $S = W_+ W_-^{-1}$.

- Disadvantage : in our case the is no natural choice of an unperturbed operator H_0 !
- **Lax–Phillips Theory** : it provides an alternative definition of the scattering operator and allows to examine some of its fine properties.
- Disadvantage : this theory applies only to a very restricted class of hyperbolic equations.

Non-stationary point of view–2

- **A suggestion of Faddeev and Pavlov** : Use the automorphic wave equation

$$u_{tt} = (\Delta + \frac{1}{4})u, \quad \Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

- **Assertion :**
  - Wave equation is equivalent to a first order system in the space of Cauchy data $\phi = (u, u_1)^t$,

$$\partial_t \phi = A\phi, \quad A = \begin{pmatrix} 0 & I \\ -H & 0 \end{pmatrix} \qquad (*)$$

Non-stationary point of view–3

- The energy form in the space of Cauchy data is positively definite in the orthogonal complement of constants.
- Let $\mathcal{H}_+$ be this orthogonal complement ; system (*) induces in this subspace a 1-parameter group $U(t)$, $t \in \mathbb{R}$ of unitary operators.
- Le spectre continu de l'opérateur A coïncide avec $\mathbb{R}$.
- Let $\mathcal{H}_d \subset \mathcal{H}_+$ be the subspace spanned by the eigenvectors of A, and $\mathcal{H}_c = \mathcal{H}_+ \ominus \mathcal{H}_d$. Then $\mathcal{H}_c$ is invariant with respect to $U(t)$ and there exists a unitary equivalence which transforms the operators $U(t)$ into translation operators acting in $L_2(\mathbb{R})$.

Non-stationary point of view–3

- This unitary equivalence is in fact not unique ; they admit a more precise description. Recall the decomposition $F = F_0 \cup F_1$ of the fundamental domain of $\Gamma = SL(2,\mathbb{Z})$,

$$F_1 = \{x + iy\,;\, -1/2 \le x \le 1/2,\ y \ge a\}\,.$$

  - Let Φ be a smooth function of one variable such that $\Phi(y) = 0$ for $y < a$; then

$$u_\pm(z,t) = e^{\pm t/2}\Phi(ye^{\mp t}),\ \text{où } y = \operatorname{Im} z,$$

    is a automorphic wave equation.
  - Let $\mathcal{D}_\pm \subset \mathcal{H}_+$ be the subspace generated by the Cauchy data of solutions of this form.

Non-stationary point of view–4

- (One has yet to impose the orthogonality condition to the constants !) Then
  - $\mathcal{D}_\pm \subset \mathcal{H}_c$.
  - $U(t)\mathcal{D}_+ \subset \mathcal{D}_+$ for all $t \geq 0$,
  - $U(t)\mathcal{D}_- \subset \mathcal{D}_-$ for all $t \leq 0$.
  - $\bigcap_{t<0} U(t)\mathcal{D}_- = \bigcap_{t>0} U(t)\mathcal{D}_+ = \{0\}$.
  - $\bigcup_{t>0} U(t)\mathcal{D}_- = \bigcup_{t<0} U(t)\mathcal{D}_+ = \mathcal{H}_c$.
  - $\mathcal{D}_- \perp \mathcal{D}_+$.
- There exist two distinguished unitary equivalences $\mathcal{W}_\pm : \mathcal{H}_c \to L_2(\mathbb{R})$ such that

$$\mathcal{W}_\pm(\mathcal{D}_\pm) = L_2(\mathbb{R}_\pm).$$

Non-stationary point of view–5

- The composition with the Fourier transform $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ yields two spectral representations $\mathbf{W}_{\pm} = F \circ \mathcal{W}_{\pm}$ such that

$$\mathbf{W}_{\pm}(\mathcal{D}_{\pm}) = H^2_{\pm}(\mathbb{R}) \quad \text{(espaces de Hardy)}$$

- These spectral representations can be compared with that associated with the continuous spectrum eigenfunctions of operator A which can easily be constructed using the Eisenstein series. Explicitly, we have :

$$\Phi(z,k) = \begin{pmatrix} \frac{1}{ik} E(z,k) \\ E(z,k) \end{pmatrix}.$$

Spectral Representations

- Notice that the normalization of the Eisenstein series introduces a slight asymmetry between the incoming and the outgoing waves !

