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Action-angle duality for a Poisson-Lie deformation of the BC_n Sutherland system

based on joint works with T.F. Görbe and I. Marshall

Consider two *Liouville integrable* Hamiltonian systems (M, ω, H) and $(\widehat{M}, \hat{\omega}, \widehat{H})$. These systems are said to be **in action-angle duality** if there exist Darboux coordinates q_i, p_i on (dense open subset of) M, and Darboux coordinates λ_k, θ_k on (dense open subset of) $\widehat{M}$, and a global symplectomorphism $\mathcal{R} : M \to \widehat{M}$ such that

$H \circ \mathcal{R}^{-1}$ depends only on λ (action variables for H) and

$\widehat{H} \circ \mathcal{R}$ depends only on q (action variables for $\widehat{H}$).

This is non-trivial if both $H(q,p)$ and $\widehat{H}(\lambda, \theta)$ are interesting.

Self-duality is the special case when H and $\widehat{H}$ have the same form.

Action-angle duality is particularly interesting when it relates two (one-dimensional) **many-body systems** in such a way that

the q_i describe particle positions for $H(q,p)$ and

the λ_i describe particle positions for $\widehat{H}(\lambda,\theta)$.

It was discovered by Ruijsenaars (1988-95) that Calogero-Moser and Toda type systems enjoy duality relations.

The simplest example is the self-dual Calogero-Moser system:

$$H_{\mathrm{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{\mu^2}{(q_k - q_j)^2}.$$

The simplest dual pair is provided by hyperbolic Sutherland

$$H_{\mathrm{hyp-Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{\mu^2}{\sinh^2(q_k - q_j)}$$

and its Ruijsenaars dual

$$\widehat{H}_{\mathrm{rat-RS}} = \sum_{k=1}^{n} (\cosh\theta_k) \prod_{j\neq k} \left[1 + \frac{\mu^2}{(\lambda_k - \lambda_j)^2}\right]^{\frac{1}{2}}$$

Action-angle dualities of many-body systems were much studied in the nineties, e.g., by Fock, Gorsky, Nekrasov, Rosly and Roubtsov. In recent years, the subject was investigated by Pusztai, and myself in collaboration with Klimcik, Ayadi, Kluck, Görbe and Marshall. Reshetikhin pointed out dualities for many-body systems extended by 'spin' degrees of freedom.

My aim is to understand all known examples of action-angle duality in group theoretic terms, using Hamiltonian reduction, and to derive new ones. Here, I focus on systems built on the BC_n root system.

In 1993, van Diejen introduced the following integrable classical Hamiltonian:

$$H_{\mathrm{vD}}(\lambda, \theta; c_0, c_1, c_2, c_1', c_2') = \sum_{j=1}^{n} (\cosh \theta_j) \mathcal{V}_j(\lambda; c)^{\frac{1}{2}} + \mathcal{U}(\lambda; c)$$

$$\mathcal{V}_j(\lambda; c) = \prod_{i=1}^{2} \left[1 + \frac{\sinh^2 c_i}{\sin^2 \lambda_j}\right] \left[1 + \frac{\sinh^2 c_i'}{\cos^2 \lambda_j}\right] \prod_{k \neq j}^{n} \left[1 + \frac{\sinh^2 c_0}{\sin^2(\lambda_j \pm \lambda_k)}\right]$$

$$\mathcal{U}(\lambda; c) = (\sinh c_0)^{-2} \prod_{i=1}^{2} \sinh(c_i) \cosh(c_i') \prod_{k=1}^{n} \left[1 + \frac{\sinh^2 c_0}{\sin^2 \lambda_k}\right]$$
$$+ (\sinh c_0)^{-2} \prod_{i=1}^{2} \cosh(c_i) \sinh(c_i') \prod_{k=1}^{n} \left[1 + \frac{\sinh^2 c_0}{\cos^2 \lambda_k}\right]$$

In the standard trigonometric case the Darboux coordinates λ_j, θ_j and the 5 'coupling constants' $c = (c_0, c_1, c_2, c_1', c_2')$ are real. The model admits a plethora of different real forms and limits.

