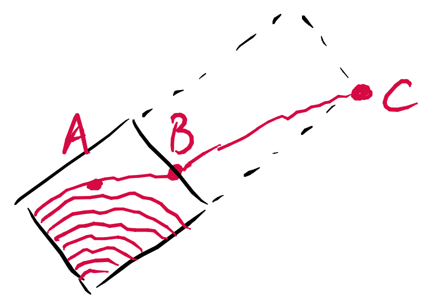
Paths and Arctic Curves: the Tangent Method at work

(P. Di Francesco, M. Lapa (UIUC Physics), E. Guitter)

0. Introduction
1. The tangent method

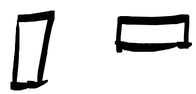
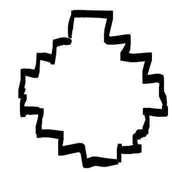


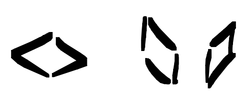
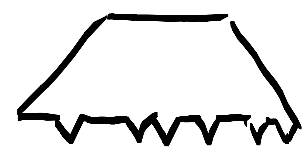
2. Domino tilings of the Aztec Diamond
3. Rhombus tilings with a special boundary
4. Alternating Sign matrices (with a symmetry axis)
5. Conclusion

0. INTRODUCTION

Combinatorial problems:

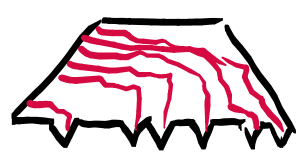
Tiling of finite domain with finite set of tiles

Ex: a. dominos 
2×1 1×2 , Aztec diamond 

b. rhombi  , "half"-hexagon 

↑ special boundary

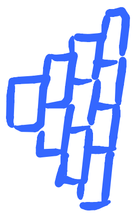
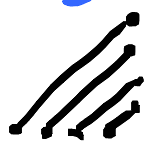
NILP = Non-intersecting lattice paths $\{a_i\} \to \{e_i\}$.



Asymptotics

- appearance of different phases
  - $\to$ order "crystal"
  - $\to$ disorder "liquid"

Ex

-  domino crystal
-  path crystal

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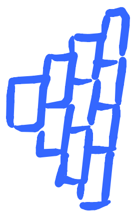
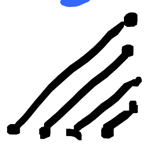
-  domino crystal
-  path crystal

- At large size ($\mathcal{D} \to N\mathcal{D}$, $N \to \infty$) there is a sharp separation between these phases

  = ARCTIC CURVE

Enumeration

- reformulate all problems in terms of NILP

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- Use Gessel-Viennot-Lindström determinant

$$\boxed{Z_{\{a_i\}\to\{e_i\}} = \det_{1\le i,j\le n}\left(Z_{a_i\to e_j}\right)}$$ = partition function of paths from $a_1 \dots a_n$ to $e_1 \dots e_n$

$$Z_{a\to b} = \sum_{\substack{\text{paths}\\ a\to b}} w(\text{path}) \; ; \quad w(\text{path}) = \prod_{\text{edges}} w(\text{edge})$$

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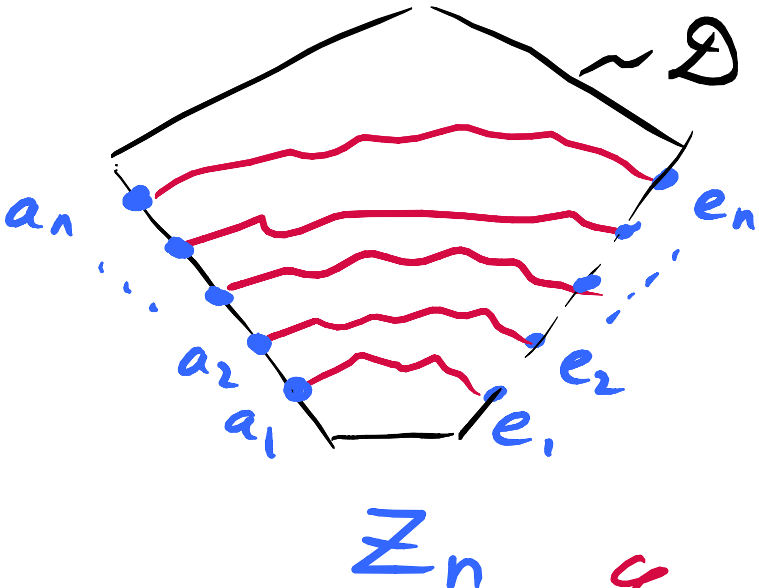
- Calculate det by LU decomposition of $A = (Z_{a_i\to e_j})$

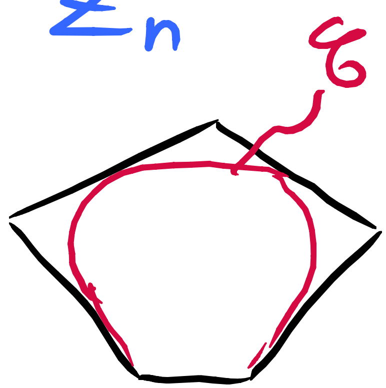
$$\det(A) = \det(U) \qquad L = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \times & & 1 \end{pmatrix} \quad U = \begin{pmatrix} \times & \cdots & \times \\ & \ddots & \vdots \\ 0 & & \times \end{pmatrix}$$

- use generating functions as a tool

1. TANGENT METHOD [Colomo–Sportiello 16]

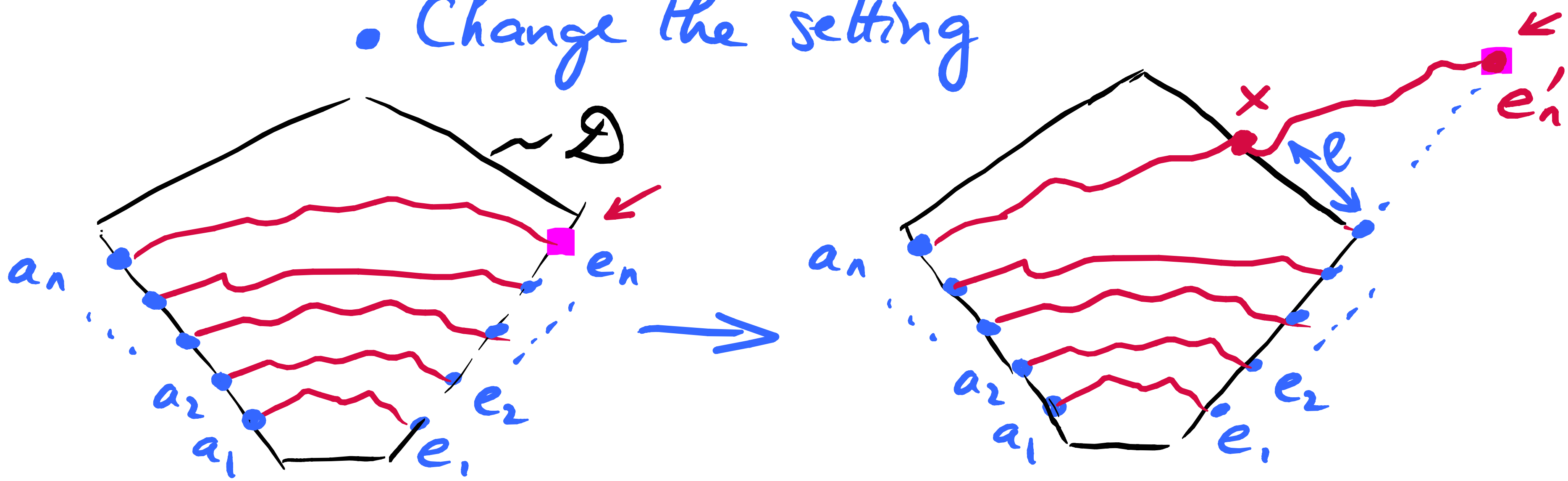


- large size



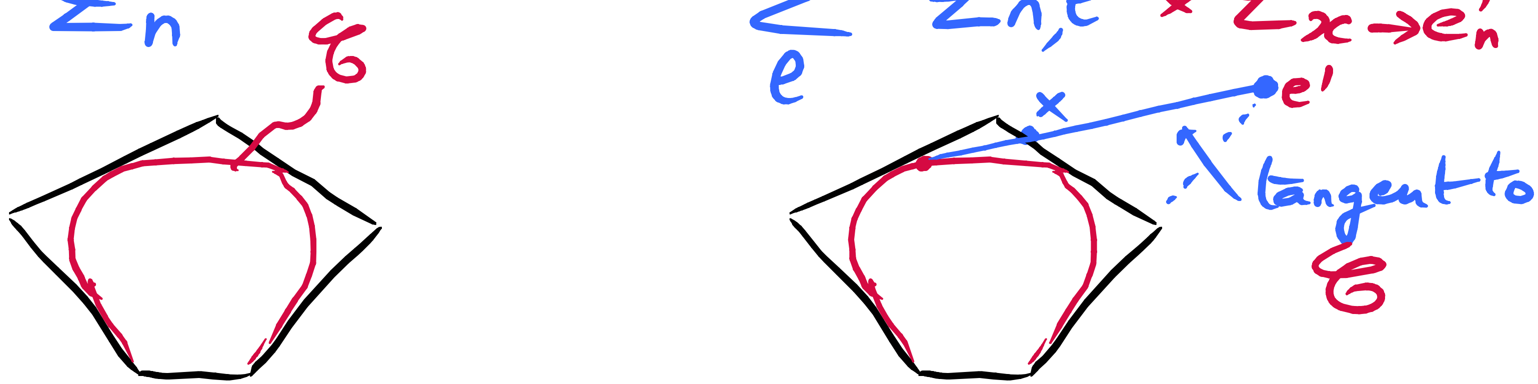
1. TANGENT METHOD [Colomo–Sportiello 16]

• Change the setting



Z_n $\qquad\qquad$ $\sum_e Z_{n,e} \times Z_{x \to e'_n}$

• large size



Relies on 2 properties :

1. "left to its own devices, a directed random path with fixed endpoints is most likely to follow a straight line"

2. "The line followed by the external path away from the others is tangent to the arctic curve $\mathcal{E}$"

1. can be proved rigorously.
2. still an assumption

APPLYING THE TANGENT METHOD:

Summary: 1. Compute the "escaping path" partition function $Z_{n,\ell}$ and "1pt function"

$$\boxed{H_{n,\ell} := \frac{Z_{n,\ell}}{Z_{n,0}}}$$

$Z_{n,\ell}$ ← escaping point at ℓ

$Z_{n,0}$ ← original partition function

2. Compute the free path partition function

$$\boxed{Y_{\ell,e'} = \text{single path from } \ell \to e'}$$

3. Scaling estimate $\sum_{\ell} H_{n,\ell}\, Y_{\ell,e'}$

$n\to\infty$ $\quad \ell = n\xi \quad e' = na$ ← new endpoint; a = free parameter then $\sum_{\ell} H_{n,\ell}\, Y_{\ell,\ell'} \sim \int d\xi\; e^{n\underbrace{(S_0(\xi)+S_1(\xi,a))}_{S}}$

by saddle point $\rightarrow$ most likely $\xi = \xi_0 = \text{fctn}(a)$

$\Rightarrow$ tangent line = thru $n\xi$ and na

$\Rightarrow$ 6 as envelope, for varying a.

