Asymptotic behaviour for homoenergetic solutions of the Boltzmann equation

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Outline

1. Introduction
2. Homoenergetic solutions of the Boltzmann equation
3. Paradigmatic examples
4. Simple shear case for Maxwell molecules ($\gamma = 0$)
5. Simple shear case for hard potentials ($\gamma > 0$)
6. Final comments

Introduction

The "Boltzmann equation" ... [1872]

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{transport op.}} = \underbrace{Q(f,f)}_{\text{collision op.}} \qquad \text{on} \quad f(t,x,v) \geq 0$$

$$Q(f,f)(v) = \int_{S^2} \int_{\mathbb{R}^3} \underbrace{B(v - v_*, \omega)}_{\text{collision kernel } (\geq 0)} \{\underbrace{f(v')f(v'_*)}_{\text{appearing}} - \underbrace{f(v)f(v_*)}_{\text{disappearing}}\} d\omega \, dv_*$$

Postulates:

- particles interact via binary collisions (dilute regime)
- collisions are localized in space and time (the duration of a collision is very small)
- collisions are elastic (momentum and kinetic energy are preserved)
- collisions are microreversible (reversibility at microscopic level)
- Boltzmann chaos (velocities of two particles about to collide are uncorrelated)

Structure of the Boltzmann equation

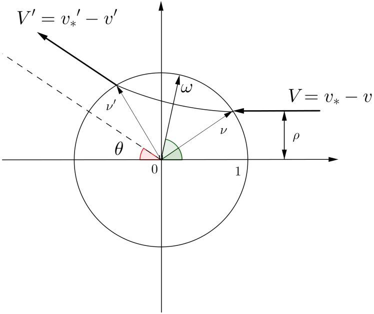
$$Q(f,f)(v) = \int_{S^2}\int_{\mathbb{R}^3} \underbrace{B(v-v_*,\omega)}_{\text{collision kernel } (\geq 0)} \{\underbrace{f(v')f(v_*')}_{\text{appearing}} - \underbrace{f(v)f(v_*)}_{\text{disappearing}}\} d\omega\, dv_*$$

$$v' = v - [(v-v_*)\cdot\omega]\,\omega\,, \quad v_*' = v_* + [(v-v_*)\cdot\omega]\omega \quad \text{(velocity collision rule)}$$

- $B(v-v_*,\omega) = B(v-v_*,\cos\theta)$

$$\cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \omega \right\rangle$$

- $B(|v-v_*|,\cos\theta) = \dfrac{\rho}{\sin\theta}\left|\dfrac{d\rho}{d\theta}\right|$
- B depends on the potential ϕ



Conservation laws for the Boltzmann equation

- Conservation of mass, momentum and kinetic energy

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} f\,dx\,dv = 0, \qquad \frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} v_i f\,dx\,dv = 0, \quad \frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f\,dx\,dv = 0$$

Macroscopic balance equations

- Define the density of mass, momentum and energy

$$\rho(x,t) = \int_{\mathbb{R}^3} f\, dv$$

$$p_i(x,t) = \int_{\mathbb{R}^3} v_i f\, dv \qquad i = 1,2,3$$

$$w(x,t) = \int_{\mathbb{R}^3} |v|^2 f\, dv$$

- Define the macroscopic average velocity, the momentum flow and the energy flow

$$\rho(x,t) V_i(x,t) = \int_{\mathbb{R}^3} v_i f\, dv$$

$$P_{ij} = \int_{\mathbb{R}^3} v_i v_j f\, dv \qquad i,j = 1,2,3$$

$$r_i(x,t) = \int_{\mathbb{R}^3} v_i |v|^2 f\, dv \qquad i = 1,2,3$$

Macroscopic balance equation

- $c := v - V$ deviation of the velocity of a single particle from the average velocity

$$\Rightarrow \quad P_{ij} = \rho V_i V_j + M_{ij} \quad \text{with} \quad M_{ij} = \int_{\mathbb{R}^3} c_i c_j f \, dv$$

$$\Rightarrow \quad w = \underbrace{\tfrac{1}{2}\rho|v|^2}_{\text{kinetic en.}} + \underbrace{\rho e}_{\text{internal en.}} \quad \text{with} \quad \rho e = \frac{1}{2}\int_{\mathbb{R}^3} |c|^2 f \, dv$$