Scaling limits are obtained by introducing two more parameters α and β and afterwards taking them to zero in $H_{\mathrm{vD}}(\alpha\lambda, \beta\theta; \alpha\beta c)$. Scaling with β leads to the trigonometric BC_n Sutherland model: $\lim_{\beta\to 0} \beta^{-2}\left(H_{\mathrm{vD}}(\lambda, \beta\theta; \beta c) - C\right) =$

$$\frac{1}{2}\sum_{j=1}^{n} \theta_j^2 + \frac{1}{2}\sum_{k\neq j} \frac{c_0^2}{\sin^2(\lambda_j \pm \lambda_k)} + \sum_{j=1}^{n}\left(\frac{a^2}{\sin^2\lambda_j} + \frac{b^2}{\cos^2\lambda_j}\right).$$

Scaling with α leads to the rational van Diejen model: $\lim_{\alpha\to 0} H_{\mathrm{vD}}(\alpha\lambda, \theta; \alpha c) =$

$$\sum_{j=1}^{n}(\cosh\theta_j)\left[\prod_{i=1}^{2}\left(1+\frac{c_i^2}{\lambda_j^2}\right)\prod_{k\neq j}^{n}\left(1+\frac{c_0^2}{(\lambda_j\pm\lambda_k)^2}\right)\right]^{\frac{1}{2}} + \frac{c_1c_2}{c_0^2}\prod_{j=1}^{n}\left[1+\frac{c_0^2}{\lambda_j^2}\right]$$

Scaling both with α and β leads to the rational Calogero model of type B_n:

$$H_{\mathrm{Cal}} = \frac{1}{2}\sum_{j=1}^{n}\theta_j^2 + \frac{1}{2}\sum_{k\neq j}\frac{c_0^2}{(\lambda_j\pm\lambda_k)^2} + \sum_{j=1}^{n}\frac{a^2}{\lambda_j^2}.$$

It is an open question to derive the 5-coupling van Diejen system by Hamiltonian reduction. Today, I report on progress for certain 3-parametric limiting cases. First, I recall that the trigonometric BC_n Sutherland system and its action-angle dual arises from a reduction of $T^*SU(2n)$ [Pusztai '12, Feher-Görbe '14].

A dual pair associated with the BC_n root system

The trigonometric BC_n Sutherland system

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\le j<k\le n}\left(\frac{\gamma}{\sin^2(q_j-q_k)}+\frac{\gamma}{\sin^2(q_j+q_k)}\right)+\sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)}+\sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)}$$

is dual to the (completed) rational Ruijsenaars-Schneider-van Diejen system

$$\widehat{H}(\lambda,\theta) = \sum_{j=1}^{n}\cos(\theta_j)\left[1-\frac{u^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{v^2}{\lambda_j^2}\right]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{\mu^2}{(\lambda_j-\lambda_k)^2}\right]^{\frac{1}{2}}\left[1-\frac{\mu^2}{(\lambda_j+\lambda_k)^2}\right]^{\frac{1}{2}}$$

$$+\frac{uv}{\mu^2}\prod_{j=1}^{n}\left[1-\frac{\mu^2}{\lambda_j^2}\right]-\frac{uv}{\mu^2}.$$

Here, the coupling constants are subject to $\gamma > 0$, $\gamma_2 > 0$, $4\gamma_1+\gamma_2 > 0$, and $\mu > 0$, $|u| \neq |v| \neq 0$. Duality holds under the relation $\gamma = \mu^2$, $\gamma_1 = \frac{uv}{2}$, $\gamma_2 = \frac{(u-v)^2}{2}$.

The Sutherland positions q live in the open polytope (Weyl alcove)

$$\mathcal{D}_q = \{q \in \mathbb{R}^n \mid \frac{\pi}{2} > q_1 > \cdots > q_n > 0\}$$

and the Sutherland actions λ (dual positions) fill the closed polyhedron

$$\mathcal{D}_\lambda = \{\lambda \in \mathbb{R}^n \mid \lambda_a - \lambda_{a+1} \ge \mu \ (a=1,\ldots,n-1),\ \lambda_n \ge \max(|u|,|v|)\}.$$

The above description of dual system is valid on a dense open submanifold parametrized by $\mathcal{D}_\lambda^o \times \mathbb{T}^n$. The torus $\mathbb{T}^n$ collapses at the boundary of $\mathcal{D}_\lambda$.

This dual pair results by reducing $T^*SU(2n) \simeq \{(k,J) \mid k \in SU(2n),\, J \in su(2n)\}$ using the symmetry group $K_+ \times K_+$, with $K_+ = S(U(n) \times U(n))$ being the block-diagonal subgroup of $K = SU(2n)$.

