Combinatorics of the ASEP on a ring and Macdonald polynomials

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based on joint work with Sylvie Corteel (Paris Diderot) and Lauren Williams (UC Berkeley)

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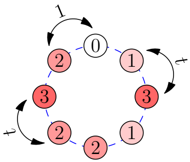
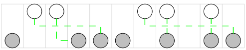
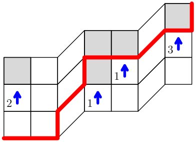
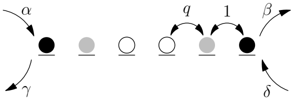




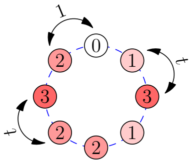


overview: asymmetric simple exclusion process (ASEP) and orthogonal polynomials

ASEP with open boundaries



ASEP on a ring



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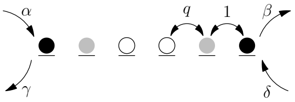

Koornwinder-Macdonald polynomials
(Corteel-Williams, Cantini, 2015)

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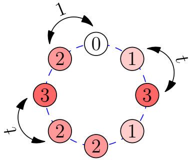

Macdonald polynomials (type A)
(Prolhac-Evans-Mallick 2009)

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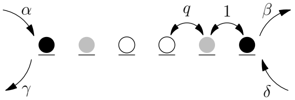

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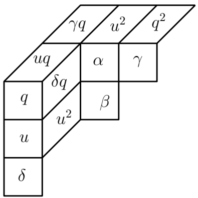

- (Corteel-M.-Williams, 2016)

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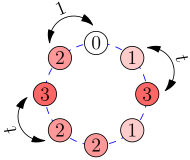

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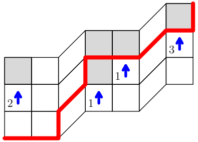
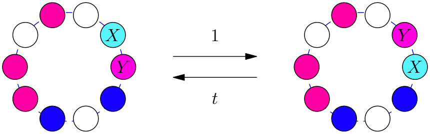


- (Corteel-M.-Williams, 2018)

2-species ASEP on a ring

- fix a ring with n sites and k, r, ℓ such that $n = k + r + \ell$.
- $ASEP(k, r, \ell)$ is a Markov chain on states $X \in \{2, 1, 0\}^n$ with k 2's, r 1's, and ℓ 0's
- possible transitions are swaps of adjacent particles:



with rate 1 if particle X has a bigger label than particle Y, and rate t otherwise. $(0 \leq t \leq 1)$

- we wish to compute the steady state probabilities $\Pr(X)$ of states of the ASEP

example for ASEP of size $(1, 1, 1)$:

$$\Pr(2,1,0) = \Pr(1,0,2) = \Pr(0,2,1) = \frac{1}{Z}(1+2t)$$

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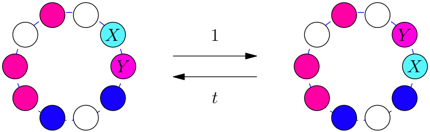

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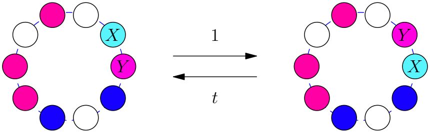

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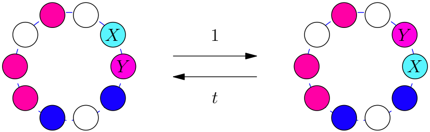

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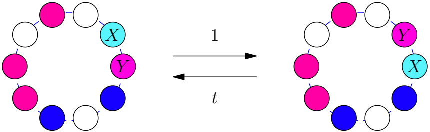

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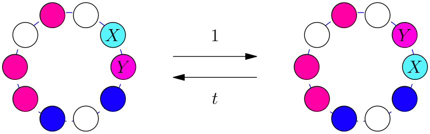

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Matrix Ansatz: 1-species ASEP with open boundaries

Theorem (Derrida-Evans-Hakim-Pasquier 1993)

