An Introduction to the BV-BFV Formalism

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Based on joint work with P. Mnëv, N. Moshayedi, N. Reshetikhin, M. Schiavina, K. Wernli

Outline

1. **Differential Graded Symplectic Manifolds**
2. **BF^{n}V structures**
   - BFV
   - BV
3. **Relaxed BF^{n}V Structures**
4. **Quantization**
   - BFV-BF^{2}V
   - BV-BFV

Graded manifolds

- A graded manifold is “like a manifold,” but we also allow odd local coordinates.
- The odd coordinates anticommute with themselves and commute with the even coordinates.
- We also assign a $\mathbb{Z}$-degree to coordinates (in physics ghost number). In this talk, I will assume

$$\textit{parity} = \mathbb{Z}\textit{-degree modulo 2}$$

- If M is a graded manifold, then $C^\infty(M)$ is a graded commutative algebra.
- Example: $M = T[1]N$, N an ordinary manifolds:
  - Coordinates q^i on N degree 0; fiber coordinates v^i degree $+1$.
  - $C^\infty(M) = \Omega^\bullet(N)$.

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Cohomological vector fields

- If M is a graded manifold, a differential on $C^\infty(M)$ has been called by Vaintrob a cohomological vector field (cvf).
- Explanation:
  - Derivation on $C^\infty(M)$ = vector field Q on M.
  - Differential = $\deg Q = 1$ and $[Q, Q] = 0$.
- Example: $M = T[1]N$, N an ordinary manifolds.
  The de Rham differential on N is a cvf on $C^\infty(M)$: $Q = \sum_i v^i \frac{\partial}{\partial q^i}$
- Example: $M = \mathfrak{g}[1]$, $\mathfrak{g}$ a Lie algebra:
  - All coordinates c_i have degree 1.
  - $C^\infty(M) = \Lambda^\bullet \mathfrak{g}^*$.
  - The Chevalley–Eilenberg differential on $\mathfrak{g}$ is a cvf:

$$Q = \frac{1}{2} \sum_{ijk} f^{ij}_k c_i c_j \frac{\partial}{\partial c_k}$$

with f^{ij}_k the structure constants. (In physics the BRST operator.)

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Graded symplectic forms

- A graded symplectic form ω of degree n is a closed, nondegenerate 2-form with internal degree equal to n.
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  - Coordinates q^i on N degree 0; fiber coordinates p_i degree 1.
  - $C^\infty(M) = \mathfrak{X}^\bullet(N)$ (multivector fields).
  - $\omega = \sum_i \mathrm{d}p_i\, \mathrm{d}q^i$ is a graded symplectic form of degree 1.
  - The Schouten–Nijenhuis bracket on multivector fields is the associated Poisson bracket.
- This may be generalized to $M = T^*[n]N$, N an ordinary manifold:
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Differential graded symplectic structure

- A differential graded symplectic structure of degree n on a graded manifold M is a pair (ω, Q) where:
  - ω is a a graded symplectic form of degree n, and
  - Q is a symplectic cvf, i.e.,

$$\mathrm{L}_Q\,\omega = 0 \quad \text{and} \quad [Q, Q] = 0$$

- A stronger version is when Q is Hamiltonian, i.e., there is a function S (necessarily of degree $n+1$) such that

$$\iota_Q\omega = \mathrm{d}S \quad \text{and} \quad \{S, S\} = 0 \ \text{(classical master equation)}$$

- As observed by Roytenberg, if $n \neq -1$, a symplectic cvf is always Hamiltonian (with a unique Hamiltonian function):

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with E the "graded Euler vector field": $E(f) = \deg f\, f$.

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Examples of differential graded symplectic structures

- Example: $M = T^*[1]N$, N an ordinary manifolds:
  - A function S of degree 2 on M is then the same as a bivector field π on N.
  - The master equation $\{S, S\} = 0$ translates to $[\pi, \pi] = 0$; i.e., π is a Poisson bivector field.
  - Q is then the Poisson–Lichnerowicz differential.
- Example: $M = \mathfrak{g}[1]$, $\mathfrak{g}$ a Lie algebra:
  - A nondegenerate symmetric bilinear form on $\mathfrak{g}$ can be viewed as a constant symplectic form of degree 2 on $\mathfrak{g}[1]$.
  - If Q corresponds to the Chevalley–Eilenberg differential, it is symplectic iff the pairing is invariant.
  - The corresponding Hamiltonian function turns out to be

$$S = \frac{1}{6} \sum f^{ijk} c_i c_j c_k$$

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BF^{n}V structures

There are three important particular cases:

- $n = -1$ This is the Batalin–Vilkovisky (BV) formalism used in QFT. The Hamiltonian function is required to exist as an extra assumption.
- $n = 0$ This is the Batalin–Fradkin–Vilkovisky (BFV) formalism used to give a cohomological resolution of symplectic reduction (see next).
- $n = 1$ If there are only coordinates of nonnegative degree, this is just the example of $M = T^*[1]N$ with a Poisson structure on N.
  More generally, it describes Poisson structures up to homotopy (i.e., the Poisson bracket is an L_∞-structure).