- We can associate with these eigenfunctions a generalized Fourier transformation $\mathcal{E}$ (defined formally by means of the energy inner product).
  - **Théorème.** For the Eisenstein representation $\mathcal{E}$ we have :
    - $\mathcal{E}(\mathcal{D}_-) = B_d^{-1} \cdot H^2_-$,
    - $\mathcal{E}(\mathcal{D}_-) = S_R^{-1} B_d \cdot H^2_+$, where S_R is the reflect ion coefficient,

$$S_R(k) = \frac{B(1/2, k)\zeta(2k)}{\zeta(2k+1)}$$

Scattering operator

- and

$$B_0(k) = \frac{k - i/2}{k + i/2} d^{ik}$$

(This Blaschke factor accounts for the orthogonality condition of the initial data to the eigenfunction of A associated with the constants.)

- Scattering operator S compares two different spectral representations $\mathbf{W}_{\pm}$. Explicitly, S is the multiplication operator by the function $S(k) = S_R^{-1}(k) B_0^2(k)$. One has :

$$S(k) = \left(\frac{k - i/2}{k + i/2}\right)^2 \frac{\Gamma(ik + 1/2)}{\Gamma(ik)\Gamma(1/2)} \frac{\zeta(2ik+1)}{\zeta(2ik)} d^{ik}.$$

The contraction semigroup

- Put $\mathcal{K} = \mathcal{H}_c \ominus (\mathcal{D}_+ \oplus \mathcal{D}_-)$, and let $\mathcal{P}$ be the projection operator from $\mathcal{H}_c$ sur $\mathcal{K}$ and $Z(t) := \mathcal{P}U(t)\mathcal{P}$, $t > 0$, the semigroup generated by the group $U(t)$.
- Heuristically, the wave equation describes the escape of energy to infinity ; the semigroup $Z(t)$ takes into account the fraction of energy which is trapped inside by multiple scattering and is thus obstructed to escape.
  - $S(k)$ is the characteristic function of $Z(t)$.
- **Théorème.** The Riemann hypothesis is equivalent to the following assertion :
  - There exists a dense subset of elements $f \in \mathcal{K}$ such that

$$\lim \sup_{t \to +\infty} t^{-1} \log ||Z(t)f|| \leq -1/4.$$

Symmetric Spaces of rank $r > 1$

- Let $G = ANK$ be the Iwasawa decomposition, $A \subset G$ the split Cartan subgroup, $\mathfrak{a}$ its Lie algebra, $r = \operatorname{rang} X = \dim \mathfrak{a}$, $W = W(G, K)$ the restricted Weyl group, $D^G(X)$ the algebra of G-invariant differential operators on $X = G/K$.
  - How to define an analogue of the wave equation for $X = G/K$ (and eventuelly for $\Gamma\backslash X$)?
    - A simple option (often used in the theory of integrable systems) : Choose a set of operators $H_1, \ldots, H_r \in D^G(X)$ and associate evolution parameters $t_1, \ldots, t_r$ to each of them.
    - A more natural choice is based on the use of *Harish-Chandra isomorphism*.

Symmetric Spaces of rank $r > 1$

- **Theorem** (Harish-Chandra). There exists a canonical isomorphism of the ring of W-invariant polynomials on $\mathfrak{a}^*$ onto the algebra $D^G(X)$ G-invariant differential operators on $X = G/K$,

$$\Delta : P^W(\mathfrak{a}^*) \to D^G(X) : \sigma \mapsto \Delta_\sigma .$$

- Key definition :
  - The generalized wave equation is written for functions on the space-time $\mathfrak{a} \times X$.
  - We denote by $\sigma(-i\frac{\partial}{\partial t})$ the differential operator on $\mathfrak{a}$ with constant coefficients with symbol σ.

