Holonomy braidings, biquandles and quantum invariants of links with SL2C flat connections.

Representation Theory, Mathematical Physics and Integrable Systems at CIRM.

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Joint work with
Christian Blanchet, Bertrand Patureau-Mirand and Nicolai Reshetikhin.

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The pivotal category of the unrestricted quantum group

Let $q = e^{\frac{2\pi\sqrt{-1}}{\ell}} \in \mathbb{C}$ be the $!^{\text{th}}$ root of unity. Set $r = \frac{\ell}{2}$ if ! is even and $r = !$ if ! is odd.

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- The unrestricted quantum group $U_q\mathfrak{sl}(2)$ is a $\mathbb{C}$-algebra given by generators $K^{\pm 1}, E, F$ and relations:

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

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• Representation theory studied by C. De Concini, V. Kac, C. Procesi, N. Reshetikhin, M. Rosso and others.

Pivotal Category

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$\mathscr{C}$ is a pivotal $\mathbb{C}$-category: $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$= \overleftarrow{\mathrm{coev}}_V : \mathbb{C} \to V \otimes V^*$ is given by $1 \mapsto \sum v_j \otimes v_j^*$,

$= \overleftarrow{\mathrm{ev}}_V : V^* \otimes V \to \mathbb{C}$ is given by $f \otimes w \mapsto f(w)$,

$= \overrightarrow{\mathrm{ev}}_V : V \otimes V^* \to \mathbb{C}$ is given by $v \otimes f \mapsto f(K^{1-r} v)$,

$= \overrightarrow{\mathrm{coev}}_V : \mathbb{C} \to V^* \otimes V$ is given by $1 \mapsto \sum v_j^* \otimes K^{r-1} v_j$,

where $\{v_j\}$ is a basis of V and $\{v_j^*\}$ is the dual basis of V^*.

Definition: modified trace

There exists a modified trace on the ideal of projective modules Proj of $\mathscr{C}$: a family of linear functions

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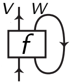
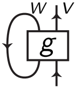
- **Partial trace properties** If $V \in \mathrm{Proj}$ and $W \in \mathscr{C}$ then for any $f \in \mathrm{End}_{\mathscr{C}}(V \otimes W)$ and $g \in \mathrm{End}_{\mathscr{C}}(W \otimes V)$ we have

$$\mathrm{t}_{V\otimes W}(f) = \mathrm{t}_V \left(\quad \right)$$



$$\mathrm{t}_{W\otimes V}(g) = \mathrm{t}_V \left(\quad \right).$$



The holonomy braiding

Theorem (Kashaev-Reshetikhin)

The conjugation by the h-adic universal R-matrix specializes at root of unity to an algebra morphism

$$\mathscr{R} : U_q\mathfrak{sl}(2) \otimes U_q\mathfrak{sl}(2) \to (U_q\mathfrak{sl}(2) \otimes U_q\mathfrak{sl}(2))[\mathcal{W}^{-1}]$$

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This map on $Z_0 \otimes Z_0$ is given by:

$$\mathscr{R}(K^r \otimes 1) = (K^r \otimes 1)\,\mathcal{W}, \qquad \mathscr{R}(1 \otimes K^r) = (1 \otimes K^r)\,\mathcal{W}^{-1}$$

$$\mathscr{R}(E^r \otimes 1) = E^r \otimes K^r \qquad \mathscr{R}(1 \otimes F^r) = K^{-r} \otimes F^r$$

$$\mathscr{R}(1 \otimes E^r) = K^r \otimes E^r + E^r \otimes 1\left(1 - (1 \otimes K^{2r})\,\mathcal{W}^{-1}\right)$$

$$\mathscr{R}(F^r \otimes 1) = F^r \otimes K^{-r} + 1 \otimes F^r\left(1 - (K^{-2r} \otimes 1)\,\mathcal{W}^{-1}\right)$$

