Cluster structure of the quantum Coxeter–Toda system

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I will talk about some joint work (`arXiv: 1806.00747`) with **Alexander Shapiro** on an application of cluster algebras to quantum integrable systems.

Setup for higher Teichmüller theory

$$S = \text{ a marked surface}$$

$$G = PGL_n(\mathbb{C})$$

$$\mathcal{X}_{G,S} := \text{moduli of framed } G\text{–local systems on } S$$

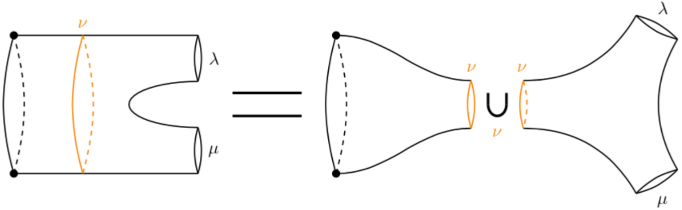
Modular functor conjecture

As Sasha explained, Fock and Goncharov constructed an assignment

$$S \rightsquigarrow \{\text{algebra } \mathcal{X}^q_{G,S}, \text{ Hilbert space representation } V \text{ of } \mathcal{X}^q_{G,S}, \\ \text{unitary rep of } \mathrm{MCG}(S) \text{ on V}\}.$$

Conjecture: (Fock–Goncharov '09) This assignment is local with respect to **gluing** of surfaces.

Modular functor conjecture



i.e. if $S = S_1 \cup_\gamma S_2$, there should be a mapping–class–group equivariant isomorphism

$$V[S] \simeq \int_\nu^\oplus V_\nu[S_1] \otimes V_\nu[S_2]$$

$\nu =$ eigenvalues of monodromy around loop γ

Relation with quantum Toda system

So we need to understand the **spectral theory** of the operators $\hat{H}_1, \ldots, \hat{H}_n$ quantizing the functions on $\mathcal{X}_{G,S}$ sending a local system to its eigenvalues around γ.

If we can find a **complete set of eigenfunctions** for $\hat{H}_1, \ldots, \hat{H}_n$ that are simultaneous eigenfunctions for the **Dehn twist** around γ, we can construct the isomorphism and prove the conjecture.

Rank 1 case

For $G = PGL_2$, this amounts to simultaneously diagonalizing the *geodesic length operator*

$$H = e^{2\pi b\hat{p}} + e^{-2\pi b\hat{p}} + e^{2\pi b\hat{x}} \quad (b \in \mathbb{R})$$

and the Dehn twist operator D.

Kashaev '00: the operators H, D have common eigenfunctions

$$\Psi_\lambda, \quad \lambda \in \mathbb{R}_{>0},$$

with eigenvalues

$$H \cdot \Psi_\lambda = 2\cosh(\lambda)\Psi_\lambda,$$

$$D \cdot \Psi_\lambda = e^{2\pi i\lambda^2}\Psi_\lambda.$$

Rank 1 case

The eigenfuctions Ψ_λ are orthogonal and complete:

$$\int_{\mathbb{R}} \Psi_\lambda(x)\overline{\Psi_\mu(x)}dx = \frac{\delta(\lambda-\mu)}{m(\lambda)}$$

$$\int_{\mathbb{R}_{>0}} \Psi_\lambda(x)\overline{\Psi_\lambda(y)}m(\lambda)d\lambda = \delta(x-y),$$

where the spectral measure is

$$m(\lambda) = \sinh(b\lambda)\sinh(b^{-1}\lambda),$$

and $q = e^{\pi i b^2}$.

Higher rank: quantum Coxeter–Toda system

How to generalize Kashaev's result to higher rank gauge groups?

First, we need to identify the operators $\hat{H}_1, \ldots, \hat{H}_n$.