- We must estimate $H_{n,\ell=n\xi}$ at large n
- Do exact enumeration first. $Z_{n,0} = \text{LGV}\det = \det_{0\le i,j\le n}(A_{ij})$

$$Z_{n,\ell} = \det_{0\le i,j\le n}(\tilde{A}_{ij}) \text{ where } \begin{cases} \tilde{A}_{ij} = A_{ij} & j<n \\ \tilde{A}_{i,n} = Z_{i\to \ell'} \end{cases}$$

only the last column differs

LU decomposition:

[DF+Lapa 17]

$$A = L \cdot \tilde{U} \qquad \det A = \det \tilde{U}$$

$\tilde{U}$: upper triangular

L: uni-lower triangular

Then $L^{-1}\tilde{A} = \tilde{U}$ with $\tilde{U}$ upper triangular

Same L!

$\tilde{U}_{ij} = U_{ij} \quad j<n$

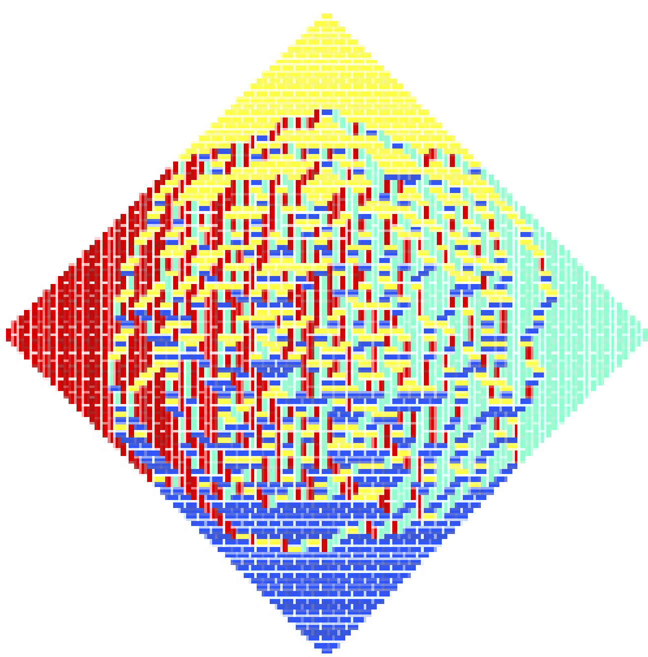
and $$\boxed{H_{n,\ell} = \frac{\det(\tilde{A})}{\det(A)} = \frac{\tilde{U}_{n,n}}{U_{n,n}}}$$

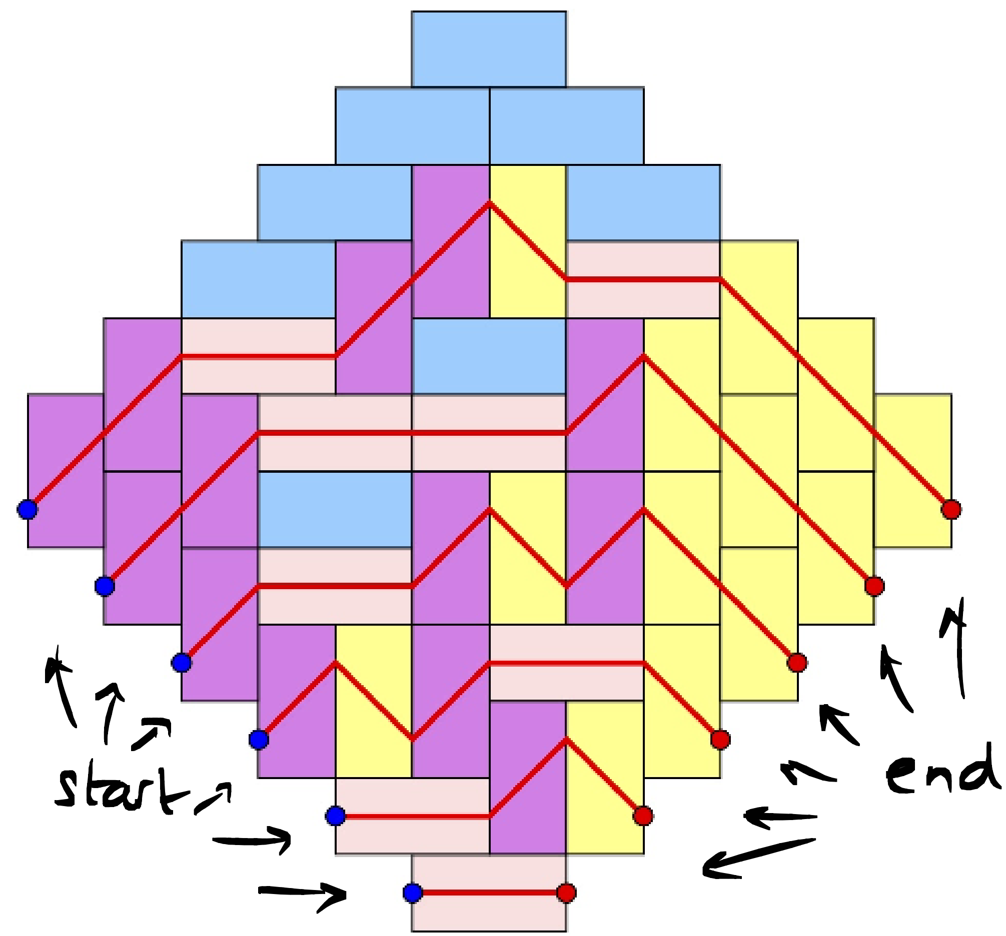
$\left(= \dfrac{\det \tilde{U}}{\det U}\right)$

It all boils down to LU decomposition

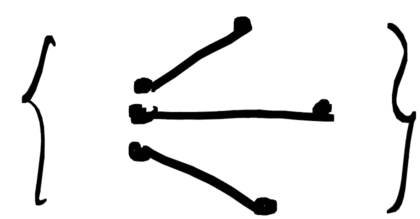
<u>NB:</u> $\tilde{U}_{n,n} = \sum_i (L^{-1})_{n,i} \cdot \tilde{A}_{i,n}$ = alternating sum, not good for large n estimates $\rightarrow$ TURN IT INTO a >0 sum!!

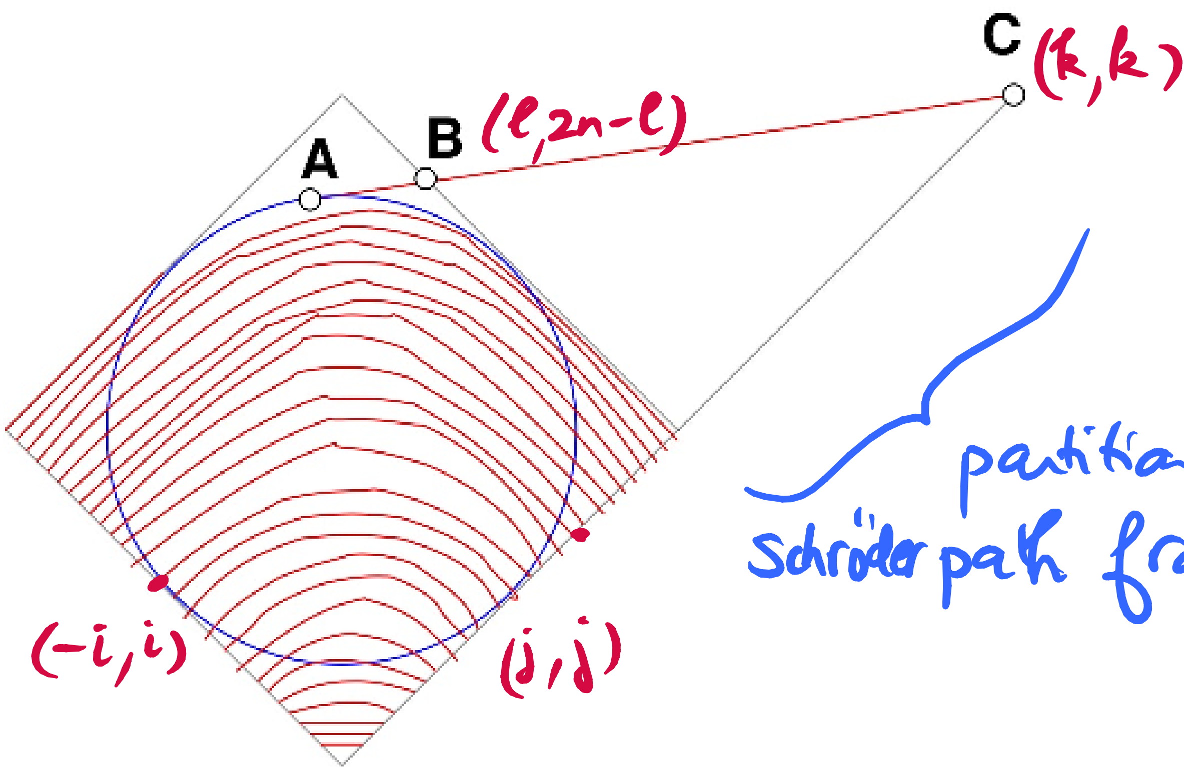
2. Domino Tilings of the Aztec Diamond





Schröder Paths:





partition function for a single Schröder path from $B \to C$ $= Y_{\ell,k}$

partition fctn w/escaping path at B

"$Z_{n,\ell}$"

$$\frac{1}{Z_n} \sum_{\ell} Z_{n,\ell} \, Y_{\ell,k} \quad ?$$

↑ most likely ℓ ?