We may call the general case of degree n a BF^{n+1}V structure.

$$\text{Poisson}_\infty = \text{BF}^2\text{V}$$

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Symplectic reduction in codimension one (BFV)

- Let (N, ω_N) be a symplectic manifold, ϕ a function and $C := \phi^{-1}(0)$ a submanifold.
- The restriction of ω_N to C is degenerate. Its kernel is generated by the Hamiltonian vector field X_ϕ of ϕ:

$$\iota_{X_\phi}\omega_N = \mathrm{d}\phi \approx 0$$

- We define $\underline{C} = C/X_\phi$. Algebraically,

$$C^\infty(\underline{C}) = (C^\infty(N)/ < \phi >)^{X_\phi} = N(< \phi >)/ < \phi >,$$

with $N(< \phi >) = \{f \in C^\infty(N) : \{\phi, f\} = g\phi,\ g \in C^\infty(N)\}$.
- Define $M = N \times T^*\mathbb{R}[1]$, $\omega = \omega_N + \mathrm{d}b\mathrm{d}c$, $S = c\phi$ a dgs manifold of degree 0. Then

$$Qb = \phi, \quad Qf = \{\phi, f\}c, \quad Qc = 0.$$

In particular, in degree 0 and -1:

$$Q(f + gcb) = \{\phi, f\}c - g\phi c, \quad Q(hb) = h\phi + \{\phi, h\}cb.$$

Hence $H^0_Q(M) = C^\infty(\underline{C})$.

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Symplectic reduction of coistropic submanifolds

- The previous slide may be generalized to the case when we have a set of r functions ϕ^i and a submanifold

$$C := \{x \in N : \phi^i(x) = 0 \; i = 1, \dots, r\}$$

- Assume that C is coisotropic, i.e., $\{\phi^i, \phi^j\}|_C = 0 \; \forall i, j$.
- The kernel of the restriction of ω_N to C is generated by the Hamiltonian vector fields of the ϕ^is. We denote by $\underline{C}$ the quotient of C by this kernel. (The reduced phase space.)
- The main result by BFV and Stasheff is that

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with: $M = N \times T^*\mathbb{R}^r[1]$, $\omega = \omega_N + \sum_{i=1}^r \mathrm{d}b^i \mathrm{d}c_i$,

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Equivariant momentum map

- A special case is when the ϕ^i are the components of an equivariant momentum map $\phi\colon N \to \mathfrak{g}^*$.
- In this case we have

$$S = \sum_{i=1}^{r} c_i \phi^i + \frac{1}{2} \sum f_k^{ij} b^k c_i c_j$$

- Q in this case is also called the BRS operator.

The BV formalism

- Let $\mathcal{F}$ be an odd finite dimensional symplectic manifold. By results of Batalin–Vilkovisky, Witten, Schwarz, Khudaverdian, Ševera. . . :
  1. Half densities on $\mathcal{F}$ naturally define densities on Lagrangian submanifolds of $\mathcal{F}$, so they can be integrated.
  2. There is a canonically defined operator Δ on half densities satisfying $\Delta^2 = 0$.
  3. $\int_{\mathcal{L}} \Delta\rho = 0$, for $\mathcal{L}$ Lagrangian, ρ half density.
  4. $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_t}\sigma = 0$ if $\Delta\sigma = 0$ and $\mathcal{L}_t$ a family of Lagrangian submanifolds.
- Main example: $\mathcal{F} = \Pi T^*M$, M manifold.
  Then half densities on $\mathcal{F}$ = differential forms on M,
  $\int_{\Pi N^*C}\rho = \int_C \rho,\qquad \Delta \equiv \mathrm{d}.$
- Application: Use $\mathcal{L}$ as a gauge fixing.
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The master equation

- Fix a half density ρ with $\Delta\rho = 0$. On functions define Δ by
$$\Delta f := \frac{\Delta(f\rho)}{\rho}.$$
- In view of applications to path integrals, consider $f = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$. Then $\Delta f = 0$ corresponds to the Quantum Master Equation (QME)
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Either working with $\hbar \to 0$ or assuming $\Delta S = 0$, we get the Classical Master Equation (CME)
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Further remarks

- It is convenient to introduce a $\mathbb{Z}$-grading (For simplicity in this talk: parity $=$ $\mathbb{Z}$-grading modulo 2.).
- One then assigns degrees so that S has degree zero and its Hamiltonian vector field Q

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  has degree 1. This forces ω to have degree -1.
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Example: gauge theories

- Suppose we have a gauge theory with space of fields F_M, a local action functional S_M^0 and a gauge group acting on it.
- We denote by ϕ the fields, by c the ghosts and by δ_{BRST} the BRST operator. We assign degrees by $|\phi| = 0$ and $|c| = 1$.
- We introduce antifields ϕ^+, c^+ with opposite nature than the field, opposite parity and degrees given by $|\phi^+| = -1$ and $|c^+| = -2$.
- We denote by $\mathcal{F}_M$ the space of all the (ϕ, c, ϕ^+, c^+)'s and we set

$$S_M = S_M^0(\phi) + \int_M (\phi^+ \, \delta_{\mathrm{BRST}}\phi + c^+ \, \delta_{\mathrm{BRST}} c),$$

$$\omega_M = \int_M (\delta\phi^+ \delta\phi + \delta c^+ \delta c)$$

- Then, if $\partial M = \emptyset$, the BV action S satisfies the CME.
- We usually consider these “BRST-like” theories, but there are more general examples.