Let $X = X_1 \dots X_n$*, with* $X_i \in \{2, 0\}$*, be a word representing a state of the 1-species ASEP. If there exist matrices* D *and* E *and vectors* $\langle w|$ *and* $|v\rangle$*, that satisfy:*

$$DE = tED + D + E,$$

$$\langle w|E = \frac{1}{\alpha}\langle w|, \qquad D|v\rangle = \frac{1}{\beta}|v\rangle,$$

then

$$\Pr(X) = \frac{1}{Z_n}\langle w|\, \mathrm{Mat}(X)|v\rangle$$

where $\mathrm{Mat}(X) = \mathrm{Mat}(X_1)\dots\mathrm{Mat}(X_n)$ *is a map defined by* $\mathrm{Mat}(2) = D$, $\mathrm{Mat}(0) = E$*, and* $Z_n = \langle w|(D+E)^n|v\rangle$.

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$\Pr(X)$ **can be expressed in terms of a product of non-commuting operators** D **and** E**, where** $2 \mapsto D$**,** $1 \mapsto A$**, and** $0 \mapsto E$

Example:

$$\Pr(200020) = \frac{1}{Z_6}\langle w|DEEEDE|v\rangle$$

combinatorial interpretation of the quadratic algebra $DE = qED + D + E$

Given a word in $\mathrm{Mat}(X)$ in D's and E's, we can apply the relation $DE = qED + D + E$ to get

$$\mathrm{Mat}(X) = \sum_{i,j} c_{i,j}(t) E^i D^j$$

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For example, let $X = 2202$. Then

$$\begin{aligned}\mathrm{Mat}(X) &= DDED = D(tED + D + E)D \\ &= t(tED + D + E)DD + DDD + (tED + D + E)D \\ &= t^2 EDDD + (t+1)DDD + 2tEDD + DD + ED.\end{aligned}$$

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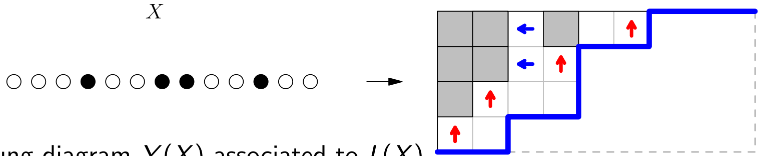
Define a Q-tableau from this algebra by assigning D to vertical edge and E to horizontal edge and applying local edge rewriting rules. (due to X. Viennot)

alternative tableaux for 1-species ASEP

- Lattice path $L(X)$ by drawing $1 \mapsto$ south edge and $0 \mapsto$ west edge



- Young diagram $Y(X)$ associated to $L(X)$
- filling of $Y(X)$ with up-arrows and left-arrows such that **any box pointed at by an arrow must be empty**
- the weight of a tableaux is a monomial: $\mathrm{wt}(T) = \alpha^{\#\uparrow(T)}\beta^{\#\leftarrow(T)}q^{\mathrm{free}(T)}$.

Note that alternative tableaux are in bijection with *permutation tableaux*.

Theorem (Corteel-Williams 2007)

For the 1-species ASEP with α, β, q

$$\Pr(X) = \frac{1}{Z_n} \sum_{T \in AT(X)} \mathrm{wt}(T)$$

where $Z_n = \sum_{T \in AT(n)} \mathrm{wt}(T)$ is the partition function.

2-species ASEP generalization: Matrix Ansatz

For ASEP with particles of type 2, 1, 0, add third operator A to get the quadratic algebra (Uchiyama '09):

$$DE = tED + D + E, \qquad DA = tAD + A, \qquad AE = tEA + A$$

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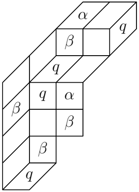
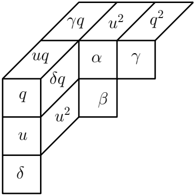
Generalization of the alternative tableaux:



(M.-Viennot '15)



(Corteel-M.-Williams '16)

Get formula for Koornwinder-Macdonald polynomials (for partitions consisting of a single row/column) at $q = t$.