Z_n)

LGV matrix:

$$A_{ij} = \frac{1}{1-z-w-zw}\Big|_{z^i w^j} = \sum_{p=0}^{\mathrm{Min}(i,j)} \frac{(i+j-p)!}{p!\,(i-p)!\,(j-p)!}$$

LU decomposition

$$L_{ij} = \frac{1}{1-z(1+w)}\Big|_{z^i w^j} = \binom{i}{j} \qquad (L^{-1})_{ij} = (-1)^{i+j}\binom{i}{j}$$

$$U_{ij} = \frac{1}{1-w(1+2z)}\Big|_{z^i w^j} = 2^i \binom{j}{i}$$

Partition function:

$$\boxed{Z_n = \det A = \prod_{0}^{n} U_{ii} = 2^{n(n+1)/2}}$$

$Z_{n,\ell}$

LGV matrix:

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & j<n \\ A_{i+\ell-n,n} & j=n \end{cases}$$

LU decomposition:

$$L^{-1}\tilde{A} = \tilde{U}$$

$$\tilde{U}_{ij} = \begin{cases} U_{ij} & j<n \\ \sum_{k}^{n} L^{-1}_{ik}\tilde{A}_{kn} & j=n \end{cases}$$

1-pt function:

$$\boxed{H_{n,\ell} = \frac{\det \tilde{A}}{\det A} = \frac{\tilde{U}_{n,n}}{U_{n,n}} = \frac{1}{2^n}\sum_{j=0}^{\ell}\binom{n}{j}}$$

Proof:

$$\tilde{U}_{n,l} = \sum_i \underbrace{L^{-1}_{n,i}}_{(-1)^{n+i}\binom{n}{i}} \tilde{A}_{i,n} = \sum_i L^{-1}_{n,i} \underbrace{A_{i+l-n,n}}_{\frac{1}{1-z-w-zw}\Big|_{z^{i+l-n}w^{n}}}$$

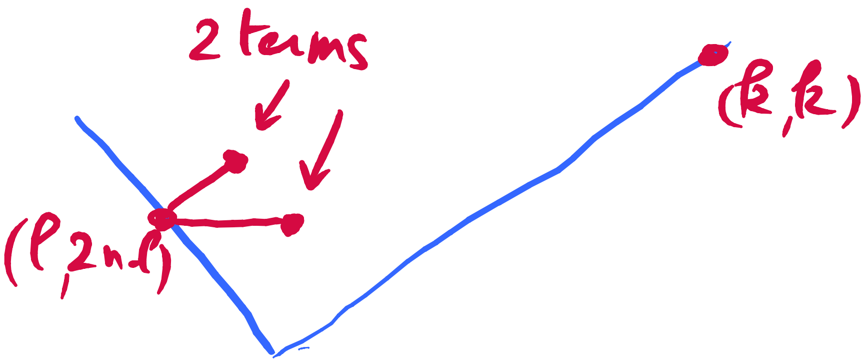
$$= \sum_i (-z)^{n-i}\binom{n}{i} \frac{1}{1-z-w-zw}\Bigg|_{z^{l}w^{n}}$$

$$= \frac{(1-z)^n}{1-z-w-zw}\Bigg|_{z^{l}w^{n}} = (1-z)^n \frac{(1+z)^n}{(1-z)^{n+1}}\Bigg|_{z^{l}} = \frac{(1+z)^n}{1-z}\Bigg|_{z^{l}}$$

$$= \sum_{0}^{l} \binom{n}{j} \qquad \text{qed.}$$

$Y_{\ell,k}$

Single path from $(\ell, 2n-\ell) \longrightarrow (k,k)$ exiting diamond



$$\boxed{Y_{\ell,k} = A_{n-\ell,\,k-n-1} + A_{n-\ell-1,\,k-n-1}}$$

Tangent method: asymptotics of $\sum_\ell H_{n,\ell} Y_{\ell,k}$

Scaling: $\boxed{n \text{ large} \quad \ell = n\xi \quad k = nz \quad \xi \in (0,1);\ z > 1.}$

$Y_{n,k}$

$$Y_{n\xi,nx} \sim 2A_{n(1-\xi),n(z-1)} \sim \int_0^{\mathrm{Min}(1-\xi,x-1)} d\theta \; e^{S_0(\theta,\xi,z)}$$

$$\boxed{\underbrace{S_0(\theta,\xi,z)}_{\text{Stirling}} = (z-\xi-\theta)\log(z-\xi-\theta) - \theta\log\theta - (1-\xi-\theta)\log(1-\xi-\theta) - (z-1-\theta)\log(z-1-\theta)}$$

$H_{n,\ell}$

$$H_{n,n\xi} \sim \frac{1}{2^n}\sum_{j=0}^{\ell}\binom{n}{j} \sim \int_0^{\xi} d\varphi \; e^{nS_1(\varphi,z)}$$

$$\boxed{S_1(\varphi,z) = -\varphi\log\varphi - (1-\varphi)\log(1-\varphi) - \log 2}$$

Saddle point:

total action: $S = S_0 + S_1 \quad (\varphi, \theta, \zeta, z)$

$$\frac{\partial S}{\partial \varphi} = 0 \quad \Rightarrow \quad \varphi_0 = \frac{1}{2}$$

$$\begin{cases} (1)\ \zeta > \frac{1}{2} & \text{then} \quad S_1(\varphi_0, z) = 0 \quad \text{and} \quad H_{n,n\zeta} \sim 1 \\ (2)\ \zeta < \frac{1}{2} & \text{then} \quad S_1(\zeta, z) \text{ dominates} = -\zeta \log \zeta - (1-\zeta) \log(1-\zeta) - \log 2 \end{cases}$$

$$\begin{cases} (1)\ \zeta > \frac{1}{2} & S = S_0(\theta, \zeta, z) \\ (2)\ \zeta < \frac{1}{2} & S = S_0(\theta, \zeta, z) + S_1(\zeta, z) \end{cases}$$

Now extremize S over θ, ζ : $\dfrac{\partial S}{\partial \theta} = \dfrac{\partial S}{\partial \zeta} = 0$

(1) no solution

(2) $(1-\zeta-\theta)(z-1-\theta) = \theta(z-\zeta-\theta)$ and $(1-\zeta-\theta)(1-\zeta) = (z-\zeta-\theta)\zeta$

$$\Rightarrow \boxed{\zeta_0(z) = \frac{1}{2z}}$$

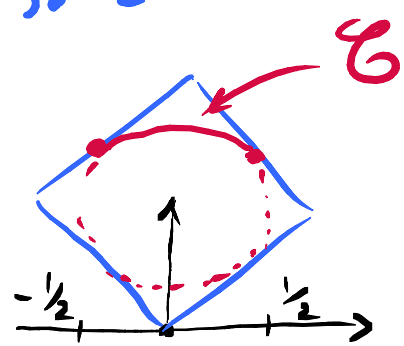
most likely exit point $= (\zeta_0 \, , 2-\zeta_0)$

Tangent Family:

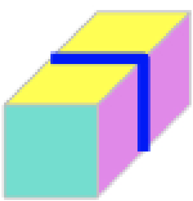
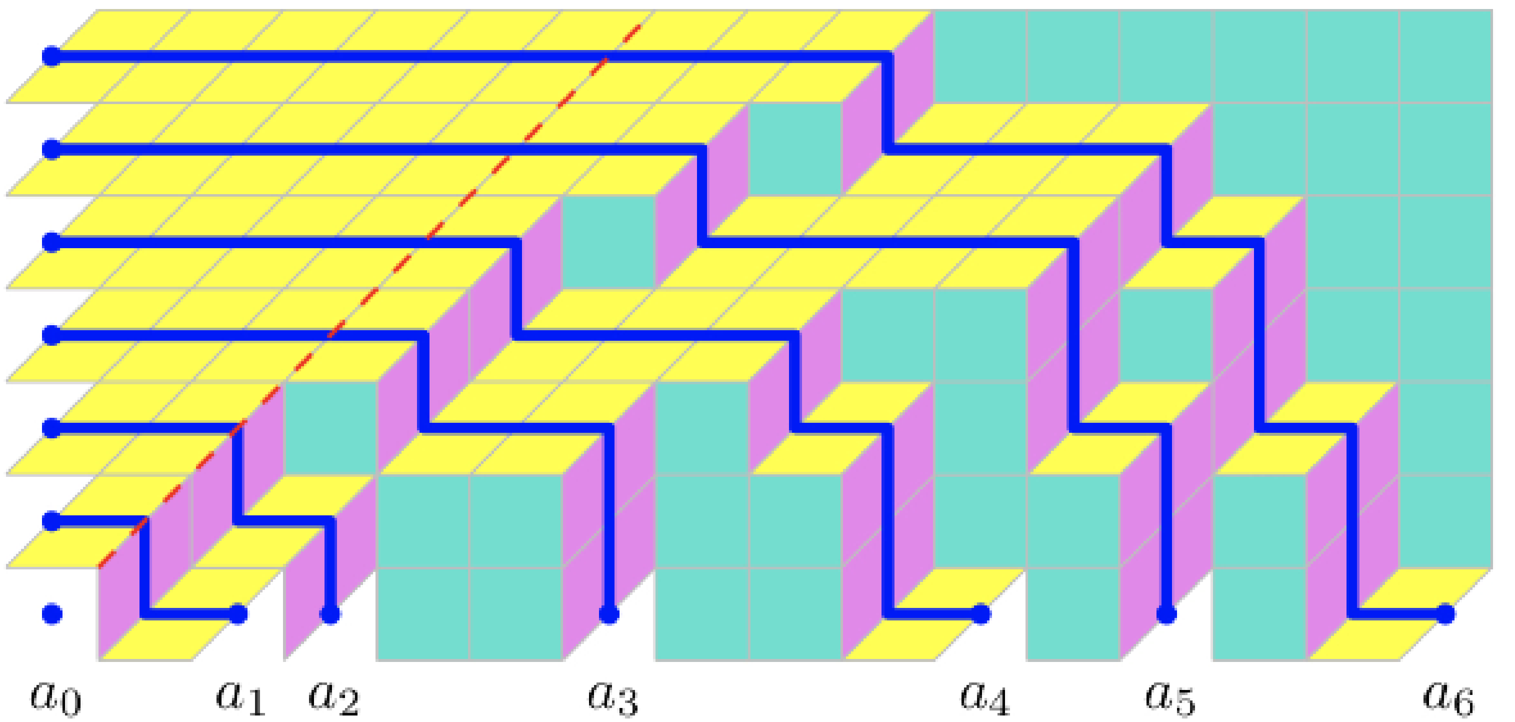
$$L(x,y) = y - \frac{2-\zeta_0-z}{\zeta_0-z}\, x + 2z\, \frac{1-\zeta_0}{\zeta_0-z} = 0$$