Theorem (S.–Shapiro '17)

When $G = PGL_n$, there exists a cluster for $\mathcal{X}^q_{G,S}$ in which the operators $\hat{H}_1, \ldots, \hat{H}_n$ are identified with the Hamiltonians of the **quantum Coxeter–Toda** *integrable system.*

Classical Coxeter–Toda system

Fix $G = PGL_n(\mathbb{C})$, equipped with a pair of opposite Borel subgroups $B_\pm$, torus H, and Weyl group $W \simeq S_n$.

We have **Bruhat cell decompositions**

$$\begin{aligned} G &= \bigsqcup_{u \in W} B_+ u B_+ \\ &= \bigsqcup_{v \in W} B_- v B_- . \end{aligned}$$

The **double Bruhat cell** corresponding to a pair $(u, v) \in W \times W$ is

$$G^{u,v} := (B_+ u B_+) \cap (B_- v B_-) .$$

Poisson structure on G

There is a natural Poisson structure on G with the following key properties:

- $G^{u,v} \subset G$ are all **Poisson subvarieties**, whose Poisson structure descends to quotient $G^{u,v}/Ad_H$.
- The algebra of conjugation invariant functions $\mathbb{C}[G]^{Ad_G}$ (generated by traces of finite dimensional representations of G) is **Poisson commutative**:

$$f_1, f_2 \in \mathbb{C}[G]^{Ad_G} \implies \{f_1, f_2\} = 0$$

Hoffmann–Kellendonk-Kutz-Reshetikhin '00:

Suppose u, v are both **Coxeter** elements in W (each simple reflection appears exactly once in their reduced decompositions)

e.g. $G = PGL_4, \quad u = s_1 s_2 s_3, \quad v = s_1 s_2 s_3.$

In particular, Coxeter elements have length

$$l(u) = \dim(H) = l(v)$$

In this Coxeter case,

$$\dim(G^{u,v}/Ad_H) = l(u) + l(v)$$
$$= 2\dim(H).$$

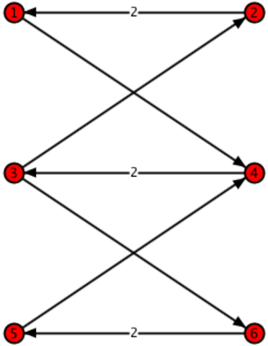
And we have $\dim(H)$–many independent generators of our Poisson commutative subalgebra $\mathbb{C}[G]^G$ (one for each fundamental weight)

$\implies$ we get an **integrable system** on $G^{u,v}/Ad_H$, called the **Coxter–Toda** system.

Double Bruhat cells

Berenstein–Fomin–Zelevinsky: the double Bruhat cells $G^{u,v}$, and their quotients $G^{u,v}/Ad_H$ are *cluster Poisson varieties.*

e.g. $G = PGL_4$, $u = s_1 s_2 s_3$, $v = s_1 s_2 s_3$. The quiver for $G^{u,v}/Ad_H$ is



Cluster Poisson structure on G

The Poisson bracket on $\mathbb{C}[G^{u,v}]$ is **compatible** with the cluster structure: if $Y_1, \ldots, Y_d$ are cluster coordinates in chart labelled by quiver Q,

$$\{Y_j, Y_k\} = \epsilon_{jk} Y_j Y_k,$$

where

$$\epsilon_{jk} = \#(j \to k) - \#(k \to j)$$

is the signed adjacency matrix of the quiver.

