Integrable Hamiltonian partial differential and difference equations and related algebraic structures

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1. Basic notions

Kostant Theorem. Any cocommutative Hopf algebra H over an algebraically closed field $\mathbb{F}$ of characteristic 0 is a smash product of $\mathcal{U}(\mathfrak{g})$ and $\mathbb{F}[\Gamma]$.

Definition. A Lie pseudoalgebra over H is an H-module L with an H-bilinear map

$$L \otimes L \to (H \otimes H) \otimes_H L, \; a \otimes b \mapsto [a * b],$$

satisfying skewsymmetry and Jacobi identity.

Example 0. $H = \mathbb{F}$ corresponds to Lie algebras.
Example 1. $H = \mathbb{F}[D]$ ($\mathfrak{g} = \mathbb{F}, \Gamma = \{e\}$) corresponds to (additive) Lie conformal algebras (LCA).
Example 2. $H = \mathbb{F}[S, S^{-1}]$ ($\mathfrak{g} = 0, \Gamma = \mathbb{Z}$) corresponds to multiplicative Lie conformal algebras (mLCA).
These three cases are the most important for the theory of Hamiltonian integrable systems.

To define these structures we shall use the equivalent language of λ-brackets.

LCA is $\mathbb{F}[D]$-module L, endowed with a λ-bracket $L \otimes L \to L[\lambda],\ a \otimes b \mapsto [a_\lambda b]$, satisfying

$$\begin{array}{rl} \text{(sesquilineariy)} & [Da_\lambda b] = -\lambda[a_\lambda b], D[a_\lambda b] = [Da_\lambda b] + [a_\lambda Db], \\ \text{(skewsymmetry)} & [b_\lambda a] = -[a_{-D-\lambda} b], \\ \text{(Jacobi)} & [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]. \end{array}$$

mLCA is $\mathbb{F}[S, S^{-1}]$-module L with λ-bracket $L \otimes L \to L[\lambda, \lambda^{-1}]$, satisfying the multiplicative version of the axioms:

$$\begin{array}{rl} \text{(sesquilineariy)} & [Sa_\lambda b] = \lambda^{-1}[a_\lambda b],\ S[a_\lambda b] = [Sa_\lambda Sb] \\ \text{(skewsymmetry)} & [b_\lambda a] = -[a_{(S\lambda)^{-1}} b] \\ \text{(Jacobi)} & [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda\mu} c] + [b_\mu [a_\lambda c]]. \end{array}$$

Let L be a Lie pseudoalgebra over H, $I \subset H$, the augmentation ideal. Let $\overline{L} = L/IL$ be the space of Hamiltonian functionals, and let $\int : L \to \overline{L}$ be the quotient map.

Key Lemma. The pseudobracket of L induces a canonical Lie algebra bracket $\{\int a, \int b\}$ on $\overline{L}$, and a representation $\{\int a, b\}$ of the Lie algebra $\overline{L}$ by derivation of L, commuting with H ("evolutionary vector fields").

Main examples.

LCA: $\overline{L} = L/DL, \{\int a, \int b\} = \int [a_\lambda b]_{\lambda=0}, \{a, \int b\} = [a_\lambda b]_{\lambda=0}$

mLCA: $\overline{L} = L/(S-1)L, \{\int a, \int b\} = \int [a_\lambda b]_{\lambda=1}, \{a, \int b\} = [a_\lambda b]_{\lambda=1}$

Definition. Given a Lie pseudoalgebra L and $\int h \in \overline{L}$ (a Hamiltonian functional), the associated Hamiltonian equation is

$$\frac{df}{dt_h} = \{\textstyle\int h, f\}, \ f \in L. \tag{1}$$

This equation is called integrable if $\int h$ is contained in an infinite-dimensional abelian subalgebra A of the Lie algebra $\overline{L}$. As $\int h$ runs over A, we obtain an integrable hierarchy of Hamiltonian equations. One says that elements of A are integrals of motion of (1) (since $\frac{d}{dt_h} \int h_1 = 0$) in involution. By key lemma, these equations are symmetries of each other:

$$\frac{d}{dt_h}\frac{d}{dt_{h_1}} = \frac{d}{dt_{h_1}}\frac{d}{dt_h} \text{ for all } h, h_1 \in A.$$

Main Problem. Construct integrable hierarchies of Hamiltonian equations.