Before reduction, we have two Abelian Poisson algebras of $(K_+ \times K_+)$-invariant Hamiltonians generated by

$$\mathcal{H}_a(k,J) = \frac{1}{2}\mathrm{tr}\,(\mathrm{i}J)^{2a} \quad \text{and} \quad \widehat{\mathcal{H}}_a(k,J) := \frac{1}{2}\mathrm{tr}\,(k^\dagger I k I)^a \quad \text{with} \quad I := \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n).$$

The moment map values for the actions of K_+ defined by left- and respectively by right-multiplications are fixed to be

$$\mathrm{diag}(C(\mu), \mathbf{0}_n) + \mathrm{i}(\mu + u)I \quad \text{and} \quad \mathrm{i}vI,$$

where $C(\mu) \in u(n)$ reads $C(\mu)_{lm} = \mathrm{i}\mu(\delta_{lm} - 1)$. Then λ and q arise as eigenvalues:

$$-\mathrm{i}J \sim \mathrm{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n), \qquad k^\dagger I k I \sim \mathrm{diag}(e^{2\mathrm{i}q_1}, \dots, e^{2\mathrm{i}q_n}, e^{-2\mathrm{i}q_1}, \dots, e^{-2\mathrm{i}q_n}).$$

The Hamiltonians $\mathcal{H}_a$ reduce to the commuting Sutherland Hamiltonians, which in their action-angle variables λ, θ become

$$\mathcal{H}_a^{\mathrm{red}}(\lambda) = \sum_{j=1}^n (\lambda_j)^{2a}, \qquad a = 1, \dots, n.$$

The Hamiltonians $\widehat{\mathcal{H}}_a$ reduce to the commuting RSvD Hamiltonians, which in their action-angle variables q, p become

$$\widehat{\mathcal{H}}_a^{\mathrm{red}}(q) = \sum_{j=1}^n \cos(2aq_j), \quad a = 1, \dots, n.$$

$\mathcal{H}_1^{\mathrm{red}}$ gives H in the q, p coordinates and $\widehat{\mathcal{H}}_1^{\mathrm{red}}$ gives $\widehat{H}$ in the λ, θ coordinates.

Deformed dual pair from a reduction of a Heisenberg double

We shall reduce the standard Poisson-Lie analogue of $T^*SU(2n)$. This is the (symplectic) Heisenberg double of Poisson $SU(2n)$, which as a manifold is the real Lie group $\mathcal{M} := SL(2n, \mathbb{C})$.

• Every $g \in \mathcal{M}$ admits the alternative Iwasawa decompositions

$$g = k_L b_R = b_L k_R, \qquad k_L, k_R \in K, \quad b_L, b_R \in B,$$

where $K := SU(2n)$ and $B := B(2n)$ consists of upper triangular matrices with positive diagonal. Using these, $\mathcal{M}$ is equipped with the Alekseev-Malkin symplectic form

$$\omega_{\mathcal{M}} = \frac{1}{2}\Im \mathrm{tr}\,(db_L b_L^{-1} \wedge dk_L k_L^{-1}) + \frac{1}{2}\Im \mathrm{tr}\,(b_R^{-1} db_R \wedge k_R^{-1} dk_R).$$

• The smooth functions depending only on b_L, or only on b_R, form two mutually commuting Poisson algebras, and similarly for k_L and k_R. These are (up to signs) the Poisson algebras of the standard Poisson groups K and B in duality.

The 'master system' $(\mathcal{M}, \omega_{\mathcal{M}}, \{\mathcal{H}_a\}, \{\widehat{\mathcal{H}}_a\})$ is now defined as follows. Using the Iwasawa decomposition of $g \in \mathcal{M} = SL(2n, \mathbb{C})$, written as $g = kb$, we introduce the 'unreduced Lax matrices'

$$\Omega(g) := bb^\dagger \quad \text{and} \quad L(g) := k^\dagger I k I, \quad \text{with} \quad I := \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n).$$

Two Abelian Poisson algebras are generated by the Hamiltonians

$$\mathcal{H}_a(g) := \frac{1}{2}\mathrm{tr}\,\Omega(g)^a \quad \text{and} \quad \widehat{\mathcal{H}}_a(g) := \frac{1}{2}\mathrm{tr}\,L(g)^a, \qquad a = 1, 2, \dots$$