Envelope $\quad \dfrac{\partial L}{\partial z} = L = 0$

$\mathcal{C}$: $\boxed{x^2 + (y-1)^2 = \frac{1}{2}}$ $\quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$



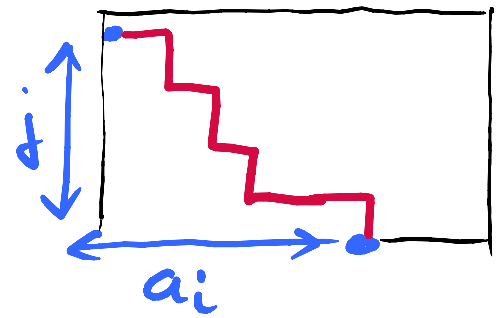
3. RHOMBUS TILING WITH SPECIAL BOUNDARY CONDITIONS

→ a_i fixed but chosen arbitrarily

Partition function

- LGV matrix: $A_{i,j} = \binom{j+a_i}{j}$



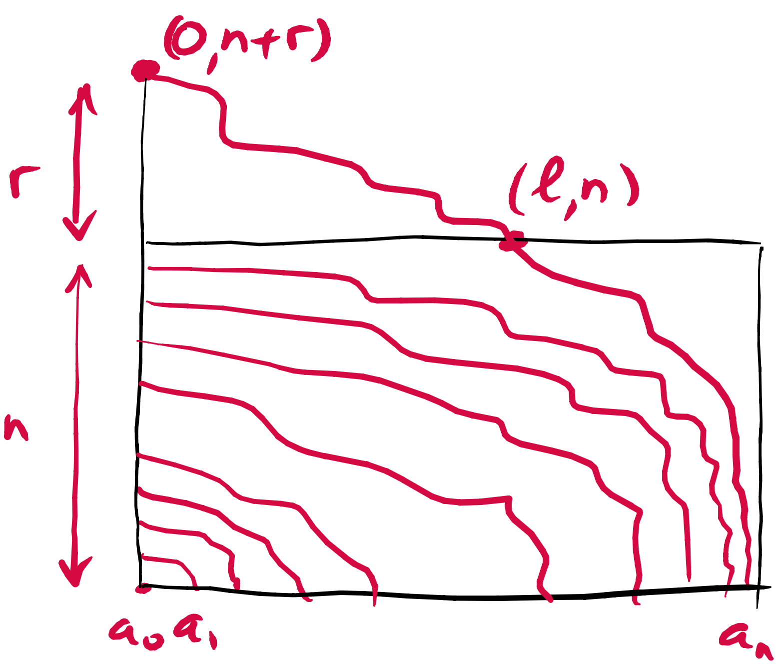
- LU decomposition
  - $$L^{-1}_{ij} = \begin{cases} \dfrac{\prod_{s=0}^{i-1}(a_i - a_s)}{\prod_{\substack{s=0 \\ s\neq j}}^{i}(a_j - a_s)} & i \leq j \\ 0 & \text{otherwise} \end{cases}$$
    (uni-lower triangular)
  - $$U_{ij} = \sum_k L^{-1}_{ik} A_{kj} = \prod_{0}^{i-1}(a_i - a_s) \oint_{a_0 \dots a_i} \frac{dt}{2i\pi} \frac{\frac{1}{j!}\prod_{0}^{j-1}(t+j-s)}{\prod_{0}^{i}(t-a_s)}$$
    UPPER!

$$U_{ii} = -\operatorname{Res}_{\infty} = \frac{1}{i!} \prod_{0}^{i-1} (a_i - a_s)$$

$$\Rightarrow \boxed{Z_n = \det(A) = \det U = \frac{\Delta(a_0, a_1, \dots a_n)}{\Delta(0, 1, 2 \dots n)}}$$

$$\Delta(x_0 \cdots x_n) = \prod_{i<j} (x_j - x_i) \quad \text{(Vandermonde)}.$$

TANGENT METHOD



$Y_{\ell,r}$ (single path partition function)

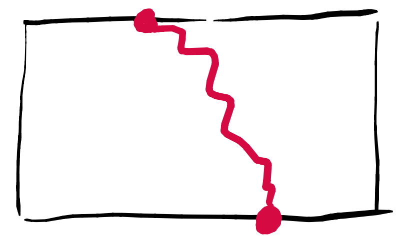
$Z_{n,\ell}$ $\quad H_{n,\ell} = \dfrac{Z_{n,\ell}}{Z_n}$ (1-pt function)

estimate $\sum_{\ell} H_{n,\ell}\, Y_{\ell,r}$ at large n

Use LU decomposition!

One-point function

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & j<n \\ \binom{a_i+n-\ell}{n} & j=n \end{cases}$$



$$H_{n,\ell} = \frac{(L^{-1}\tilde{A})_{n,n}}{(L^{-1}A)_{n,n}} = \frac{\tilde{U}_{n,n}}{U_{n,n}}$$

$$\tilde{U}_{n,n} = \sum L^{-1}_{n,k}\tilde{A}_{k,n} = \prod_0^{n-1}(a_n-a_s) \oint \frac{dt}{2i\pi} \frac{\frac{1}{n!}\prod_0^{n-1}(t+n-\ell-s)}{\prod_0^{n}(t-a_s)}$$

$a_s \geq \ell$

Single Path partition function

$$Y_{\ell,r} = \binom{\ell+r-1}{\ell}$$

Asymptotics and tangent method

$\ell = n\xi$; $r = nz$; $a_i = n\,\alpha(i/n)$ $\begin{cases} \alpha \text{ piecewise } C^1 \\ \alpha' \geq 1 \end{cases}$

THM [DF+Guitter 18] The tangent method leads to the following parametric arctic curve:

$$\boxed{\begin{cases} X(t) = t - \dfrac{x(t)(1-x(t))}{x'(t)} \\ Y(t) = (1-x(t))^2/x'(t) \end{cases} \quad (t \in \mathbb{R}) \qquad x(t) = e^{-\int_0^1 \frac{du}{t-\alpha(u)}} \quad \text{moment g.f.}}$$

<u>Proof:</u>

$$\begin{cases} H_{n,\ell=n\zeta} \sim \oint \frac{dt}{2i\pi}\, e^{n S_0(t,\zeta)} & \text{(up to } t \to nt\text{)} \\ Y_{\ell=n\zeta,\, r=nz} \sim e^{n S_1(z,\zeta)} \end{cases}$$

$$S_{tot} = S_0 + S_1 = \int_0^1 du \log\left(\frac{t+u-\zeta}{t-\alpha(u)}\right) + (\zeta+z)\log(\zeta+z) - \zeta\log\zeta$$
$$- z\log z$$

Proof: $\begin{cases} H_{n,\ell=n\zeta} \sim \oint \frac{dt}{2i\pi} e^{n S_0(t,\zeta)} & \text{(up to } t \to nt\text{)} \\ Y_{\ell=n\zeta, r=nz} \sim e^{n S_1(z,\zeta)} \end{cases}$

$$S_{tot} = S_0 + S_1 = \int_0^1 du \log\left(\frac{t+u-\zeta}{t-\alpha(u)}\right) + (\zeta+z)\log(\zeta+z) - \zeta\log\zeta - z\log z$$

Saddle-point:
$$\frac{t+1-\zeta}{t-\zeta} \underbrace{e^{-\int_0^1 \frac{du}{t-\alpha(u)}}}_{=: x(t)} = 1; \quad \frac{(\zeta+z)(t-\zeta)}{\zeta(t+1-\zeta)} = 1$$

Proof:

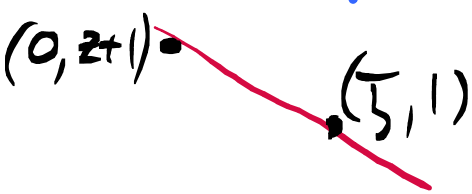
$$\begin{cases} H_{n,\ell=n\xi} \sim \oint \frac{dt}{2i\pi}\, e^{n S_0(t,\xi)} \\[2ex] Y_{\ell=n\xi,\, r=nz} \sim e^{n S_1(z,\xi)} \end{cases}$$

(up to $t \to nt$)

$$S_{tot}=S_0+S_1=\int_0^1 du \log\left(\frac{t+u-\xi}{t-\alpha(u)}\right)+(\xi+z)\log(\xi+z)-\xi\log\xi$$
$$-z\log z$$

Saddle-point:

$$\frac{t+1-\xi}{t-\xi}\underbrace{e^{-\int_0^1\frac{du}{t-\alpha(u)}}}_{=:\,x(t)}=1;\qquad \frac{(\xi+z)(t-\xi)}{\xi(t+1-\xi)}=1$$