The key observation is that Lie conformal algebras is the right framework for the theory of Hamiltonian PDE, while multiplicative Lie conformal algebras is the right framework for the theory of Hamiltonian differential-difference equations.

A structure theory of LCA and subsequently of Lie pseudoalgebras was developed at the end of last century: [K.96], [D'Andrea-K.98], [Bakalov-K.-Voronov99] on classification and cohomology of LCA, and [Bakalov-D'Andrea-K.01] on structure and classification of Lie pseudoalgebras (after learning the definition from [Beilinson-Drinfeld04]).

Applications of Lie conformal algebras to Hamitonian PDE were developed in [Barakat-De Sole-K.09], [De Sole-K.-Wakimoto 10], [De Sole-K. 12-13], [De Sole-K.-Valeri.13-18].

2. A construction of integrable Hamiltonian PDE's

One of the highlights of my talk (joint work with A. De Sole and D. Valeri, 2016)

Theorem

Let N be an integer ≥ 2. Then for each partition of N in a sum of positive integers we construct an integrable hierarchy of Hamiltonian PDE of Lax type.

Example 1: $N = 2$

partition $2 = 2$ corresponds to the **KdV** hierarchy of evolution equations, the simplest equation being ($u = u(x,t), u' = \frac{\partial u}{\partial x}$)):

$$\frac{du}{dt} = u''' + uu' \qquad (1895), (1877)$$

The first important discovery of the theory of integrable systems: KdV is integrable! (Gardner-Green-Kruskal-Miura, 1967)

partition $2 = 1 + 1$ corresponds to the **NLS** hierarchy (=AKNS), the simplest equation being

$$\begin{cases} \frac{du}{dt} = u'' + ku^2 v \\ \frac{dv}{dt} = -v'' - kuv^2 \end{cases} \qquad (1964)$$

Example 2, $N = 3$

partition $3 = 3$ corresponds to the **Boussinesq** hierarchy, the simplest equation being the Boussinesq equations (1872)

$$\frac{du}{dt} = v', \ \frac{dv}{dt} = u''' + uu'$$

partition $3 = 1 + 1 + 1$ corresponds to the **3 wave equation** hierarchy

partition $3 = 2 + 1$ corresponds to the **Yajima-Oikawa** hiearchy, the simplest equation describing sonic-Langmuir solitons:

$$\begin{cases} \frac{du}{dt} = -u'' + uw \\ \frac{dv}{dt} = v'' - vw \\ \frac{dw}{dt} = (uv)' \end{cases} \qquad (1976)$$

Example 3, general N
partition $N = N$ corresponds to the **N-th Gelfand-Dickey** hierarchy (1975)

partition $N = 2 + 1 + \cdots + 1$ corresponds to the $\mathbf{N-2}$**-component Yajima-Oikawa** hierarchy

partition $N = p + p + \cdots + p$ (r times) corresponds to the **p-th** $\mathbf{r \times r}$**-matrix Gelfand-Dickey hierarchy**

3. Lax equations

Definition (P. Lax 1968)

Let $L = L(t)$ and $P = P(t)$ be some linear operators, depending on a parameter t (time). Lax equation is

$$\frac{dL}{dt} = [P, L] \tag{2}$$

Theorem (Lax "theorem")

For all $n \geq 1$, $\frac{d}{dt}\mathrm{tr}\, L^n = 0$, *i.e. all* $\mathrm{tr} L^n$ *are integrals of motion of equation (2).*

"Proof." (2) easily implies that $\frac{dL^n}{dt} = [P, L^n]$. Take trace of both sides:

$$\frac{d}{dt}\mathrm{tr}\, L^n = \mathrm{tr}\,[P, L^n] = 0$$

since trace of any commutator is 0.

Lax main example (I ignore coefficients when they can be changed by rescaling):

$$L = D^2 + u, \quad P = D^3 + 2uD + u'.$$

Then $[P, L] = u''' + uu'$, hence

$$\frac{dL}{dt} = [P, L] \Leftrightarrow \text{KdV} : \ \frac{du}{dt} = u''' + uu'.$$

Unfortunately, the proof makes no sense in the infinite-dimensional setting: Let $P = \frac{d}{dx}$, $L = x$, then $[P, L] = I$ and $\mathrm{tr}[P, L] = \infty$, not 0.
But this can be fixed! (Gelfand-Dickey 1975)
Fractional powers: If $L = D^n + ...$, take $P = (L^{k/n})_+$ in (2), then $\int \mathrm{Res}_D\, L^{k/n}$ is an integral of motion of the Lax equation (2).