They are invariant with respect to the action of the symmetry group $K_+ \times K_+$ defined by

$$K_+ \times K_+ \times \mathcal{M} \ni (\eta_L, \eta_R, g) \mapsto \eta_L g \eta_R^{-1} \in \mathcal{M}, \qquad K_+ = \{k \in K \mid IkI = k\}.$$

$K_+ < K$ is a Poisson subgroup and the action of $K_+ \times K_+$ is a Poisson action generated by the following moment map (in Lu's sense)

$$\mathcal{M} \in g \mapsto (\pi_N(b_L), \pi_N(b_R^{-1})) \in B/N \times B/N,$$

where $\pi_N : B \to B/N$ is the projection associated with the normal subgroup $N < B = B(2n)$ of elements having $\mathbf{1}_n$ as $n \times n$ diagonal blocks.

The constraints and the key spectral invariants

Inspired by experience, we consider the moment map 'constraint surface'

$$\mathcal{M}_0 := \left\{ g \in \mathcal{M} \;\middle|\; b_R := b = \begin{pmatrix} e^{-v}\mathbf{1}_n & * \\ 0 & e^{v}\mathbf{1}_n \end{pmatrix},\; b_L = \begin{pmatrix} e^{u}\boldsymbol{\sigma} & * \\ 0 & e^{-u}\mathbf{1}_n \end{pmatrix} \right\},$$

where $\sigma := \sigma(\mu) \in B(n)$ satisfies $\sigma\sigma^\dagger = e^{-2\mu}\mathbf{1}_n + \widehat{V}\widehat{V}^\dagger$ with fixed $\widehat{V} \in \mathbb{C}^n$ verifying $|\widehat{V}|^2 = e^{-2\mu}(e^{2n\mu} - 1)$. Here, u, v and $\mu > 0$ are real constants ($|u| \neq |v|$).

Our task is to construct two suitable models of the reduced phase space

$$\mathcal{M}_{\text{red}} = \mathcal{M}_0/(K_+(\sigma) \times K_+) \quad \text{where} \quad K_+(\sigma) = \{\eta \in K_+ \mid \eta\sigma\sigma^\dagger\eta^{-1} = \sigma\sigma^\dagger\}.$$

$\mathcal{M}_0$ is a principal bundle over $\mathcal{M}_{\text{red}}$. It inherits a symplectic structure and two reduced Abelian Poisson algebras.

For any $g = kb \in \mathcal{M}_0$, $L(g) = k^\dagger I k I$ and $\Omega(g) = bb^\dagger$ are conjugate to unique diagonal matrices of the following form:

$$L(g) \sim \text{diag}(e^{2\mathrm{i}q_1}, \ldots, e^{2\mathrm{i}q_n}, e^{-2\mathrm{i}q_1}, \ldots, e^{-2\mathrm{i}q_n}) \quad \text{with} \quad \frac{\pi}{2} \geq q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$$

and

$$\Omega(g) \sim \text{diag}(e^{2\lambda_1}, \ldots, e^{2\lambda_n}, e^{-2\lambda_1}, \ldots, e^{-2\lambda_n}) \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq |v|.$$

The respective 'spectral invariants' q_i and λ_i descend to functions on $\mathcal{M}_{\text{red}}$. Naturally, they (or their suitable functions) give rise to action variables. A crucial problem is to find their range of values.

Darboux coordinates and reduced Hamiltonians: Act I

We proved that the domain of the λ-variables is

$$\mathcal{D}_\lambda = \big\{\lambda \in \mathbb{R}^n \mid \lambda_i - \lambda_{i+1} \geq \mu \ (i = 1, \dots, n-1),\ \lambda_n \geq \max(|v|, |u|)\big\}.$$

We can parametrize a dense open subset $\mathcal{M}^0_{\text{red}} \subset \mathcal{M}_{\text{red}}$ by Darboux coordinates λ_i, θ_j varying in $\mathcal{D}^0_\lambda \times \mathbb{T}^n = \{(\lambda, e^{\mathrm{i}\theta}\}$, where $\mathcal{D}^0_\lambda \subset \mathcal{D}_\lambda$ is the interior. In these coordinates $\widehat{\mathcal{H}}^{\text{red}}_1$ becomes the RSvD type Hamiltonian