2-ASEP on a ring

Matrix ansatz (DEHP):

$$
\begin{aligned}
DE &= tED + (1-t)D + E,\\
DA &= tAD + (1-t)A,\\
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then

$$\Pr(X) = \frac{1}{Z_{(k,r,\ell)}} \operatorname{tr}(\mathrm{Mat}(X))$$

where Mat : $\{2,1,0\} \mapsto \{D, A, E\}$ and X has k 2's, r 1's, and ℓ 0's.

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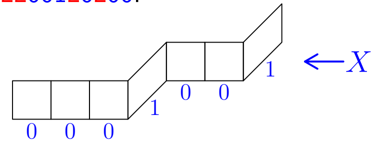
we will split the infinite tableaux into classes and define finite **cylindric tableaux**, with each tableau representing a class and having a weight which is a generating function (instead of a polynomial)

cylindric rhombic tableaux (CRT)

Let $X \in ASEP(k, r, \ell)$.

An X-strip is a strip of squares and rhombi corresponding to the 0,1-sub-word of X. For example, $X = 12200120200$:



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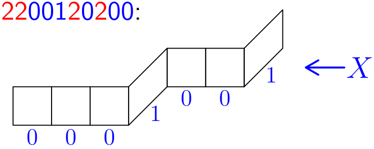
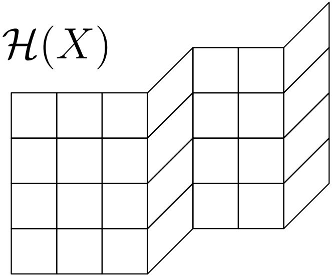


Join k X-strips to obtain the shape $\mathcal{H}(X)$ and superimpose a path $P(X)$:



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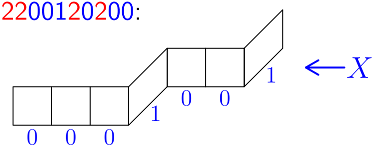

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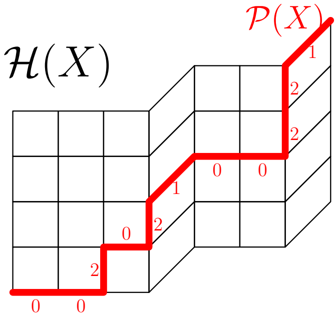
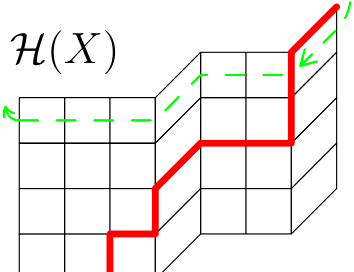


cylindric rhombic tableaux

- Identify the left and right boundaries of $\mathcal{H}(X)$, making it a cylinder.
- Horizontal strips start from $P(X)$ and wrap around to the left.
- An up-arrow can be in a square tile, killing the cells above it. (So, there can be at most one per column.)
- at $t = 0$, these tableaux are in bijection with **multiline queues**, which are an object due to Ferrari-Martin from 2005 that has been used to study the m-TASEP (M. 2018)
- choose any placement of up-arrows and some order σ assigned to the arrows in each row.



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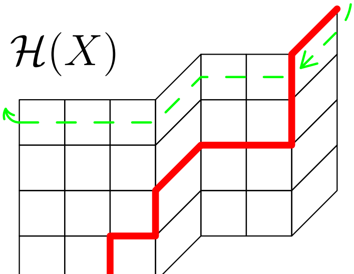

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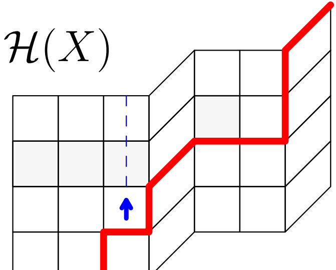

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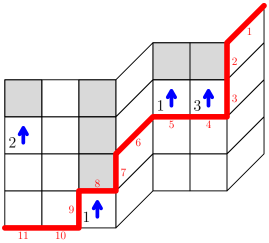