Another drawback, which hasn't been resolved completely to this day is that (2) should be selfconsistent.

Example

$L = D^3 + u, \ P = (L^{k/3})_+$. Then (2) for $k = 1$ is $\frac{du}{dt_1} = u'$, but for $k = 2$ it is $\frac{du}{dt_2} = 2u'D + u''$, inconsistent.

Theorem (Gelfand-Dickey 1975)

Let $n \geq 2$ *and*

$$L_n = D^n + u_1 D^{n-2} + \cdots + u_{n-1}.$$

Then one has the n*-th KdV hierarchy of compatible evolution equations of Lax type:*

$$\frac{dL_n}{dt_k} = [(L_n^{k/n})_+,\, L_n],\; k = 1, 2, \ldots \tag{3}$$

Moreover,

$$\int Res_D\, L_n^{k/n} \tag{4}$$

are integrals of motion in involution for all k*.*
Consequently, all equations (3) *form an integrable hierarchy of Hamiltonian PDE.*

Example 1. Lax operator for KdV:

$$L = D^2 + u$$

Example 2. Lax operator for Boussinesq:

$$L = D^3 + uD + v$$

Example 3. Lax operator for NLS:

$$L = D + uD^{-1}v$$

Example 4. Lax operator for Yajima-Oikawa:

$$L = D^2 + u + vD^{-1}w.$$

For all of them (3) is a hierarchy of compatible evolution equations, with inifinitely many integrals of motion (4).

In order to prove our theorem (the highlight), we construct, for each partition $\vec{p} = (p_1 \geq p_2 \geq \ldots > 0)$ of an integer $N \geq 2$, a Lax $r_1 \times r_1$ matrix (pseudo)differential operator $L_{\vec{p}}$ (where $r_1 =$ multiplicity of the largest part p_1 of $\vec{p}$) with the leading term D^{p_1}, such that

$$\frac{dL_{\vec{p}}}{dt_k} = \left[\left(L_{\vec{p}}^{k/p_1} \right)_+ , \; L_{\vec{p}} \right] \quad (k \in \mathbb{Z})$$

is an integrable hierarchy of compatible evolution equations with infinitely many integrals of motion

$$\int \mathrm{Res}_D \, \mathrm{tr} \, L_{\vec{p}}^{k/p_1} \quad (k \in \mathbb{Z}).$$

4. Hamiltonian formalism for PVA

Our proof uses the Hamiltonian formalism for PDE.

What is a Hamiltonian ODE?

$$\frac{du_i}{dt} = \{h, u_i\},\ i = 1, ..., \ell,$$

where $u_i = u_i(t), h = h(u_1, \ldots, u_\ell)$ is a Hamiltonian function, and $\{.\,,.\}$ is a Poisson bracket on the space of functions in the u_i i.e. $\{.\,,.\}$ is a Lie algebra bracket satisfying the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + f\{g, h\}.$$

Hamiltonian formalism for PDE was introduced by Gardner and Faddeev-Zakharov in 1971 in coordinate form.

More recently it was introduced in coordinate free form [Barakat-DeSole-K. 09].

The key idea: lift the Lie algebra bracket $\{.,.\}$ on the space of Hamiltonian functionals to a λ-bracket $\{._\lambda.\}$ on the algebra of functions.

Definition

A Poisson vertex algebra (PVA) is a differential algebra $\mathcal{V}$ (a unital commutative associative algebra with a derivation D), endowed with a structure of a Lie conformal algebra $\{\cdot_\lambda\cdot\}$, related to the product by the Leibniz rule $\{f_\lambda gh\} = \{f_\lambda g\}h + g\{f_\lambda h\}$.