$$e^{u-v}\,\widehat{\mathcal{H}}^{\text{red}}_1(\lambda,\theta) = V(\lambda) + \sum_{k=1}^n \frac{\cos\theta_k}{\cosh^2\lambda_k}\left[1 - \frac{\sinh^2 v}{\sinh^2\lambda_k}\right]^{1/2}\left[1 - \frac{\sinh^2 u}{\sinh^2\lambda_k}\right]^{1/2}$$
$$\times \prod_{\substack{l=1\\(l\neq k)}}^n \left[1 - \frac{\sinh^2\mu}{\sinh^2(\lambda_k - \lambda_l)}\right]^{1/2}\left[1 - \frac{\sinh^2\mu}{\sinh^2(\lambda_k + \lambda_l)}\right]^{1/2} \quad \text{with}$$

$$V(\lambda) = \frac{\sinh(v)\sinh(u)}{\sinh^2\mu}\prod_{k=1}^n\left[1 - \frac{\sinh^2\mu}{\sinh^2\lambda_k}\right] - \frac{\cosh(v)\cosh(u)}{\sinh^2\mu}\prod_{k=1}^n\left[1 + \frac{\sinh^2\mu}{\cosh^2\lambda_k}\right] + C.$$

We also have the reduced Lax matrix $L_{\text{red}}(\lambda,\theta)$ generating $\widehat{\mathcal{H}}^{\text{red}}_j$ for $j = 1, \dots, n$.

Even globally on $\mathcal{M}_{\text{red}}$, the other family $\mathcal{H}^{\text{red}}_j$ of reduced Hamiltonians reads

$$\mathcal{H}^{\text{red}}_j = \sum_{i=1}^n \cosh(2j\lambda_i).$$

In this way, the λ_i play the double role of positions and actions.

Darboux coordinates and reduced Hamiltonians: Act II

The action variables (2π-periodic flows) corresponding to eigenvalues of $L(g)$ are

$x_i := \log \sin q_i$ and their domain is proved to be

$$\mathcal{D}_x = \left\{x \in \mathbb{R}^n \mid x_1 \leq s := \min(0, v-u),\ x_j - x_{j+1} \geq \mu\ (j = 1, \dots, n-1)\right\}.$$

The pair $(x, e^{\mathrm{i}y}) \in \mathcal{D}_x^0 \times \mathbb{T}^n$ gives Darboux coordinates on a dense open subset $\mathcal{M}'_{\text{red}} \subset \mathcal{M}_{\text{red}}$. In these coordinates $\mathcal{H}_1^{\text{red}}$ becomes the RSvD type Hamiltonian

$$\mathcal{H}_1^{\text{red}}(x,y) = U(x) - \sum_{j=1}^{n} \cos(y_j) U_1(x_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ (k \neq j)}}^{n} \left[1 - \frac{\sinh^2(\mu)}{\sinh^2(x_j - x_k)}\right]^{\frac{1}{2}}$$

$$U(x) = \frac{e^{-2u} + e^{2v}}{2} \sum_{j=1}^{n} e^{-2x_j}, \quad U_1(x_j) = \left[1 - (1 + e^{2(v-u)}) e^{-2x_j} + e^{2(v-u)} e^{-4x_j}\right].$$

We also have the reduced Lax matrix $\Omega_{\text{red}}(x,y)$ generating $\mathcal{H}_j^{\text{red}}$ for $j = 1, \dots, n$.

The other family $\widehat{\mathcal{H}}_j^{\text{red}}$ of reduced Hamiltonians takes the following form:

$$\widehat{\mathcal{H}}_j^{\text{red}} = \sum_{i=1}^{n} \cos(2jq_i), \qquad (\cos(2jq_i) \text{ is a polynomial in } \sin q_i = e^{x_i}).$$

Therefore the x_i also play the double role of positions and actions.

Consequence: Each reduced Hamiltonian $\widehat{\mathcal{H}}_j^{\text{red}}$ and $\mathcal{H}_j^{\text{red}}$ is non-degenerate and possesses a unique equilibrium point (which is shared by its family).