Generalized wave equation

- Put :

$$\sigma'(-i\frac{\partial}{\partial t})u(t,x) = \Delta_\sigma u(t,x), \quad \sigma \in P^W(\mathfrak{a}^*). \qquad (**)$$

  - This is an over-determined system of equations (which is compatible, since $D^G(X)$ is commutative !).
  - One of its equations is ultra-hyperbolic (for $l = \operatorname{rank} X > 1$) :

$$\left(\frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_l^2}\right) u = (\Delta + \langle\rho,\rho\rangle)\, u.$$

  - Nevertheless this system has got excellent properties :

Generalized wave equation

In particular,

- The initial problem is well-posed.
- There exists a natural conserved energy form which is positive and defines an inner product in the space of initial data.
- The propagation speed is finite.
- There is a natural definition of scattering operators for this system which generalizes the Lax–Phillips theory.
  - The key definition of the space of initial data is based on the properties of the algebraic extension $P^W(\mathfrak{a}^*) \subset P(\mathfrak{a}^*)$.
    - Let Q, Q^W be the corresponding fields of fractions. Then :

Generalized wave equation

- $Q^W \subset Q$ is a Galois extension with Galois group W.
- P is a free P^W-module of dimension $N = \operatorname{card} W$.
- The space of initial data may be identified with $\mathcal{H} := C_0^\infty(x) \otimes_{P^W} \operatorname{Hom}_{P^W}(P, P^W)$
- Let $\mathcal{S}$ be the space of solutions of system (**) ; the natural mapping $\mathcal{S} \to \mathcal{H}$ is defined by

$$\langle i_0(u), p\rangle = p\left(-i\frac{\partial}{\partial t}\right) u(t,\cdot)\Big|_{t=0}, \quad \text{where} \quad u \in \mathcal{S},\ p \in P.$$

Generalized wave equation

- We can identify the dual space Q^* with Q with the help of the relative trace tr_{Q/Q^W}.
  - Put
$$D = \left\{ s \in Q;\ \mathrm{tr}\, s \cdot \sigma \in P^W \forall \sigma \in P \right\}.$$
  - It is well known that $D = \pi^{-1} \cdot P$, where π is the "product of all roots of $(\mathfrak{g}, \mathfrak{a})$".
  - We can identify the space $\mathcal{H}$ with $C_0^\infty(x) \otimes_{\mathbb{C}} D$ and define the energy form in the space of initial data by
$$e(\phi \otimes x, \psi \otimes y) = \left(\Delta_{\pi^2 \mathrm{tr} xy} \phi \right) \cdot \psi.$$
  - **Theorem.** The energy form is symmetric and positive definite.

Generalized wave equation

- Let $\mathbb{H}$ be the completion of $\mathcal{H}$ with respect to the energy norm.
  - **Theorem.** System (**) is globally solvable on $\mathfrak{a} \times X$; it gives rise to an Abelian group of unitary operators $U(t), t \in \mathfrak{a}$ acting in $\mathbb{H}$.
- Recall that the geodesics in X/K which pass through $x_0 = eK$ have the form $x_t(\alpha) = ke^{\alpha t} \cdot x_0$, where the elements $k \in K$ and $t \in \mathfrak{a}$ are fixed and $\alpha \in \mathbb{R}$. The set

$$B_R = \left\{x = ke^t \cdot x_0; k \in K, t \in \mathfrak{a}, |t| \leq R\right\}$$

is a geodesic ball in X of radius R.

Generalized wave equation

- **Theorem.** Let $u \in \mathcal{S}$ be a solution such that $\operatorname{supp} i_0(u) \subset B_R$. Then $\operatorname{supp} u(t, \cdot) \subset B_{R+|t|}$ for $\forall t \in \mathfrak{a}$.

- To decribe scattering for our system we introduce wave operators which describe the asymptotic behavior of solutions along the "light-like" geodesics in space-time.
  - Put

$$W_+ u(\tau, k) = \lim_{t \to \infty} \pi(-i\partial/\partial t) e^{\rho(t)} u(t + \tau, ke^t \cdot x_0).$$

  - **Theorem.** This limit exists when $t \to \infty$ along any ray which lies inside the positive Weyl chamber (and does not depend on its choice).