Explanation of terminology. PVA $\mathcal{V}$ is a quasicalssical limit (associated graded) of a vertex algebra V (a quantization of $\mathcal{V}$). Vertex algebra is defined via the operator product expansion of quantum fields

$$a(z)b(w) = \sum_{j\in\mathbb{Z}} \frac{c^j(w)}{(z-w)^{j+1}},$$

by defining LCA structure $[a(w)_\lambda b(w)] = \sum_{j\geq 0} \frac{\lambda^j}{j!} c^j(w), D = \partial_w$, and product $: a(w)b(w) := c_{-1}(w)$. In the quasiclassical limit product becomes commutative and associative and "Wick formula" turns into Leibniz rule.

Example

In coordinates $\mathcal{V} = \mathbb{F}[u, u', u'', \dots]$, $Du^{(n)} = u^{(n+1)}$. Define the λ-bracket on $\mathcal{V}$ by

$$\{u_\lambda u\} = \lambda \cdot 1$$

and extend by sesquilinearity and Leibniz rules.

This induces on $\overline{\mathcal{V}} = \mathcal{V}/D\mathcal{V}$ the Gardner-Faddeev-Zakharov bracket:

$$\{\textstyle\int f, \int g\} = \displaystyle\int \frac{\delta f}{\delta u} D \frac{\delta g}{\delta u},$$

where

$$\frac{\delta f}{\delta u} = \sum_{n \geq 0} (-D)^n \frac{\partial f}{\partial u^{(n)}}$$

is the variational derivative.

(Jacobi identity is very hard to check!)

Example. KdV is a Hamiltonian equation with this λ-bracket and Hamiltonian functional

$$h_2 = \int \tfrac{1}{2}(u^3 + uu''),$$

i.e. the equation $\frac{du}{dt} = \{h_2, u\}$ for the order 1 Poisson λ-bracket $\{u_\lambda u\} = \lambda$ and the Hamiltonian functional h_2 is precisely

$$\frac{du}{dt} = u''' + 3uu'.$$

The first integrals of motion are:

$$h_1 = \int \tfrac{1}{2}u^2,\ h_2,\ h_3 = \int (\tfrac{5}{8}u^4 + \tfrac{5}{3}u^2u'' + \tfrac{5}{6}uu'^2 + \tfrac{1}{2}uu^{(4)}), \ldots.$$

The corresponding symmetries are $\frac{du}{dt_n} = X_n = D\frac{\delta h_n}{\delta u}$, where:

$$X_1 = u',\ X_2 = u''' + 3uu',\ X_3 = u^{(5)} + 10u'u'' + 5uu''' + \tfrac{15}{2}u^2u', \ldots$$

5. Classical Hamiltonian reduction of PVA

Let $\mathcal{V}_0 \xrightarrow{\varphi} \mathcal{V}$ be a homomorphism of PVA's, and let $I_0 \subset \mathcal{V}_0$ be a PVA ideal. The classical Hamiltonian reduction for the data $\{\mathcal{V}, \mathcal{V}_0, \varphi, I_0\}$ is the following PVA:

$$W(\mathcal{V}, \mathcal{V}_0, \varphi, I_0) = (\mathcal{V}/\underbrace{\mathcal{V}\varphi(I_0)}_{I})^{\varphi(\mathcal{V}_0)_\lambda}$$

with the λ-bracket:

$$\{(f+I)_\lambda (g+I)\} = \{f_\lambda g\} + I[\lambda]$$

It is not well defined on $\mathcal{V}/I$, but is well defined on $(\mathcal{V}/I)^{\varphi(\mathcal{V}_0)_\lambda}$

Basic example: affine PVA $\mathcal{V}(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra with an invariant symmetric bilinear form $(.\,|\,.)$:

$$\mathcal{V}(\mathfrak{g}) = S(\mathbb{F}[D]\mathfrak{g});\ [a_\lambda b] = [a,b] + \lambda(a|b)1,\ a, b \in \mathfrak{g},$$

extended to $\mathcal{V}(\mathfrak{g})$ by sesquilinearity and Leibniz rules.
Note that $\mathcal{V}(\mathfrak{g})$ is an algebra of differential polynomials in $\dim \mathfrak{g}$ variables. This is the PVA analogue of the Kirillov-Kostant bracket.