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Sideway invertible means:

$$\text{if } B(x_1, x_2) = (x_4, x_3) \text{ then } S(x_4, x_1) = (x_3, x_2).$$

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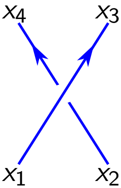
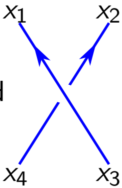
We use B to color edges of a tangle diagram with color in $\mathbf{X}$:



and



Reidemeister moves

Theorem

Any tangle isotopy from T_1 to T_2 induce a canonical bijection between $\mathbf{X}$*-coloring of the diagrams of their regular projections D_1 and D_2.*

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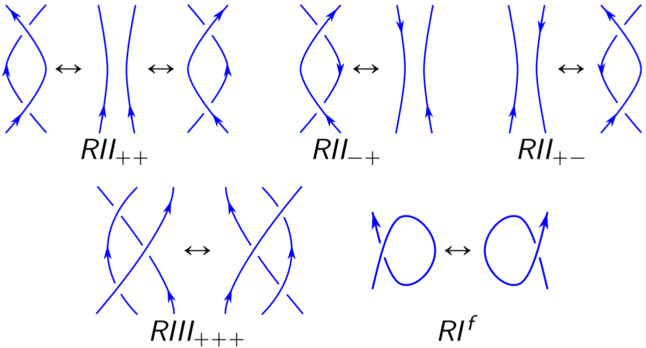
*Any tangle isotopy from T_1 to T_2 induce a canonical bijection between **X**-coloring of the diagrams of their regular projections D_1 and D_2. The bijection is obtained by a sequence of colored Reidemeister moves.*



Colorings

Theorem (V. Lebed, L. Vendramin)

Any biquandle $(\mathbf{X}, B)$ *induces a "quandle"* Q *and there is a bijection between* $\mathbf{X}$*-colorings and* Q*-colorings of diagrams.*

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Fundamental quandle: Given a tangle Γ, let $Q(\Gamma, *)$ be the set of homotopy classes of continuous paths $\# : [0,1) \to M_\Gamma$ such that $\#(0) = *$ and $\lim_{t\to 1} \#(t)$ exist and is equal to some point of the tangle Γ.

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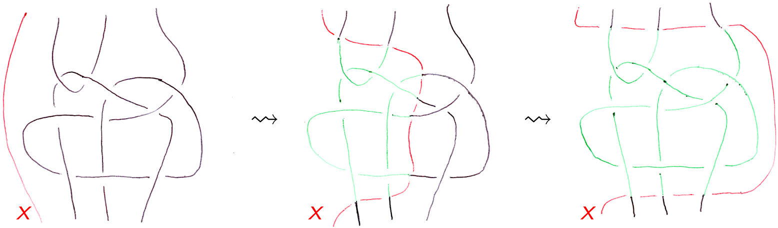
If Q is a quandle then a Q-tangle is a quandle morphism $Q(\Gamma, *) \to Q$.

Key lemma

Lemma

For any B-colored diagram and for generic $x \in \mathbf{X}$, this chain of B-colored Reidemeister moves is not broken:



Proof of the invariance

Let D and D' be two B-colored diagrams representing isotopic Q-tangles. Then it is possible that D and D' are not related by sequence of colored Reidemeister moves but:

Proposition

For generic $x \in \mathbf{X}$, $\mathrm{id}_{(x,+)} \otimes D$ and $\mathrm{id}_{(x,+)} \otimes D'$ are related by a sequence of B-colored Reidemeister moves.

Corollary

$F(D) = F(D')$.

Furthermore, we can prove that the modified trace is gauge invariant so that the property also hold for closed diagrams.

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- V. Lebed, L. Vendramin - Homology of left non-degenerate set-theoretic solutions to the Yang-Baxter equation, Advances Math. 304 (2017), 1219-1261.

Thank you and Happy Birthday Kolya!