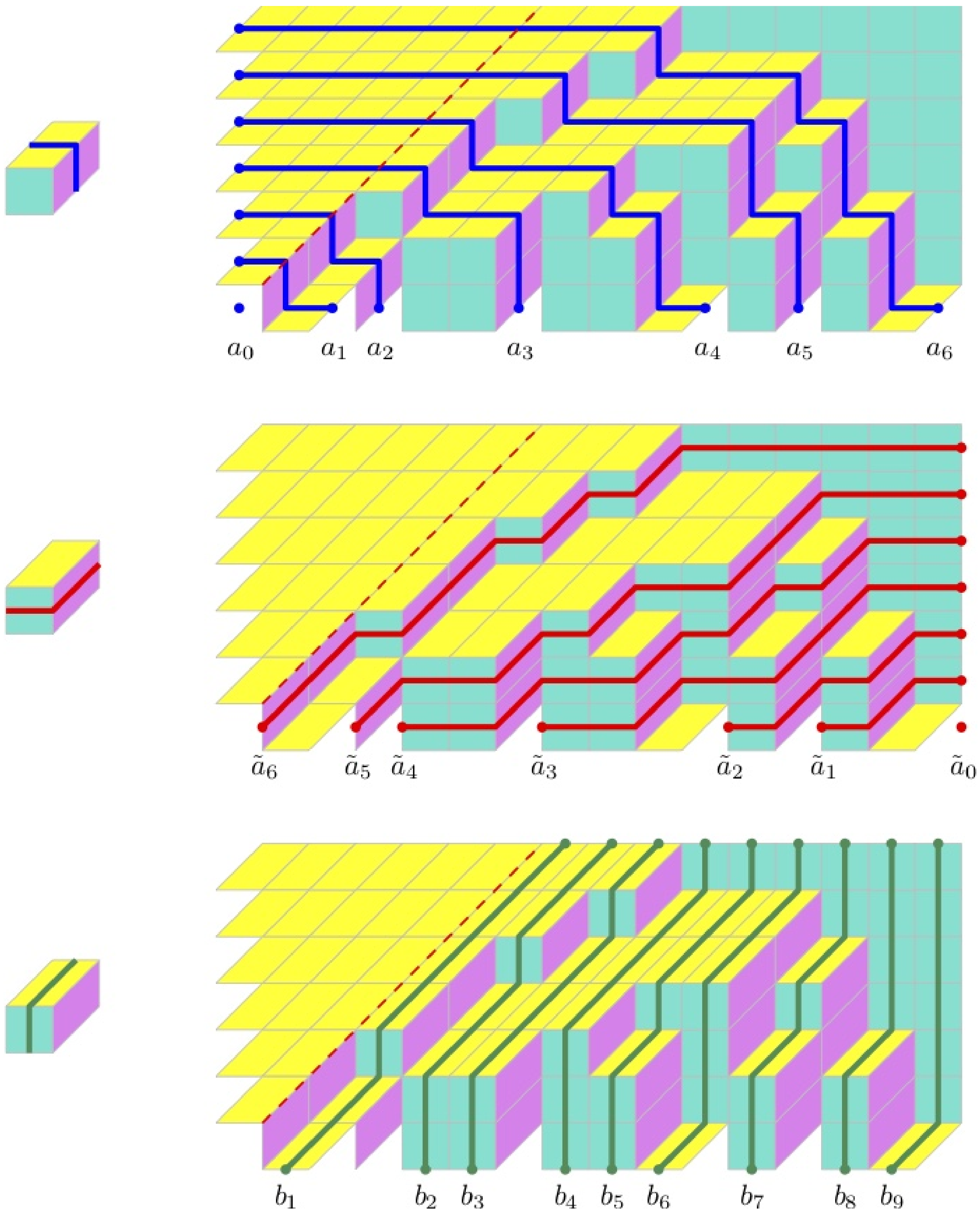
Tangent line

Envelope

slope $\frac{x(t)-1}{x(t)}$

intercept x-axis

$$\left.\begin{aligned} x(t)\,Y+(1-x(t))(X-t)&=0\\ x'(t)(Y-X+t)+x(t)-1&=0\end{aligned}\right\}\Rightarrow \text{THM}$$



3 equivalent problems

$\Downarrow$

allows to patch up various pieces of the arctic curve

(different domains of $t \in \mathbb{R}$)

Examples

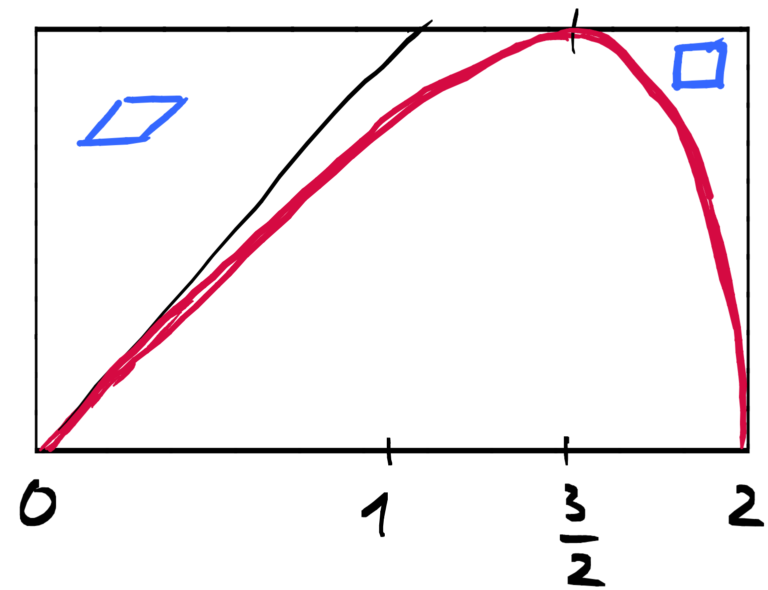
$\alpha(u)$ codes the boundary condition

- pure case $a_i = p i$.

density $1/p$

$\alpha(u) = p u$ $\quad p > 1$

$$x(t) = \left(1 - \frac{p}{t}\right)^{\frac{1}{p}}$$



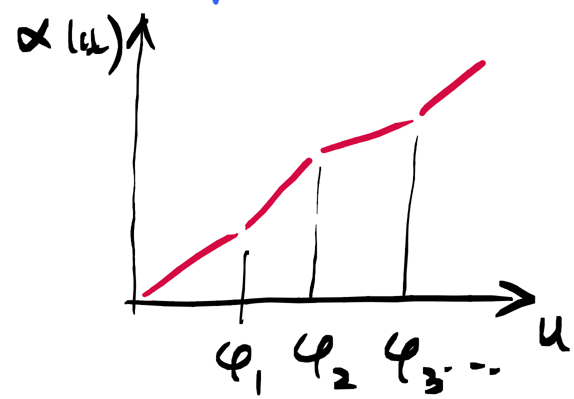
$p = 3$

Case $p \in \mathbb{N}$: arctic curve is algebraic, of degree $2p-2$

• piecewise linear case

$$\varphi_i = \gamma_1+\gamma_2+\dots+\gamma_i \qquad \theta_i = p_1\gamma_1+p_2\gamma_2+\dots+p_i\gamma_i$$

$$\alpha'(u)=p_i \qquad u\in[\varphi_{i-1},\varphi_i]$$



$$x(t) = \prod_{i=1}^{k}\left(1-\frac{\theta_i}{t}\right)^{\frac{1}{p_i}-\frac{1}{p_{i+1}}} \qquad (p_{k+1}=\infty)$$

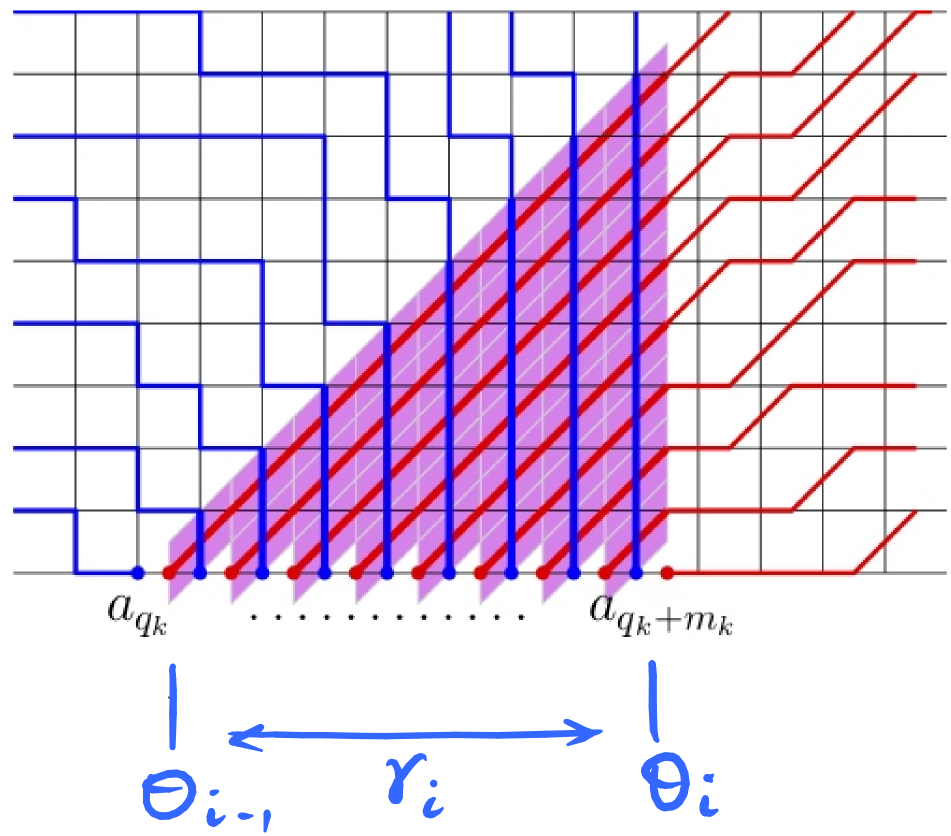
Particular cases: "freezing boundaries".

1. FREEZING

$$\boxed{p_i = 1 \qquad r_i > 0}$$

→ zig-zag boundary

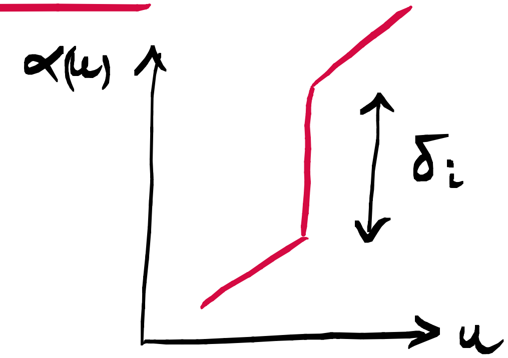
→ induces a frozen triangle inside the domain.