Two global models of $\mathcal{M}_{\text{red}}$

The action-angle variables λ_i, θ_j are not good coordinates on $\mathcal{M}_{\text{red}}$ where λ reaches the boundary of the polyhedron $\mathcal{D}_\lambda$. To describe the global structure of $\mathcal{M}_{\text{red}}$, we introduce the complex variables

$$\zeta_j = \sqrt{\lambda_j - \lambda_{j+1} - \mu} \prod_{k=1}^{j} e^{-\mathrm{i}\theta_k}, \quad j = 1, \dots, n-1, \quad \zeta_n = \sqrt{\lambda_n - |u|} \prod_{k=1}^{n} e^{-\mathrm{i}\theta_k}.$$

The boundary of $\mathcal{D}_\lambda$ is characterized by the vanishing of some ζ_k, and for the dense open part $\mathcal{M}^0_{\text{red}}$ we have

$$\mathcal{D}^0_\lambda \times \mathbb{T}^n \Longleftrightarrow (\mathbb{C}^*)^n, \quad \text{with} \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

The complex variables remain valid when we 'add the zeros', and the standard symplectic vector space $(\widehat{M}, \widehat{\omega}) = (\mathbb{C}^n, \mathrm{i} \sum_{j=1}^n d\zeta_j \wedge d\zeta_j^*)$ gives a global model of $\mathcal{M}_{\text{red}}$. The point $\zeta = 0$ corresponds to the common equilibrium of the reduced Hamiltonians $\mathcal{H}_j^{\text{red}}$.

Analogously, we combine the variables x_i, y_j into complex coordinates

$$\mathcal{Z}_j = \sqrt{x_j - x_{j+1} - \mu} \prod_{k=j+1}^{n} e^{\mathrm{i}y_k}, \quad j = 1, \dots, n-1, \quad \mathcal{Z}_n = \sqrt{s - x_1} \prod_{k=1}^{n} e^{\mathrm{i}y_k}.$$

Using these, the symplectic manifold $(M, \omega) = (\mathbb{C}^n, \mathrm{i} \sum_{j=1}^n d\mathcal{Z}_j \wedge d\mathcal{Z}_j^*)$ represents an alternative model of $\mathcal{M}_{\text{red}}$. The point $\mathcal{Z} = 0$ corresponds to common equilibrium of the dual Hamiltonians $\widehat{\mathcal{H}}_j^{\text{red}}$.

We have the reduced Lax matrices, generating the reduced Hamiltonians, in terms of both global models of $\mathcal{M}_{\text{red}}$ explicitly.

Remark: In $\mathcal{Z}_n$ we have $s = \min(0, v-u)$, and in the formula of ζ_n we assumed that $|u| > |v|$.

Consequence: The identity map of the reduced phase space $\mathcal{M}_{\text{red}}$ translates into a (very non-trivial) symplectomorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$, which parametrizes both models M and $\widehat{M}$ of $\mathcal{M}_{\text{red}}$. This is the duality map $\mathcal{R}$ that produces action-angle variables for our pair of integrable systems obtained by Hamiltonian reduction. **The Hamiltonian flows of the action variables are equivalent to the standard torus action on the symplectic vector space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.**

In summary, we have generalized the reduction treatment of the duality between the trigonometric BC_n Sutherland system

$$H(q,p) = \frac{1}{2}\sum_{j=1}^n p_j^2 + \sum_{1\leq j<k\leq n}\left(\frac{\gamma}{\sin^2(q_j-q_k)} + \frac{\gamma}{\sin^2(q_j+q_k)}\right) + \sum_{j=1}^n \frac{\gamma_1}{\sin^2(q_j)} + \sum_{j=1}^n \frac{\gamma_2}{\sin^2(2q_j)}$$

and the rational Ruijsenaars-Schneider-van Diejen system

$$\widehat{H}(\lambda,\theta) = \sum_{j=1}^n \cos(\theta_j)\left[1-\frac{u^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{v^2}{\lambda_j^2}\right]^{\frac{1}{2}} \prod_{\substack{k=1\\(k\neq j)}}^n \left[1-\frac{\mu^2}{(\lambda_j-\lambda_k)^2}\right]^{\frac{1}{2}}\left[1-\frac{\mu^2}{(\lambda_j+\lambda_k)^2}\right]^{\frac{1}{2}}$$
$$+\frac{uv}{\mu^2}\prod_{j=1}^n\left[1-\frac{\mu^2}{\lambda_j^2}\right] - \frac{uv}{\mu^2}.$$

It is well-known that the cotangent bundle $T^*SU(2n)$ can be recovered as a limit of the Heisenberg double $SL(2n,\mathbb{C})$. Next, I outline how the limit appears for the corresponding dual pairs of integrable Hamiltonians.