Quantization of Coxeter–Toda system

Consider the Heisenberg algebra $\mathcal{H}_n$ generated by

$$x_1, \ldots, x_n; \quad p_1, \ldots, p_n,$$

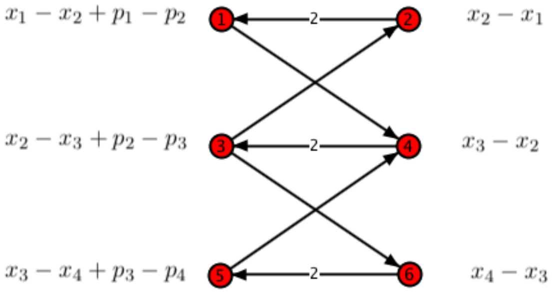
$$[p_j, x_k] = \frac{\delta_{jk}}{2\pi i}$$

acting on $L^2(\mathbb{R}^n)$, via

$$p_j \mapsto \frac{1}{2\pi i}\frac{\partial}{\partial x_j}$$

Quantization of Coxeter–Toda system

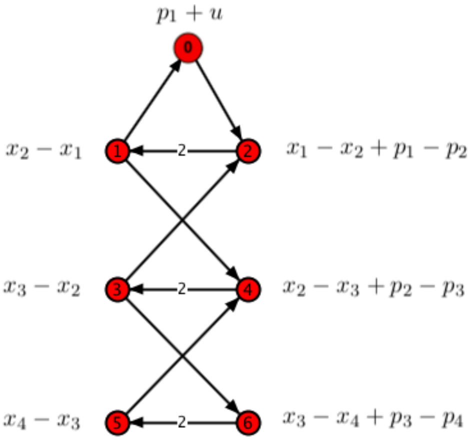
A representation of the quantum torus algebra for the Coxeter–Toda quiver:



e.g. $\hat{Y}_2$ acts by multiplication by positive operator $e^{2\pi b(x_2 - x_1)}$.

Cluster construction of quantum Hamiltonians

Let's add an extra node to our quiver, along with a "spectral parameter" u:



Construction of quantum Hamiltonians

Theorem (S.–Shapiro)

Consider the operator $Q_n(u)$ obtained by mutating consecutively at $0, 1, 2, \dots, 2n$. Then

1. *The quiver obtained after these mutations is isomorphic to the original;*
2. *The unitary operators $Q_n(u)$ satisfy*

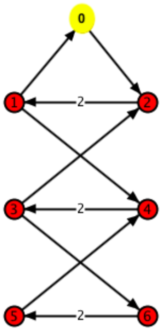
$$[Q_n(u), Q_n(v)] = 0,$$

3. *If $A(u) = Q_n(u - ib/2)Q_n(u + ib/2)^{-1}$, then one can expand*

$$A(u) = \sum_{k=0}^{n} H_k U^k, \quad U := e^{2\pi bu}$$

and the commuting operators $H_1, \dots, H_n$ quantize the Coxeter–Toda Hamiltonians.

Example: $G = PGL_4$



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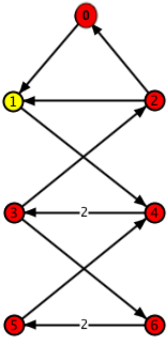

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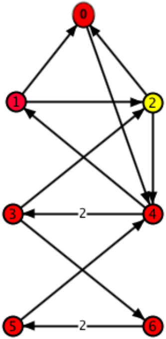

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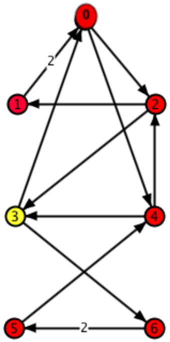

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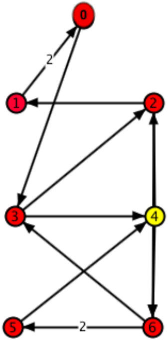

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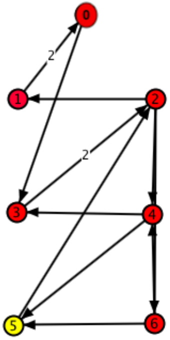

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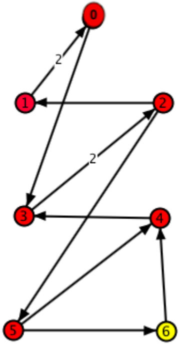