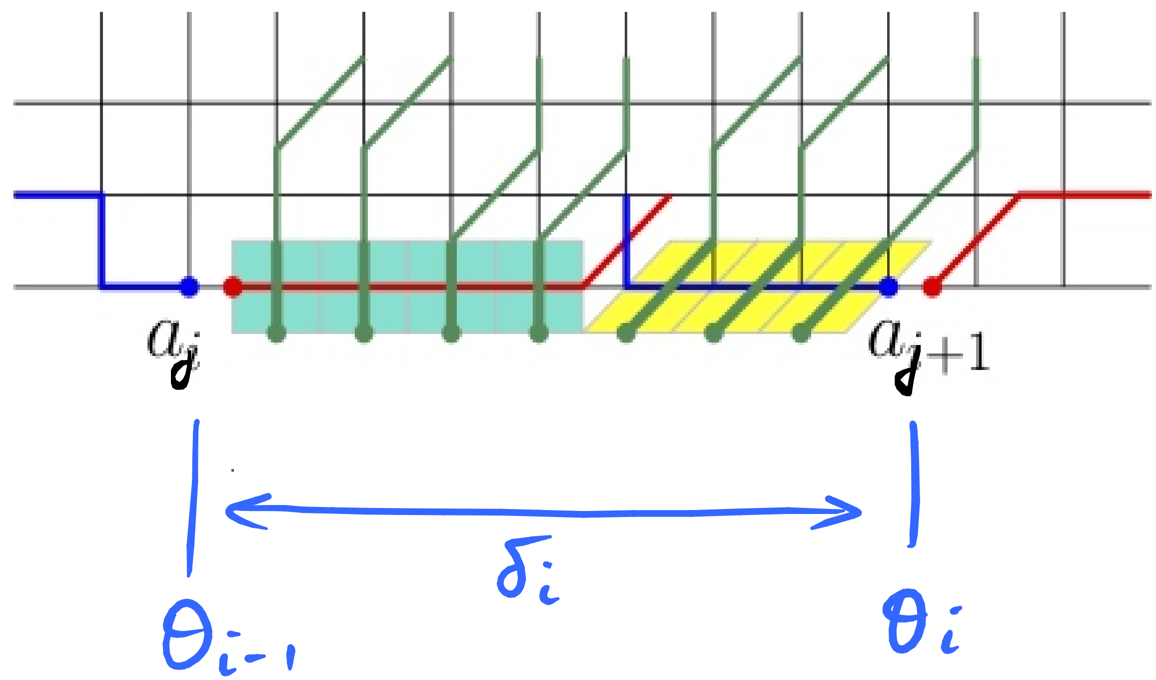
Particular cases: "freezing boundaries".

2. GAPS

$p_i = \infty$ $\quad r_i = 0$ $\quad \delta_i = r_i p_i > 0$

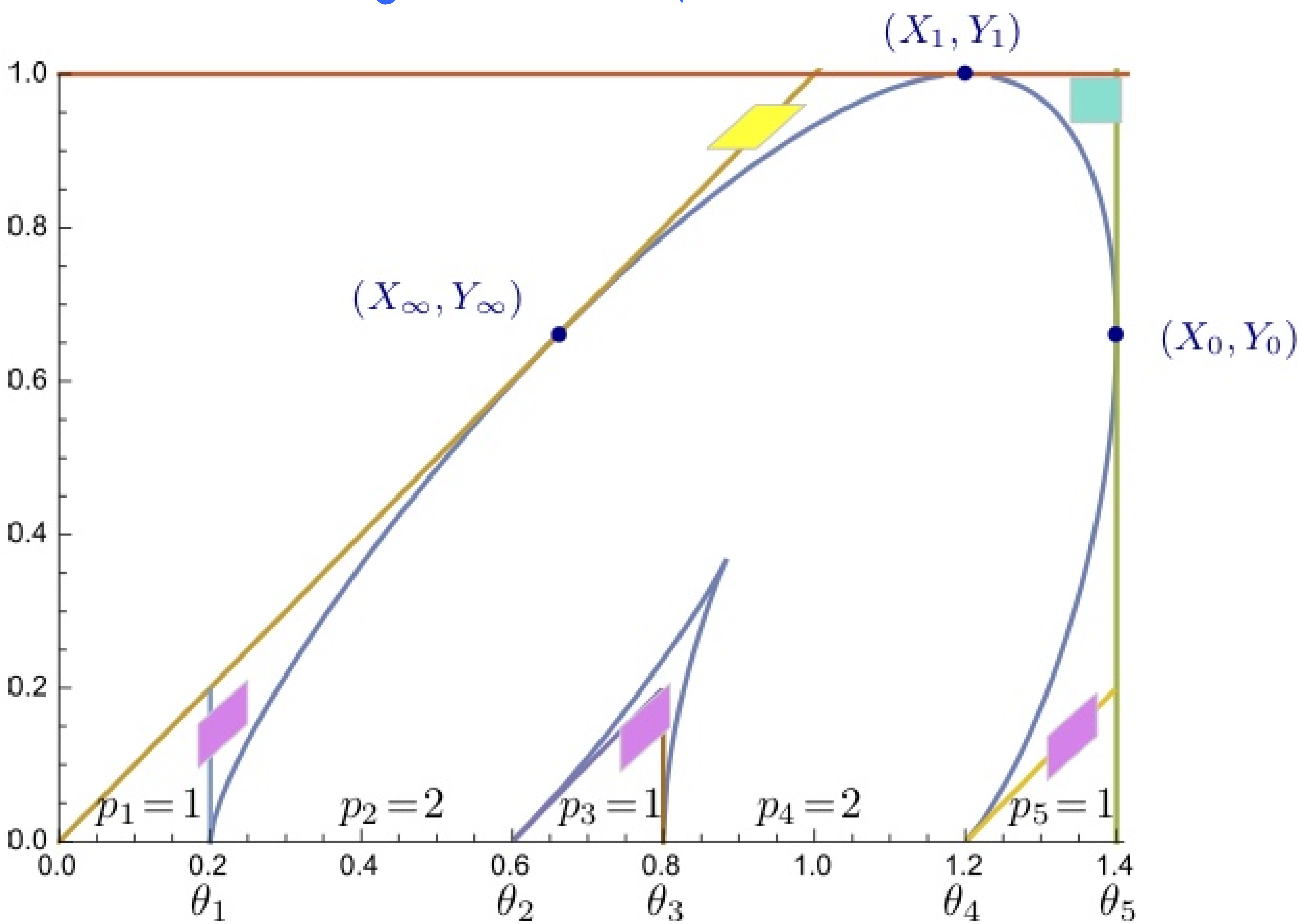


→ flat boundary



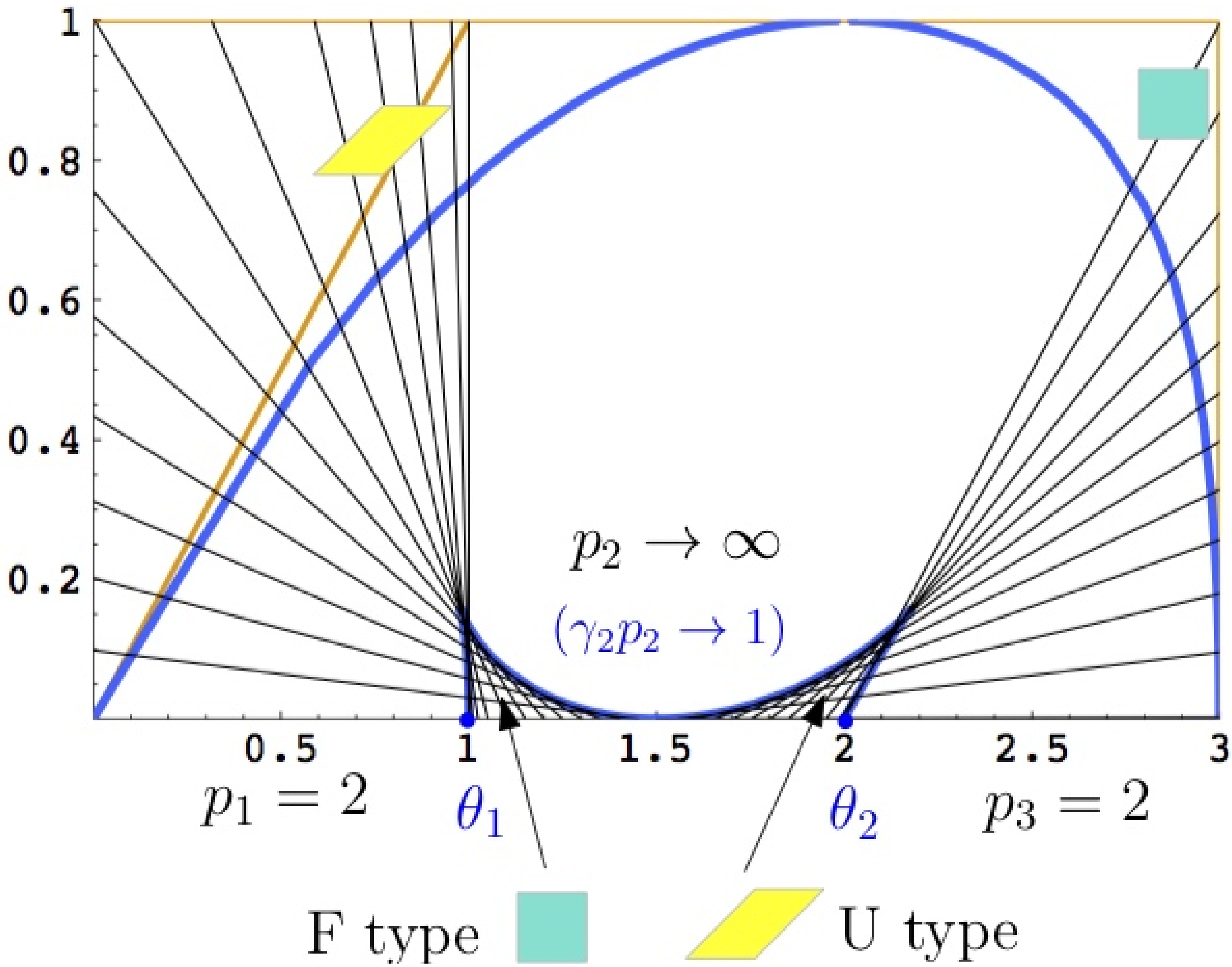
- partially frozen boundaries (portions w/ $p_i = 1$)