The cotangent bundle limit

• Limit of $\widehat{\mathcal{H}}_1^{\text{red}}$ to dual BC_n Sutherland: Introduce positive scaling parameter α. Then, on the domain $\mathcal{D}_\lambda^0 \times \mathbb{T}^n$, we obtain

$$\lim_{\alpha\to 0} \widehat{\mathcal{H}}_1^{\text{red}}(\alpha\lambda, \theta; \alpha\mu, \alpha u, \alpha v) = \widehat{H}(\lambda, \theta; \mu, u, v),$$

where $\widehat{H}$ is the dual of the BC_n Sutherland Hamiltonian. The symplectic form is also rescaled during the limit. This is very similar to the α-scaling limit of H_{vD}.

• Limit of $\mathcal{H}_1^{\text{red}}$ to BC_n Sutherland: Define new Darboux coordinates q_i, p_i by

$$\exp(x_i) = \sin(q_i) \quad \text{and} \quad y_i = p_i \tan(q_i),$$

and then make the substitution

$$u \to \beta u, \quad v \to \beta v, \quad \mu \to \beta\mu, \quad p \to \beta p, \quad \omega_{\text{red}} \to \beta\omega_{\text{red}}, \quad \text{using a parameter } \beta > 0.$$

Then $\mathcal{H}_1^{\text{red}}(x, y; u, v, \mu)$ admits the scaling limit

$$\lim_{\beta\to 0} \frac{\mathcal{H}_1^{\text{red}}(\log \sin q, \beta p \tan q; \beta\mu, \beta u, \beta v) - n}{\beta^2} = H_{\text{Suth}}(q, p; \gamma, \gamma_1, \gamma_2),$$

with $\gamma = \mu^2$ etc. The domain of x and correspondingly that of q depends on β, and in the $\beta \to 0$ limit we recover the usual BC_n domain (Weyl alcove) for q.

The second limit is rather singular: $e^{\mathrm{i}y_j} \in U(1)$ and p_j runs over $\mathbb{R}$ in the limit. This limit is similar to the β-scaling limit of H_{vD}.

Our 3-parametric systems appear to be half way in between the 5-parametric H_{vD} and the dual pair H_{Suth}, $\widehat{H}$ coming from $T^*SU(2n)$.

How to recover our 3-parametric systems from the 5-parametric H_{vD}?

First of all, we replace λ, θ by iλ, iθ in $H_{\mathrm{v}D}$, which gives

$$H'_{\mathrm{vD}}(\lambda,\theta;c_0,c_1,c_2,c'_1,c'_2)=\sum_{j=1}^{n}(\cos\theta_j)\mathcal{V}'_j(\lambda;c)^{\frac{1}{2}}+\mathcal{U}'(\lambda;c)$$

$$\mathcal{V}'_j(\lambda;c)=\prod_{i=1}^{2}\left[1-\frac{\sinh^2 c_i}{\sinh^2\lambda_j}\right]\left[1+\frac{\sinh^2 c'_i}{\cosh^2\lambda_j}\right]\prod_{k\neq j}^{n}\left[1-\frac{\sinh^2 c_0}{\sinh^2(\lambda_j\pm\lambda_k)}\right],$$

$$\mathcal{U}'(\lambda;c)=(\sinh c_0)^{-2}\prod_{i=1}^{2}\sinh(c_i)\cosh(c'_i)\prod_{k=1}^{n}\left[1-\frac{\sinh^2 c_0}{\sinh^2\lambda_k}\right]$$

$$+(\sinh c_0)^{-2}\prod_{i=1}^{2}\cosh(c_i)\sinh(c'_i)\prod_{k=1}^{n}\left[1+\frac{\sinh^2 c_0}{\cosh^2\lambda_k}\right]$$

Then we put $c_0=\mu$, $c_1=u$, $c_2=v$ and recover $\widehat{\mathcal{H}}_1^{\mathrm{red}}$ as follows:

$$\frac{e^{u-v}}{4}\widehat{\mathcal{H}}_1^{\mathrm{red}}(\lambda,\theta;\mu,u,v)-n=\lim_{\substack{c'_1\to-\infty\\ c'_2\to+\infty}}e^{c'_1-c'_2}H'_{\mathrm{vD}}(\lambda,\theta;c).$$

By a different limit, we can also recover $\mathcal{H}_1^{\mathrm{red}}$.