The PVA's that we will use to construct integrable Hamiltonian hierarchies of PDE's are classical affine W-algebras $W(\mathfrak{g}, f)$, constructed as follows. Let $\{e, h, f\}$ be an $s\ell_2$-triple in $\mathfrak{g}$, and consider the $\frac{1}{2} ad\, h$-eigenspace decomposition:

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \ . \tag{5}$$

To get $W(\mathfrak{g}, f)$, apply the classical Hamiltonian reduction to:

$$\mathcal{V} = \mathcal{V}(\mathfrak{g}),\ \mathcal{V}_0 = \mathcal{V}(\mathfrak{g}_{>0}),\ I_0 = \langle m - (f|m) \,|\, m \in \mathfrak{g}_{\geq 1} \rangle \mathcal{V}_0.$$

This is an algebra of differential polynomials in $\dim \mathfrak{g}^f$ variables. To prove our theorem we will use $\mathfrak{g} = s\ell_N$, $(a|b) = \text{tr}\ ab$, $f =$ the nilpotent element of $\mathfrak{g}$, corresponding to the partition $\vec{p}$.

6. The ancestor Lax operator and its descendants.

Let $\psi : \mathfrak{g} \to \text{End } V$ be a finite-dimensional representation of $\mathfrak{g}$ s.t. $(a|b) = \text{tr}_V \ \psi(a)\psi(b)$ is non-degenerate. Choose a basis $\{u_i\}_{i \in B}$ of $\mathfrak{g}$, compatible with (5) and let $\{u^i\}_{i \in B}$ be the dual basis. The associated ancestor Lax operator is

$$L_V(D) = D + \sum_{i \in B} u_i \psi(u^i).$$

(It is independent of the choice of basis).

The descendant Lax operator $L_{V,f}(D)$ for the PVA $W(\mathfrak{g}, f)$ is constructed as follows:

Let $J : V \to V$ [max eigenvalue for $\varphi(x)$] be the projection and $I : V[\max] \hookrightarrow V$ the inclusion. Let $\rho : \mathcal{V}(\mathfrak{g}) \to W(\mathfrak{g}, f)$ be the differential algebra homomorphism defined by:

$$\rho(a) = \pi_{\leq \frac{1}{2}}(a) + (f|a), \quad a \in \mathfrak{g}.$$

Then $L_{V,f}(D)$ is the generalized quasi-determinant:

$$L_{V,f}(D) = (J(\rho(L_V(D))^{-1}I))^{-1}$$

First main theorem. The descendant Lax operator $L_{V,f}(D)$ has its coefficients in $W(\mathfrak{g}, f)$.

Second main theorem. Let $\mathfrak{g} = s\ell_N$, ψ be its standard representation in $V = \mathbb{F}^N$, f be a nilpotent element of $\mathfrak{g}$, associated to the partition $\vec{p} = (p_1 \geq p_2 \geq \ldots > 0)$ and let r_1 be the multiplicity of p_1. Then $L_{V,f}(D)$ is an $r_1 \times r_1$ matrix pseudo-differential operator with leading term D^{p_1}. Let $B(D) = (L_{V,f}(D))^{\frac{1}{p_1}}$. Then

(a) $h_n = \int \mathrm{Res}_D \, \mathrm{tr} B(D)^n$ are in involution w.r. to the Poisson bracket on $\overline{W(\mathfrak{g}, f)}$.

(b) $\frac{du}{dt_n} = \{\int h_n, u\}$ are compatible Hamiltonian equations. These equations imply Lax equations:

$$\frac{dL(D)}{dt_n} = [B(D)^n_+, L(D)].$$

A similar theorem holds for so_N and sp_N, $V = \mathbb{F}^N$. [DSKV18]

Remark

Drinfeld and Sokolov [1985] constructed an integrable Hamiltonian hierarchy of PDE for any simple Lie algebra $\mathfrak{g}$ and its principal nilpotent element f, using Kostant's cyclic elements. In [DSKV15] we extended their method for any simple Lie algebra $\mathfrak{g}$ and its nilpotent elements f of "semisimple type". There are very few such elements in classical $\mathfrak{g}$, but about $\frac{1}{2}$ of nilpotents in exceptional $\mathfrak{g}$ are such: 13 out of 20 in E_6, 21 out of 44 in E_7, 27 out of 69 in E_8, 11 out of 15 in F_4, 3 out of 4 in G_2 [Elashvili-K.-Vinberg 13]).
For classical Lie algebras $\mathfrak{g}$ and nilpotent elements f of semisimple type the DS approach is equivalent to the Lax operator approach.