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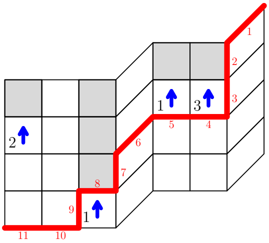
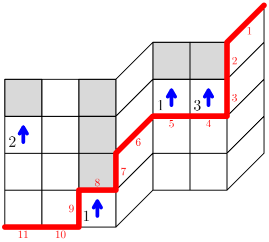


cylindric rhombic tableaux

- choose any placement of up-arrows and some order σ assigned to the arrows in each row.
- t keeps track of how many available locations are "skipped" with a **disorder** statistic
- Example:

$$\mathrm{wt}(T) = \frac{t^{10}}{[7]_t[6]_t[5]_t[4]_t},$$

where $[j]_t = (1-t^j)/(1-t)$.



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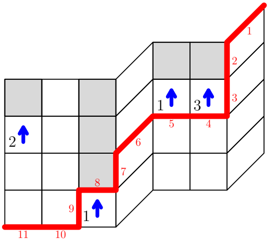

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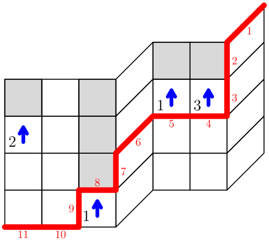

probabilities of the 2-ASEP on a ring

Theorem (Corteel-M.-Williams 2018)

Let $X \in \text{States}(k, r, \ell)$ *be a state of the 2-ASEP on a ring. Then*

$$\Pr(X) = \frac{1}{Z_{(k,r,\ell)}} \sum_{(T,\sigma)\in \text{CRT}(X)} \text{wt}(T, \sigma).$$

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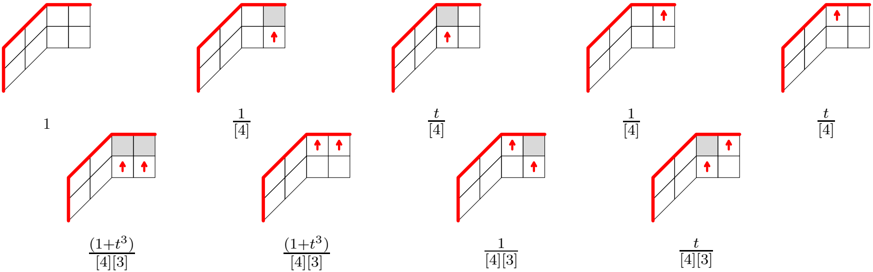

Example: $\pi(001122) = \frac{1}{Z_{(2,2,2)}[4]_t[3]_t}(6 + t + 6t^2 + t^3 + t^4)$

Nonsymmetric Macdonald polynomials

- symmetric Macdonald polynomials $P_\lambda(\mathbf{x}; q, t)$ are q, t generalizations of Schur polynomials. Can be defined via symmetrization of *nonsymmetric Macdonald polynomials* E_μ.
- E_μ introduced in 1995 (Opdam, Cherednik), can be computed from Yang-Baxter graphs, with combinatorial formula due to Haiman-Haglund-Loehr '05
- recent work of Cantini-de Gier-Wheeler computes matrix product for polynomials $f_\mu(\mathbf{x}; q, t)$ which are solutions to a certain affine Hecke algebra.
- $f_\mu(1, \ldots, 1; 1, t)$ is the steady state probability of state μ of the ASEP on a ring.
- f_μ are related to the E_λ's:

$$E_\lambda = \sum_{\mu \geq \lambda} a_{\lambda,\mu}(q, t) f_{\mu^c}$$

and the P_λ's, summed over permutations μ of the parts in λ:

$$P_\lambda(\mathbf{x}; q; t) = \sum_\mu f_\mu(\mathbf{x}; q, t).$$

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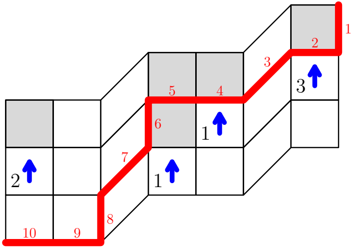
enhanced weights on CRT to compute the f_μ