Conclusion and open problems

The examples presented illustrate how Hamiltonian reduction leads to integrable many-body systems enjoying action-angle duality. This framework is useful for studying several other systems as well.

We can show that by applying suitable analytic continuations, specializations and limits, van Diejen's 5-coupling many-body Hamiltonian H_{vD} reproduces both of our reduced Hamiltonians $\mathcal{H}_1^{\mathrm{red}}$ and $\widehat{\mathcal{H}}_1^{\mathrm{red}}$. (The details are given in our papers, using slightly different notations.)

The main open problem related to this talk:

Can one generalize our 3-parametric reduction in such a way to accommodate 5-parameters and yield the 5-coupling systems of van Diejen?

Another long standing open problem: Is there a reduction picture behind the (real, repulsive) hyperbolic RS system?

$$H_{\mathrm{hyp-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 \mu}{\sinh^2 (q_k - q_j)} \right]^{\frac{1}{2}}$$

The general idea: Starting with Abelian algebras $\mathfrak{H}^1$ and $\mathfrak{H}^2$ on a master phase space $\mathcal{M}$, the reduced Abelian algebras are defined by $\mathfrak{H}^i_{\mathrm{red}} \circ \pi_0 = \mathfrak{H}^i \circ \iota_0$ for $i = 1, 2$. They turn into Abelian algebras of the models M and $\widehat{M}$ of $\mathcal{M}_{\mathrm{red}}$ according to $\mathfrak{H} \circ \Psi = \mathfrak{H}^1_{\mathrm{red}} = \hat{\mathfrak{Q}} \circ \hat{\Psi}$ and $\mathfrak{Q} \circ \Psi = \mathfrak{H}^2_{\mathrm{red}} = \hat{\mathfrak{H}} \circ \hat{\Psi}$.

$$
\begin{tikzcd}
 & \mathcal{M}_0 \arrow[r, hook, "\iota_0"] \arrow[dl, "\psi"'] \arrow[d, "\pi_0"] \arrow[dr, "\widehat{\psi}"] & \mathcal{M} \\
M \arrow[d, "q"] & \mathcal{M}_{\mathrm{red}} \arrow[l, "\Psi"'] \arrow[r, "\widehat{\Psi}"] & \widehat{M} \arrow[d, "\lambda"] \\
\mathbb{R}^n & & \mathbb{R}^n
\arrow[from=2-1, to=2-3, bend right, "\mathcal{R}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \iota_0^*(\mathfrak{H}^1) \times \iota_0^*(\mathfrak{H}^2) & \\
\mathfrak{H} \times \mathfrak{Q} \arrow[ur, "\psi^*"] \arrow[r, "\Psi^*"] & \mathfrak{H}^1_{\mathrm{red}} \times \mathfrak{H}^2_{\mathrm{red}} \arrow[u, "\pi_0^*"] & \hat{\mathfrak{Q}} \times \hat{\mathfrak{H}} \arrow[ul, "\widehat{\psi}^*"'] \arrow[l, "\widehat{\Psi}^*"'] \arrow[ll, bend left, "\mathcal{R}^*"]
\end{tikzcd}
$$

References

This talk was mainly based on the following papers:

L.F. and T.F. Görbe, Duality between the trigonometric BC(n) Sutherland system and a completed rational Ruijsenaars-Schneider-van Diejen system, Journ. Math. Phys. 55, 102704 (2014)

L.F. and T.F. Görbe, On a Poisson-Lie deformation of the BC(n) Sutherland system, Nucl. Phys. B 901, 85-114 (2015)

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L.F. and I. Marshall, Global description of action-angle duality for a Poisson-Lie deformation of the trigonometric BC_n Sutherland system, arXiv:1710.08760

Predecessors of the joint works with Görbe and Marshall are:

L.F. and B.G. Pusztai, A class of Calogero type reductions of free motion on a simple Lie group, Lett. Math. Phys. 79, 263-277 (2007)

B.G. Pusztai, The hyperbolic BC(n) Sutherland and the rational BC(n) Ruijsenaars-Schneider-van Diejen models: Lax matrices and duality, Nucl. Phys. B 856, 528-551 (2012)

I. Marshall, A new model in the Calogero-Ruijsenaars family, Commun. Math. Phys. 338, 563-587 (2015)

References to the fundamental papers of Ruijsenaars, van Diejen, Fock-Gorsky-Nekrasov-Roubtsov, and others, as well as to the reduction techniques used, can be found in the above sources.