The proof of our second main theorem uses the notion of an Adler type operator, introduced in [DSKV15] (based on the famous Adler's map, 1979) , further studied in [DSKV16], and generalized in [DSKV18]. Below we give its definition that works for type A; its generalization that works also for types B, C, D can be found in [DSKV18].

Given a PVA $\mathcal{V}$ and a vector space V, an Adler type operator is a pseudodifferential operator $L(D) \in \mathcal{V}((D^{-1})) \otimes \operatorname{End} V$, satisfying the following identity:

$$
\begin{aligned}
&\{L(z)_\lambda L(w)\} = \\
&(1 \otimes L(w+\lambda+D)) i_z (z-w-\lambda-D)^{-1} (L^*(\lambda-z) \otimes 1)\Omega \\
&-\Omega(L(z) \otimes i_z (z-w-\lambda-D)^{-1} L(w)),
\end{aligned}
$$

where i_z stands for the geometric series expansion for large z, and $\Omega \in \operatorname{End}(V \otimes V)$ is the permutation of factors.

The main property of the (generalized) Adler type operators is that they satisfy (a) and (b) of the Second Main theorem. Hence the latter follows if we succeed in proving that the descendent Lax operator $L_{V,f}(D)$ is a (generalized) Adler type operator. We show that this is indeed that case for all classical Lie algebras and their standard representations.

In [DSKV18] we discovered some further ways of constructing integrable Hamiltonian hierarchies using generalized Adler type operators. First, we classify all scalar constant coefficients pseudodifferential operators of generalized Adler type. Remarkably, it turns out that, under certain conditions, products of generalized Adler type operators produce compatible hierarchies of Lax type Hamiltonian equations.

The simplest examples of integrable equations thus obtained:

Example 1. $L(D) = L_{\mathrm{sl}_2}(D)D = D^3 - uD$
(Sawada-Kotera) $\frac{du}{dt} = u^{(5)} - 5u'u'' - 5uu''' + 5u^2u'$

Example 2. $L(D) = L_{\mathrm{so}_3}(D) = D^3 - uD - u'/2$
(Kaup-Kupershmidt) $\frac{du}{dt} = u^{(5)} - \frac{25}{2}u'u'' - 5uu''' + 5u^2u'$

Drinfeld, Sokolov and Svinolupov proved that an integrable evolution PDE in u of order 5 with polynomial RHS, which is not a symmetry of an equation of order 3, is one of these two.

The list of the resulting integrable hierarchies of Hamiltonian equations associated to all $\mathcal{W}$-algebras for classical Lie algebras and their pairwise tensor products contains all Drinfeld-Sokolov integrable type hierarchies that they attach to classical affine Lie algebras (including the twisted ones) and a node on the Dynkin diagram. At the same time we discover many new integrable Hamiltonian systems, the simplest of which, corresponding to $L(D) = L_{sp_4}(D)D^{-1}$, after Dirac reduction becomes:

$$\frac{du}{dt} = u''' - 6uu' - \frac{3}{4}(v_1 v_2'' - v_2 v_1''), \quad \frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1''' - 3(uv_1)' \\ v_2''' - 3(uv_2)' \end{pmatrix}.$$

7. Definition and classification of mPVA

The notion of mLCA was introduced in [Golenishcheva-Kutuzova-K. 98] in the study of "shifted" operator product expansions of quantum fields

$$a(z)b(w) = \sum_{j\in\mathbb{Z}} \frac{c^j(w)}{z - \gamma^j \cdot w}$$

The corresponding multiplicative λ-bracket, defined by $[a_\lambda b] = \sum_j \lambda^j c^j$, satisfies the axioms of mLCA with $S = \gamma$.

A multiplicative PVA is defined as a unital commutative associative algebra $\mathcal{V}$ with an automorphism S, endowed with a structure of mLCA, such that (the same) Leibniz rule holds:

$$\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}.$$

The notion of mPVA is equivalent to the notion of a "local" Poisson algebra, defined as a Poisson algebra $\mathcal{V}$ with an automorphism S, such that for any $a, b \in \mathcal{V}$,

$$\{S^n a, b\} = 0 \text{ for all but finitely many } n \in \mathbb{Z}.$$

Indeed $\{a_\lambda b\} = \sum_{n\in\mathbb{Z}} \lambda^n \{S^n a, b\}$ defines an mLCA structure on $\mathcal{V}$. (Converse is left to the audience.)