Generalized wave equation

- In addition,

$$||u||_E = ||W_+ u||_{L_2(\mathfrak{a} \times K/M)}.$$

- In a similar way, one can define wave operators $W_s, s \in W$, associated with other Weyl chambers ; scattering operators are defined by

$$\mathcal{S}^w = W_+ W_w^{-1}, \quad \mathbf{S}^w = F \circ \mathcal{S}^w \circ F^{-1},$$

where F is the classical Fourier transform.

  - The explicit computation of scattering operators is based on the properties of the principal series representations of G.

Wave packets

- The principal series representations are realized in the space $\mathfrak{H}_\lambda \simeq L_2(K/M)$ (where $M \subset K$ is the centralizer of $\mathfrak{a}$ in K) :

$$T_\lambda(g)a(k) = e^{\langle i\lambda - \rho, H(g^{-1}k)\rangle} a(\varkappa(g^{-1}k)),$$

where for $g = ke^H n$ we write $H = H(g), k = \varkappa(g)$.

- The general solution of system (**) is represented as a " superposition of plane waves" :

$$u(t,x) = \int_{\mathfrak{a}^*} e^{-i\lambda(t)} \langle T_\lambda(g)a_\lambda, \phi_0\rangle_{\mathfrak{H}_\lambda}\, d\lambda,$$

where ϕ_0 is the spherical vector (in our realization, $\phi_0 = 1 \in L_2(K/M)$).

Scattering of wave packets

- The asymptotics of the wave packet can be computed explicitly :

$$W_+u(\tau,k) = \int_{\mathfrak{a}^*} e^{-i\lambda(\tau)} b(-\lambda) a_\lambda(k)\, d\lambda,$$

where $b(\lambda) = \pi(\lambda)c(\lambda)$,

$$c(\lambda) = \int_{\bar{N}} e^{-\langle i\lambda+\rho, H(\bar{n})\rangle}\, d\bar{n}$$

is the *Harish-Chandra function* which is explicitly given by the famous Gindikin–Karpelevich formula,

$$c(\lambda) = \prod_{\alpha\in\Delta_+} c_\alpha(\lambda), \quad c_\alpha(\lambda) = B\left(\frac{1}{2}m_\alpha, \frac{1}{4}m_{\alpha/2} + \frac{\langle i\lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle}\right).$$

Intertwining operators

- The principal series representations $T_\lambda, T_{w\cdot\lambda}$ are unitarily equivalent for all $w \in W$.

$$T_\lambda(g)S^w_\lambda = S^w_\lambda T_{w\lambda}(g) \quad \text{for all} \quad g \in G.$$

- Explicitly in our realization :

$$(S^w_\lambda a)(k) = \frac{1}{c_w(\lambda)} \int_{\bar{N}_w} e^{-\langle i\lambda+\rho, H(\bar{n}_w)\rangle} a(km_w \varkappa(\bar{n}_w))\, d\bar{n}_w.$$

where m_w is an element of the normalizer of $\mathfrak{a}$ in K which represents the element w of the Weyl group and

$$c_w(\lambda) = \int_{\bar{N}_w} e^{-\langle i\lambda+\rho, H(\bar{n})\rangle}\, d\bar{n}_w$$

is the partial c-function.

Intertwining operators

- Here $\bar{N}$ is the unipotent subgroup which is opposite to $\bar{N}_w = \bar{N} \cap m_w^{-1} N m_w$.
- In our realization, the restriction of the principal series representations to K is the standard action in $L_2(K/M)$ by left translations ; hence S^w preserves the spherical vector (up to a scalar factor). We normalize S^w in such a way that $S^w \cdot 1 = 1$.
- Cocycle relation :

$$S^{wv} = S^w \cdot R_w^{-1} S^v R_w.$$

- Scattering operators as defined above are proportional to intertwining operators :

$$\mathbf{S}_\lambda^w = \left[\frac{b(-w\lambda)}{b(-\lambda)} \right] S_\lambda^w.$$