Example: $k=5$ $\gamma_i = \frac{1}{5}$; $p_1 = 1$, $p_2 = 2$, $p_3 = 1$, $p_4 = 2$, $p_5 = 1$

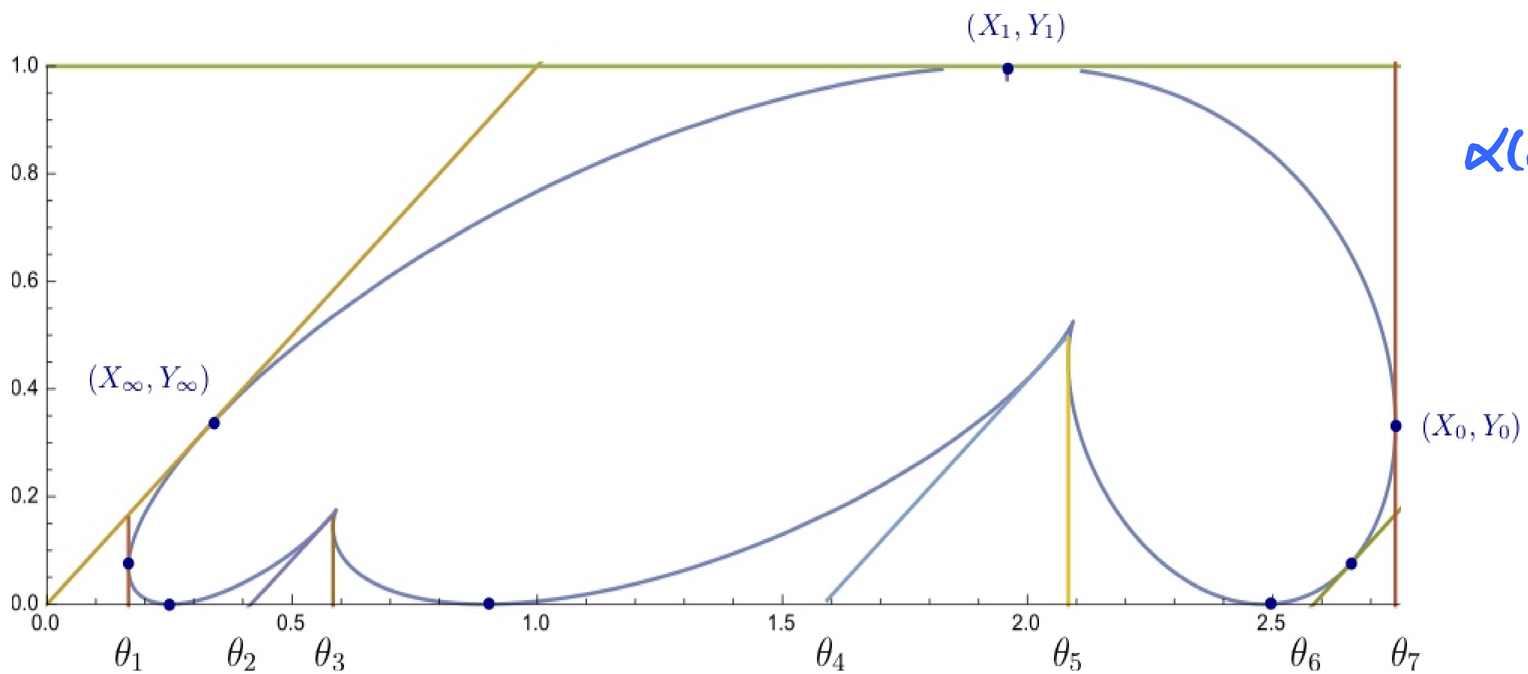


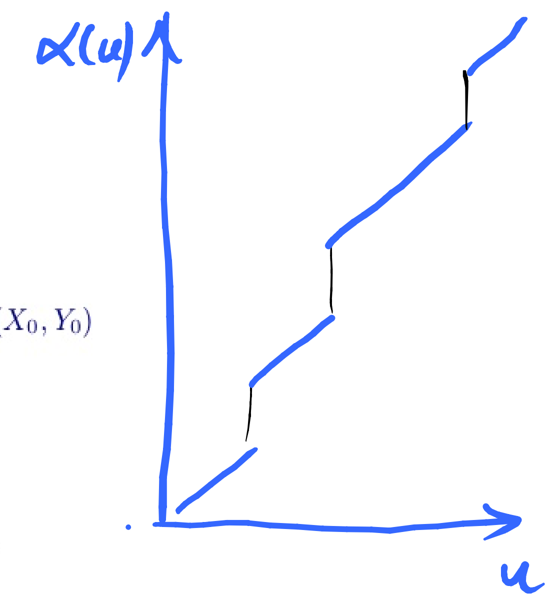
• partially frozen boundaries (portions w/gaps)

Example: $k=3$ $p_1=p_3=2$ gap $p_2=\infty$ $\delta_2=1$; $\gamma_1=\gamma_3=\frac{1}{2}$.
$\gamma_2=0$



- fully frozen boundaries (only $\alpha'=1$ or gaps)





$$\begin{cases} \alpha(u) = u + \sum_{1}^{i-1} \delta_j & u \in [\varphi_{i-1}, \varphi_i] \\ x(t) = \prod_{1}^{k} \dfrac{t-\theta_{2j-1}}{t-\theta_{2j}} \end{cases}$$

($\alpha'=1$ is $p=1$
gap is $p \to \infty$)

• non-linear cases

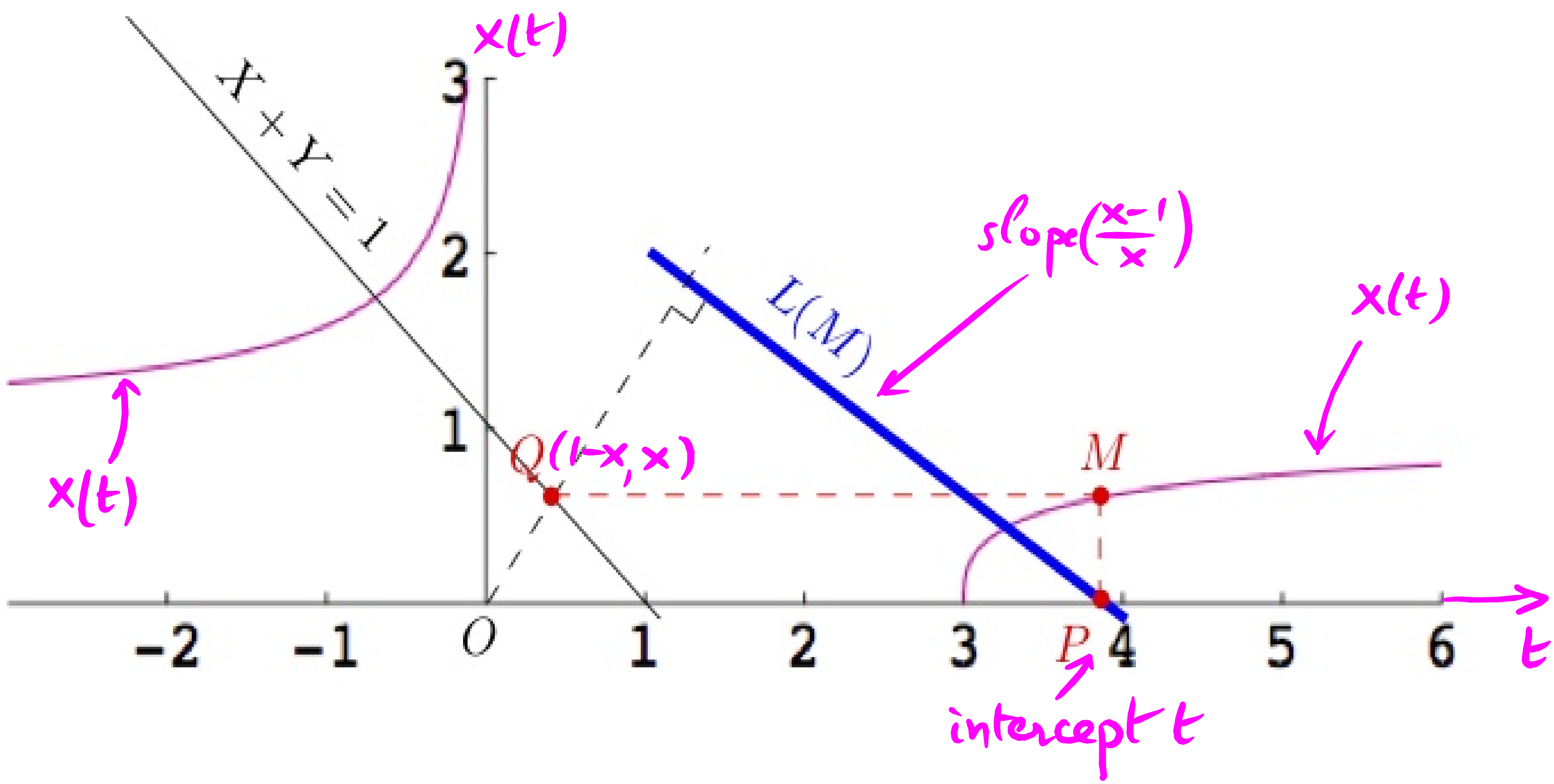
1. $\alpha(u) = pu + qu^2$ $\quad (p \geq 1 ; q > 0)$

$$x(t) = \left(\frac{p - 2t + \sqrt{p^2 + 4qt}}{p - 2t - \sqrt{p^2 + 4qt}} \right)^{\frac{1}{\sqrt{p^2+4qt}}}$$