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Relaxed structures

- Suppose we have a graded manifold M with a cohomological vector field Q and a closed 2-form ω of degree n. We set

$$\breve{\alpha} := \iota_Q \omega - \mathrm{d}S$$

and

$$\breve{\omega} := \mathrm{d}\breve{\alpha} = -\mathrm{L}_Q\,\omega$$

- It turns out that $\breve{\omega}$ is a closed, Q-invariant 2-form ω of degree $n+1$.
- We denote by $\underline{M}$ the quotient of M by the kernel of $\breve{\omega}$ (assume it is smooth). We denote by $\underline{\omega}$ its symplectic form of degree $n+1$.
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Field theory

- Suppose that M is a space of fields on some compact manifold Σ.
- Suppose we have a BF^{n-1}V structure on M with ω, Q, and S local.
- This allows us to write ω, Q, and S also on some Σ with boundary.
- If S contains derivatives of the fields, there will be boundary terms that spoil the structure.
- This relaxed structure will however induce a BF^{n}V structure on the fields on $\partial\Sigma$ (the kernel of $\breve{\omega}$ contains in particular fields in the bulk).

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An application

- Suppose (M, ω, Q, S) is a local BF^{n}V structure on a space of fields on Σ.
- Then on the space of fields $\underline{M}$ on $\partial\Sigma$ we get a BF^{n+1}V structure.
- If $\partial\Sigma = \emptyset$, we expect to quantize M to some graded vector space $\mathcal{H}$ (with S some operator).
- Example $n = 0$. On Σ we have a (relaxed) BFV structure describing the reduced phase space of some field theory. On $\partial\Sigma$ we have a BF^{2}V structure describing a Poisson structure (possibly up to homotopy).
- Example $n = -1$. On Σ we have a (relaxed) BV structure describing the symmetry content of some field theory. On $\partial\Sigma$ we have a BFV structure describing its reduced phase space (possibly up to homotopy).

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Quantization of the example $n = 0$

- If $\partial\Sigma = \emptyset$, we expect to get a vector space $\mathcal{H}$ by geometric quantization of M together with a coboundary operator Ω quantizing S. Its cohomology in degree zero describes a quantization of the reduced phase space.
- If $\partial\Sigma \neq \emptyset$, we expect $\mathcal{H}$ to be a representation of a quantization of $\underline{M}$.
- For example, we may consider the deformation quantization ot the Poisson structure described by $\underline{M}$.
- If $\Sigma = \Sigma_1 \cup_D \Sigma_2$ with D a common boundary component for Σ_1 and Σ_2, we expect $\mathcal{H}$ for Σ to be recovered as the tensor product of the $\mathcal{H}$s for Σ_1 and Σ_2 over the algebra associated to D.

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- Let Σ be a 2-manifold and $\mathfrak{g}$ a quadratic Lie algebra.
- Let N be the space of $\mathfrak{g}$-valued 1-forms A (connections) on Σ with the Atiyah–Bott symplectic structure $\omega = \frac{1}{2}\int_\Sigma \delta A \delta A$.
- We let C denote the space of flat connections. Then $\underline{C}$ turns out to be the quotient by gauge transformations.
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Quantization of the example $n = -1$

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- If $\Sigma = \Sigma_1 \cup_D \Sigma_2$ with D a common boundary component for Σ_1 and Σ_2, we expect the state for Σ to be recovered as the pairing of the states for Σ_1 and Σ_2 in the Hilbert space associated to D.
- We produced a rather general construction, which relies on the existence of a “nice” polarization of $\underline{M}$. The construction also keeps track of “residual fields" (e.g., zero modes).
- We have successfully applied this construction to abelian BF theories:

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- By perturbing abelian *BF* theory, we have extended the construction to other theories like
  1. Quantum mechanics and topological quantum mechanics
  2. Split Chern–Simons theory
  3. 2D Yang–Mills theory
  4. Poisson sigma model
- In the last example, one can e.g. recover the associativity of Kontsevich's star product from the composition of states.
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Final remarks

- Other theories like scalar field, spinor field, Yang–Mills can be treated alike (but one has to take renormalization into account).
- Classical BV-BFV gravity theories (in the Einstein–Hilbert as well as in the Palatini–Cartan version) have also been studied.
- A discretized version of nonabelian *BF* theory has also been studied: in this setting, all spaces are finite dimensional and all the quantum BV-BFV results are rigorous from the start.

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Happy Birthday Kolya