Light cone

- Set

$$C+ = \{(t,x) \in \mathfrak{a}_+ \times X \; ; \; |t(x)| < |t|\} .$$

- We define the Hardy space H_2^- of functions sur $\mathfrak{a}_{\mathbb{C}}^*$ with values in $L_2(K/M)$ which are regular in the tubular domain $\operatorname{Im}\lambda \in \mathfrak{a}_-^*$.
- The outgoing solutions are characterized by the condition :

$$[b(-\lambda)]a \in H_2^- \quad \Rightarrow \quad u|_{C_+} = 0.$$

  - **Casuality :** $S^w H_2^- \subset H_2^-$ for all $w \in W$; moreover, if $w \geq v$ (with respect to the Bruhat partial order), we have $S^w H_2^- \subset S^v H_2^-$.

Huygens Principle

- In contrast with the normalized intertwining operators, scattering operators $\mathbf{S}^w$ in general *do not satisfy* casuality condition because of the scalar factor $\frac{b(-w\lambda)}{b(-\lambda)}$.

- **Huygens Principle.** All solutions with compact support are eventually outgoing, i.e., they disappear identically after a while in any given compact domain.
- Why this property does not hold ?
  - Because of the “mass gap” in the space of non-zero curvature.
  - Because of “wave diffusion”.
- The first possibility is avoided by the good choice of the wave equation (based on the Harish-Chandra isomorphism !)

Huygens Principle

- The second one deserves a thorough study.
- **Theorem.** The following assertions are equivalent :
  - Operators $\mathbf{S}^w$ are causal.
  - All roots of (G, K) have even multiplicities.
  - Huygens Principle for the generalized wave equation is valid.
- The difficulties with the Huygens Principle are due to the poles of $\frac{b(-w\lambda)}{b(-\lambda)}$ which lie in the wrong domain (unless $b(\lambda)^{-1}$ is polynomial, which happens exactly when all roots of (G, K) have even multiplicities. A similar phenomenon is well known for the ordinary wave equation in $\mathbb{R}^{2k}$.

Huygens Principle

- Wave equation $\mathbb{R}^n$ is a special case of generalized wave equation in symmetric spaces of zero curvature. This time it is an overdetermined system of differential equations with constant coefficients ; its construction is parallel to the one described above. The analogue of the Harish-Chandra isomorphism is the isomorphism $D^{G_0}(X_0) \simeq P^K(\mathfrak{p}^*) \simeq P^W(\mathfrak{a}^*)$
  - **Theorem.** Huygens Principle is valid for $X = G/K$ if and only if it is valid for the associated symmetric space of zero curvature.
  - The analytic properties scattering operators in these cases are quite different :

Huygens Principle

- For $X = G/K$ scattering operators are meromorphic (with poles in the 'wrong' half-planes). When the curvature tends to zero, these poles condense to give cuts ; hence in this case scattering operators are close to Hilbert transform.

- Why the violation of Huygens Principle is unfavorable ?
- In this case :
  - The subspaces $\mathcal{D}_+, \mathcal{D}_-$ are not orthogonal.
  - Scattering operators are not holomorphic in the upper half-plan (or in the tubular domain $\mathrm{Im}\, k \in \mathfrak{a}_+$).

Huygens Principle

  - there is no immedate way to define the contraction semigroup associated with the group of evolution operators.

- A failed (but very seducing) project :
  - To treat reducible representations of the principal series in the framework of Scattering theory.
- A curious fact :
  - Huygens Principle does not hold for the wave equation in the Poincaré half-plane.
  - But it holds for the automorphic wave equation !

Factorized Scattering

- Until now I did not mention the key property of Scattering operators in the case of symmetric spaces of rank $r > 1$: they are actually reduced to the rank 1 case, due to the factorization formulae which generalize the formula of Gindikin–Karpelevich. This property matches perfectly the non-stationnary approach described above :
  - Namely, on has to study the asymptotic behavior of solutions along singular geodesics associated with the walls of Weyl chambers.
    - This study gives rise to a whole tower of Plancherel type theorems and allows to decompose Scattering operators into product of operators of the same type type associated to symmetric spaces of lower rank.