2. $\alpha(u) = \frac{1}{a} u^a$ $\quad a \in (0,1)$

$$x(t) = e^{-{}_2F_1\left(1, \frac{1}{a}; 1+\frac{1}{a} \middle| \frac{1}{at}\right)/t}$$

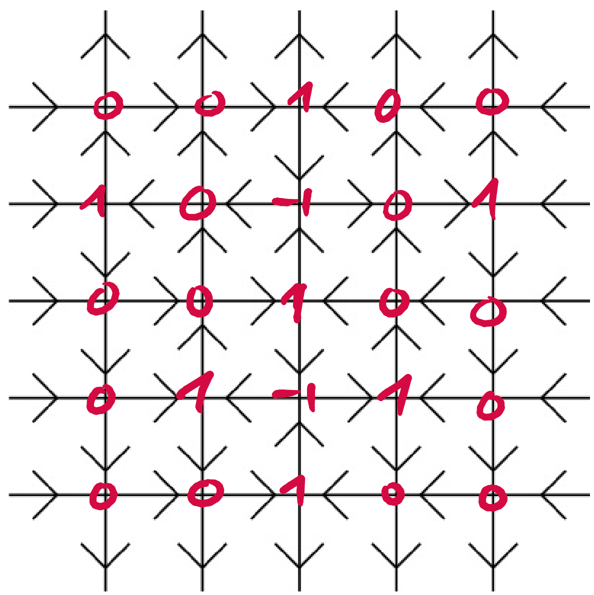
A Geometric Construction of the Arctic curve

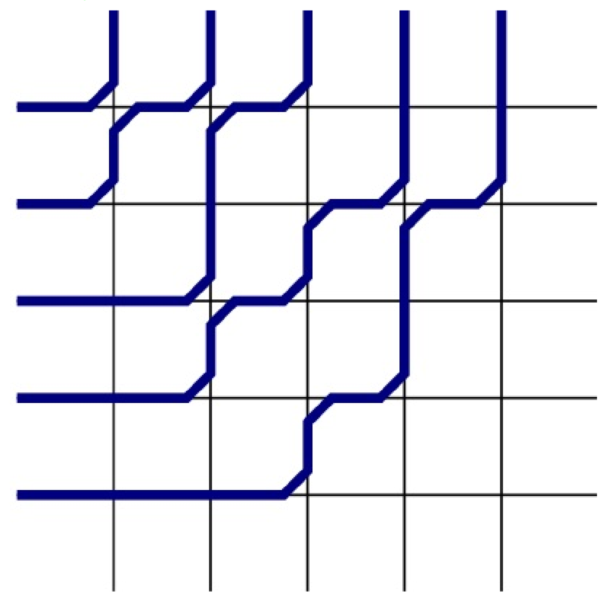


4. Vertically Symmetric Alternating sign matrices

[DF+Lapa 17]

ASM
|||
6 Vertex



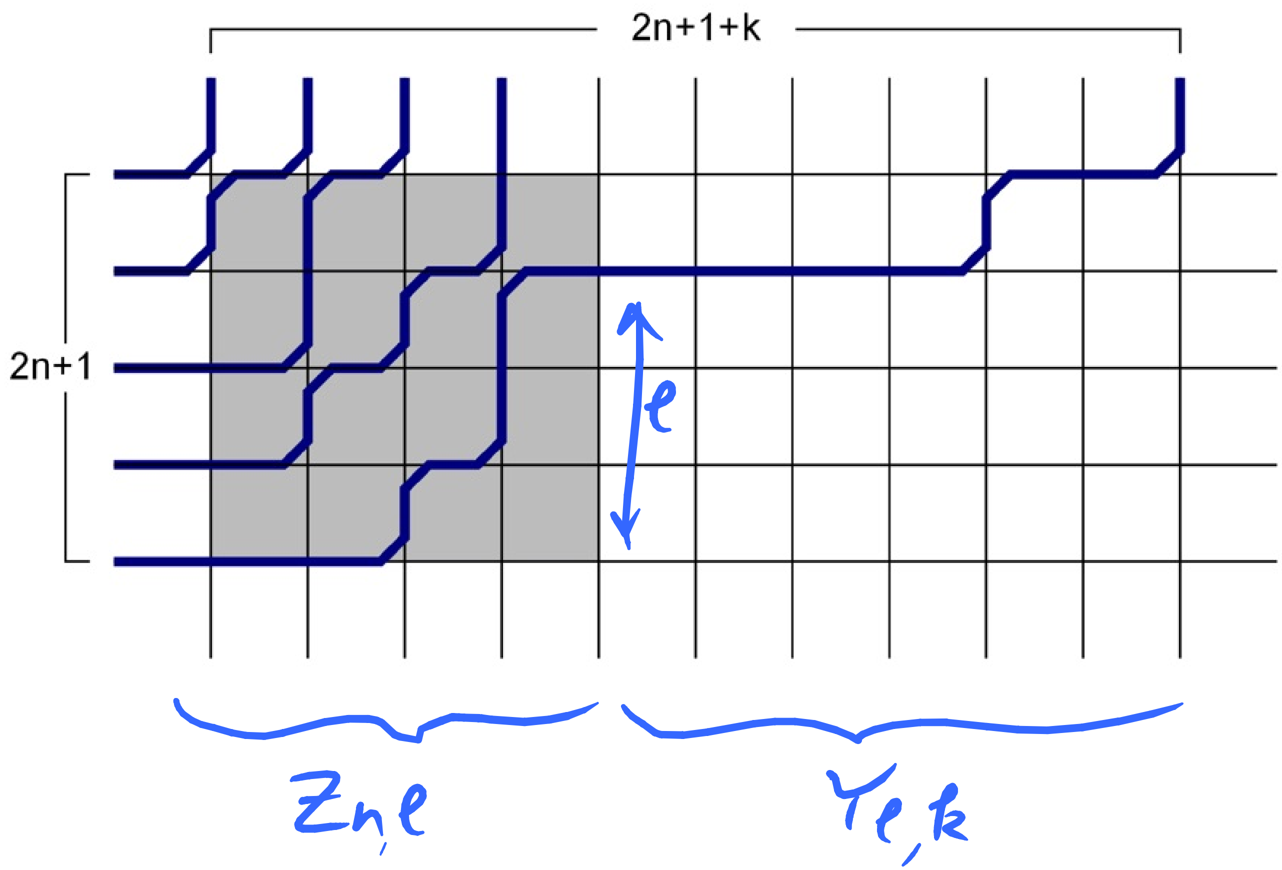


VSASM

symmetric wrt vertical line.

OSCULATING PATHS

TANGENT METHOD



Crucial relation [Razumov-Stroganov 04]

$$\frac{1}{N_{VSASM}(2n+1)} \sum_{\ell=1}^{2n} N_{VSASM}(2n+1,\ell)\, t^{\ell-1}$$

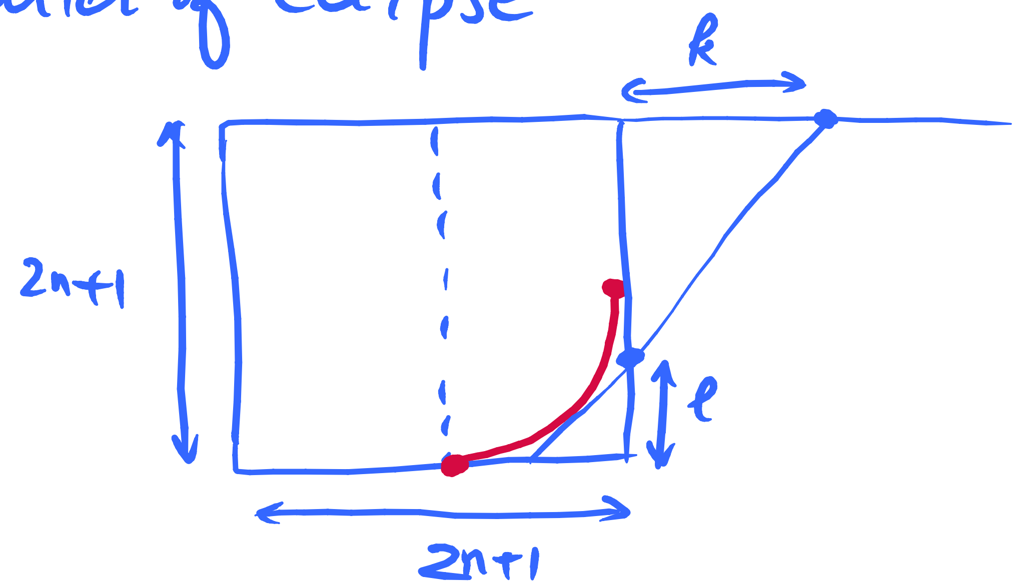
← position of 1 in last column

$$= \frac{1}{N_{ASM}(2n-1)} \frac{t}{1+t} \sum_{i=1}^{2n} N_{ASM}(2n,i)\, t^{i-1}$$

$$H_{n,\ell} = \frac{N_{VSASM}(2n+1,\ell)}{N_{VSASM}(2n+1)} \qquad Y_{\ell,k} = \sum_{i=0}^{\min(k-1,2n+1-\ell)} \binom{k-1}{i}\binom{2n+1-\ell}{i}$$

(8) $$\boxed{4\left((x-1)^2 + y^2 - xy\right) + 4(x-1) + 8y + 1 = 0}$$

$\Rightarrow$ quarter of ellipse



complete by symmetry = <u>same result as ASMs</u>

CONCLUSION

- It works, but why?

→ must show tangency to $\mathcal{C}$

- Beyond NILP = it still works. Why?

and what kind of interaction can we allow

- many other examples
  - Osculating Schröder
  - inhomogeneous weights
  - fused 6V ...

Merci !

[1] P. Di Francesco and M. Lapa, "Arctic curves in Path models from the tangent method" J Phys A: Math Theor 51 (2018) 155202. ArXiv 1711.03182 [math-ph].

[2] P. Di Francesco and E. Guitter, "Arctic curves for paths with arbitrary starting points: a tangent method approach" ArXiv 1803.11463 [math-ph].

Bon Anniversaire

Kolya !