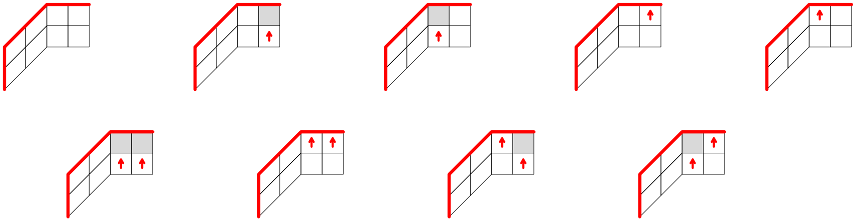
To a CRT with arrow placement and order (T, σ), we can assign a weight $\mathrm{wt}_{qtx}(T, \sigma)(\mathbf{x}; q, t)$ as follows:

- t's count the disorder statistic of (T, σ), and $[j]_{qt} = (1 - qt^j)/(1 - t)$
- associate x_i's to the edges of $P(X)$ and have them keep track of which strips contain the *last arrow* in a given row according to σ
- have q count the number of times we "cross the boundary" of the CRT to traverse the arrows in order σ



$$\mathrm{wt}_{qtx}(T, \sigma) = \frac{t^{12} q^3}{[7]_{qt}^{(4)}!} x_2 x_3 x_4^2 x_5 x_6 x_7 x_8 x_9^2 x_{10}^2$$

example for $X = 001122$



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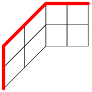
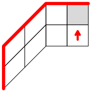
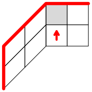


$x_1^2x_2^2x_3x_4$



$\frac{qx_1x_2^2x_3x_4x_6}{[4]_{qt}}$



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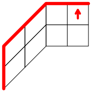
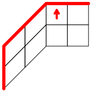
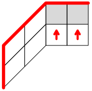


$\frac{qx_1x_2^2x_3x_4x_5}{[4]_{qt}}$



$\frac{qtx_1^2x_2x_3x_4x_5}{[4]_{qt}}$



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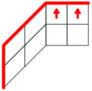

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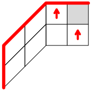

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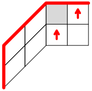

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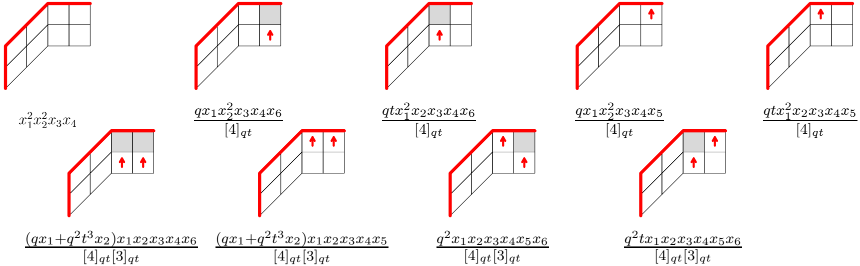
example for $X = 001122$



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In particular, when λ is a partition with parts of size at most 2, we obtain:

Theorem (Corteel-M.-Williams '18)

$$E_\lambda = \sum_{(T,\sigma)\in \mathrm{CRT}(\lambda^C)} \mathrm{wt}_{qtx}(T,\sigma).$$

e.g. nonsymmetric Macdonald poly. $E_{(2,2,1,1,0,0)}$ equals $f_{(0,0,1,1,2,2)}$ above.

summary

For 2-ASEP, we defined cylindric rhombic tableaux (CRT) with q, t, x weights that give formula $f_\mu(x_1, \ldots, x_n; q, t)$ such that:

- $\Pr(\mu) = f_\mu(1, \ldots, 1; 1, t)$ is the probability of the 2-ASEP on a ring
- $P_\lambda = \sum_\mu f(x_1, \ldots, x_n; q, t)$ is the partition function over the CRT when λ has parts of size 0, 1, 2
- when μ is a partition, $f_{\mu^C}(x_1, \ldots, x_n; q, t) = E_\mu$, the non-symmetric Macdonald polynomial (e.g. if $\mu = 211100$, then $\mu^C = 011122$).
- there is a bijection to *multiline queues* and a solution-in-progress for higher species.

Happy birthday Kolya!!!