The most famous example of a "local" PA is the Faddeev-Takhtajan-Volkov algebra [1986]: $\mathcal{V} = \mathbb{F}[u_n | n \in \mathbb{Z}]$ with the automorphism $S(u_n) = u_{n+1}$, and the Poisson bracket

$$\{u_m, u_n\} = u_m u_n \left((\delta_{m+1,n} - \delta_{m,n+1}) (1 - u_m - u_n) \right.$$
$$\left. - u_{m+1}\delta_{m+2,n} + u_{n+1}\delta_{m,n+2} \right).$$

The meaning of this algebra from the mPVA viewpoint is explained below.

The first result on mPVA is a classification up to order $N = 5$ in one variable u [K.-Wakimoto 18].
First we show that one has for general $N \geq 1$:

$$\{u_\lambda u\} = \sum_{j=1}^{N} (\lambda^j - (\lambda S)^{-j}) f_j(u, u_1, \dots, u_j),\ u_i := S^i(u).$$

Jacobi identity is equivalent to a system of N^2 PDE on N functions f_j in N variables u_j. The two most important examples:
Example 1 (general type) $f_j = c_j g(u) g(u_j)$, where $g = g(u) \in \mathcal{V},\ c_j \in \mathbb{F},\ j = 1, \dots, N$.
Example 2 (complementary type) for $N = 2$ (and similarly for $N \geq 3$):

$$f_1 = g(u)g(u_1)(F(u) + F(u_1)),\ f_2 = g(u)g(u_2)F(u_1),$$

where $F(u) = g(u)F'(u)$.

Theorem

Any mPVA in one variable u of order $N \leq 3$ is either of general type, or of complementary type, or a linear combination of the order N complementary type and order 1 general type for the same g. For orders $N = 4$ and 5 there are also exceptional types.

Example

FTV algebra corresponds to the difference between complementary order 2 mPVA and general order 1 mPVA with $c_1 = 1,\ g = F = u$.

8. Volterra lattice: the simplest example of a differential-difference equation

Let $\mathcal{V} = \mathbb{F}[u_n | n \in \mathbb{Z}], S(u_n) = u_{n+1}$ with the multiplicative λ-bracket

$$\{u_\lambda u\}_1 = \lambda u u_1 - \lambda^{-1} u u_{-1}. \tag{6}$$

The Volterra lattice

$$\frac{du_n}{dt} = u_n(u_{n+1} - u_{n-1}), \; n \in \mathbb{Z} \tag{7}$$

is a Hamiltonian differential-difference equation with λ-bracket (6) and Hamiltonian functional $h_1 = \int u$:

$$\frac{du}{dt} = \{h_1{}_\lambda u\}_1\big|_{\lambda=1} = uu_1 - uu_{-1}.$$

Applying S^n to this equation, we get (7).

Volterra lattice is also Hamiltonian with respect to the order 2 complementary λ-bracket $\{._\lambda.\}_2$ and the Hamiltonian functional $h_0 = \frac{1}{2}\int \log u$, i.e. it is bi-Hamiltonian.
It follows that it is integrable. In fact it is the first equation of a hierarchy of Lax type differential-difference equations for the pseudodifference operator $L = S + uS^{-1}$:

$$\frac{dL}{dt_n} = [(L^{2n})_+, L], \; n = 1, 2, \dots,$$

with integrals of motion

$$h_m = \int \operatorname{Res} L^{2m}, m = 1, 2, \dots,$$

where Res $\sum_j a_j S^j = a_0$.

The compatible pair of multiplicative λ-brackets of orders 1 and 3 produces another bi-Hamiltonian integrable differential-difference equation:

$$\frac{du}{dt} = e^{iu+u_1} + ie^{iu_1+u_2} - e^{iu_{-2}+u_{-1}} - ie^{iu_{-1}+u},$$

which seems to be new.
The well-known Toda lattice hierarchies and Bogoyavlensky lattices can be treated along the same lines.
The general theory is work in progress.