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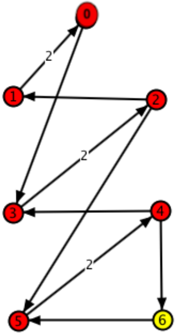
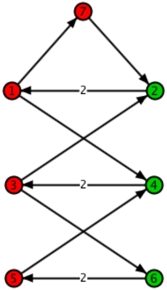


The Dehn twist

The Dehn twist operator D_n from quantum Teichmüller theory corresponds to mutation at all even vertices $2, 4, \ldots, 2n$:



We have

$$[D_n, Q_n(u)] = 0$$

Quantum Coxeter–Toda eigenfunctions

Kharchev–Lebedev–Semenov-Tian-Shansky '02: recursive construction of common eigenfunctions for $\mathfrak{gl}_n$ Hamiltonians $H_1, \ldots, H_n$ via *Mellin–Barnes* integrals:

$$\Psi^{\mathfrak{gl}_{n+1}}_{\lambda_1,\ldots,\lambda_{n+1}}(x_1,\ldots,x_{n+1}) = \int_{\mathbb{R}^n} \mathcal{K}(\lambda,\gamma|x_{n+1})\Psi^{\mathfrak{gl}_n}_{\gamma_1,\ldots,\gamma_n}(x_1,\ldots,x_n)d\gamma.$$

The b–Whittaker functions $\Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n}$ satisfy

$$H_k \cdot \Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n} = e_k(\lambda)\Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n},$$

where $e_k(\lambda)$ is the k–th elementary symmetric function in

$$e^{2\pi b\lambda_1}, \ldots, e^{2\pi b\lambda_n}.$$

Cluster construction of quantum Toda eigenfunctions

Using the cluster realization of the Toda Hamiltonians, we give a **dual** recursive construction of the b–Whittaker functions: we expand

$$\Psi^{\mathfrak{gl}_{n+1}}_{\lambda_1,\ldots,\lambda_{n+1}}(x_1,\ldots,x_{n+1}) = \int_{\mathbb{R}^n} \mathcal{L}(\mathbf{x},\mathbf{y}|\lambda_{n+1})\Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n}(y_1,\ldots,y_n)d\mathbf{y}.$$

Using this construction, we can

- compute the eigenvalues of the Dehn twist D_n; and
- establish orthogonality and completeness of the b–Whittaker functions.

Construction of the eigenfunctions

Problem: Construct complete set of joint eigenfunctions for operators $Q_n(u), D_n$.

e.g. $n = 1$.

$$Q_1(u) = \varphi(p_1 + u)$$

If $\lambda \in \mathbb{R}$,

$$\Psi_\lambda(x_1) = e^{2\pi i \lambda x_1}$$

satisfies

$$Q_1(u)\Psi_\lambda(x_1) = \varphi(\lambda + u)\Psi_\lambda(x_1).$$

(equivalent to formula for Fourier transform of $\varphi(z)$)

Recursive construction: we want to find an operator $\mathcal{R}_n^{n+1}(\lambda)$ such that

$$\mathcal{R}_n^{n+1}(\lambda_{n+1}) \cdot \Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n} = \Psi^{\mathfrak{gl}_{n+1}}_{\lambda_1,\ldots,\lambda_{n+1}}$$

Set

$$\mathcal{R}_n^{n+1}(\lambda) = Q_n\left(\lambda^*\right) \frac{e^{2\pi i \lambda x_{n+1}}}{\varphi(x_{n+1} - x_n)},$$

where $\lambda^* = \frac{i(b+b^{-1})}{2} - \lambda$.

Pengaton identity $\implies$

$$Q_{n+1}(u)\ \mathcal{R}_n^{n+1}(\lambda) = \varphi(u+\lambda)\ \mathcal{R}_n^{n+1}(\lambda)\ Q_n(u).$$

Recursive construction of the eigenfunctions

So given a $\mathfrak{gl}_n$ eigenvector $\Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n}(x_1,\ldots,x_n)$ satisfying

$$Q_n(u)\Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n} = \prod_{k=1}^{n} \varphi(u+\lambda_k)\ \Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n},$$

we can build a $\mathfrak{gl}_{n+1}$ eigenvector

$$\Psi^{\mathfrak{gl}_{n+1}}_{\lambda_1,\ldots,\lambda_{n+1}}(x_1,\ldots,x_{n+1}) := \mathcal{R}_n^{n+1}(\lambda_{n+1}) \cdot \Psi^{\mathfrak{gl}_n}_{\lambda_1,\ldots,\lambda_n}$$

satisfying

$$Q_{n+1}(u)\Psi^{\mathfrak{gl}_{n+1}}_{\lambda_1,\ldots,\lambda_{n+1}} = \prod_{k=1}^{n+1} \varphi(u+\lambda_k)\ \Psi^{\mathfrak{gl}_{n+1}}_{\lambda_1,\ldots,\lambda_{n+1}}.$$

Dehn twist eigenvalue

Similarly, the pentagon identity implies

$$D_{n+1} R_n^{n+1}(\lambda) = e^{\pi i \lambda^2} R_n^{n+1}(\lambda) D_n,$$

so by the recursion we derive the Dehn twist spectrum

$$D_n \cdot \Psi_{\lambda_1,\ldots,\lambda_n} = e^{\pi i (\lambda_1^2 + \cdots + \lambda_n^2)} \Psi_{\lambda_1,\ldots,\lambda_n}.$$

A modular b–analog of Givental's integral formula

Writing all the $R_n^{n+1}(\lambda)$ as integral operators, we get an explicit Givental–type integral formula for the eigenfunctions:

$$\Psi_\lambda^{(n)}(x) = e^{2\pi i \lambda_n \underline{x}} \int \prod_{j=1}^{n-1} \left(e^{2\pi i \underline{t}_j(\lambda_j - \lambda_{j+1})} \prod_{k=2}^{j} \varphi(t_{j,k} - t_{j,k-1}) \right.$$
$$\left. \prod_{k=1}^{j} \frac{dt_{j,k}}{\varphi(t_{j,k} - t_{j+1,k} - c_b)\varphi(t_{j+1,k+1} - t_{j,k})} \right),$$

where $t_{n,1} = x_1, \ldots, t_{n,n} = x_n$.
e.g. $n = 4$ we integrate over all but the last row of the array

$$\begin{array}{llll} t_{11} & & & \\ t_{21} & t_{22} & & \\ t_{31} & t_{32} & t_{33} & \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

Orthogonality and completeness

Using the cluster recursive construction of $\Psi^{\mathfrak{gl}_n}$, we can prove the orthogonality and completeness relations

$$\int_{\mathbb{R}^n} \Psi_\lambda^{\mathfrak{gl}_n}(x)\overline{\Psi_\mu^{\mathfrak{gl}_n}(x)}dx = \frac{\delta(\lambda-\mu)}{m(\lambda)},$$

$$\int_{\mathbb{R}^n} \Psi_\lambda^{\mathfrak{gl}_n}(x)\overline{\Psi_\mu^{\mathfrak{gl}_n}(y)}m(\lambda)d\lambda = \delta(x-y),$$

with spectral measure

$$m(\lambda) = \prod_{j<k} \sinh(\pi b(\lambda_j-\lambda_k))\sinh(\pi b^{-1}(\lambda_j-\lambda_k)).$$

Unitarity of the b–Whittaker transform

Theorem (S.–Shapiro)

The b–Whittaker transform

$$(\mathcal{W}[f])(\lambda) = \int_{\mathbb{R}^n} \Psi_\lambda^{\mathfrak{gl}_n}(x) f(x) dx$$

is a unitary equivalence.

This completes the proof of the Fock–Goncharov conjecture for $G = PGL_n$.