- System *(not closed)* of 5 scalar conservation laws [mass, momentum and kinetic energy]

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho V_i) = 0; \qquad \frac{\partial}{\partial t}(\rho V_i) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho V_i V_j + M_{ij}) = 0 \quad i = 1,2,3$$

$$\frac{\partial}{\partial t}(\rho \frac{|v|^2}{2} + \rho e) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[\rho V_i \left(\frac{|v|^2}{2} + e\right) + q_i + \sum_{i=1}^{3} V_j M_{ij}\right] = 0$$

M_{ij} and q_i depend on higher order moments of f !

Boltzmann's H theorem

- Boltzmann's H theorem

$$\frac{d}{dt}\mathcal{H}(t,x) = \frac{d}{dt}\int_{\mathbb{R}^3} f \log f \, dv = -D(f) \le 0$$

- The entropy production is

$$\begin{aligned} D(f) &= -\int_{\mathbb{R}^3} Q(f,f) \log f \, dv \\ &= \int_{S^2\times\mathbb{R}^3\times\mathbb{R}^3} B(v-v_*,\omega)[f'f'_* - ff_*] \log\left(\frac{f'f'_*}{ff_*}\right) d\omega \, dv_* \, dv \ge 0 \end{aligned}$$

- Cancellation at $ff_* = f'f'_*$: local Maxwellian equilibrium

$$M_f(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-|v-V|^2/2T} \qquad \rho \ge 0,\ V \in \mathbb{R}^3,\ T > 0$$

- Any nonequilibrium state of an isolated gas evolves irreversibly towards the equilibrium state (Second law of thermodynamics)

Homogeneous Boltzmann

Particles described by their space homogeneous distribution density $f = f(t, v)$

$$\partial_t f(t,v) = \int_{S^2}\int_{\mathbb{R}^3} B(v-v_*,\omega)\{f(v')f(v'_*)-f(v)f(v_*)\}d\omega\, dv_*$$

- Conservation of mass, momentum and kinetic energy

$$\forall\, t \geq 0, \quad \int_{\mathbb{R}^3} f(t,v)\,\varphi(v)\,dv = \int_{\mathbb{R}^3} f_0(v)\,\varphi(v)\,dv \quad \text{for} \quad \varphi(v) = 1, v, |v|^2$$

- Entropy functional $\quad H(f) = \displaystyle\int_{\mathbb{R}^3} f\,\log(f)\,dv$

Properties of homogeneous Boltzmann

- Boltzmann's H theorem

$$\frac{d}{dt}H(f) = -D(f)$$
$$= \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} B(v-v_*,\omega)\,(f'f'_* - ff_*)\log\frac{f'f'_*}{ff_*}\,d\omega\,dv_*\,dv \le 0$$

- Any equilibrium is a Maxwellian distribution

$$M_f(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-|v-V|^2/2T} \qquad \rho > 0,\, u \in \mathbb{R}^3 \text{ and } T > 0$$

ρ: macroscopic density, V : macroscopic bulk velocity

$$T := \frac{1}{3\rho}\int_{\mathbb{R}^3} f(v)|v-V|^2\,dv \qquad \text{macroscopic temperature}$$

- A solution f is expected to relax towards equilibrium:

  f converges to the Maxwellian distribution when $t \to +\infty$

Homoenergetic solutions of the Boltzmann equation

Homoenergetic solutions

$$f(x,v,t) = g(\underbrace{v-\xi(x,t)}_{w},t) \qquad x\in\mathbb{R}^3,\ v\in\mathbb{R}^3,\ t>0$$

- $g(\cdot,t)$ characterizes the dispersion of velocities for the molecules at a given point (Truesdell, Galkin,1960's)
- f solves the Boltzmann equation $\Rightarrow$

$$\frac{\partial \xi_k}{\partial x_j} \text{ constant in } x \quad \& \quad \partial_t\xi + \xi\cdot\nabla\xi = 0 \qquad \Rightarrow \qquad \xi(x,t) = M(t)\,x$$

$$\frac{dM}{dt}(t) + (M(t))^2 = 0,\ M(0) = A \quad \Leftrightarrow \quad M(t) = (I+tA)^{-1}A = A(I+tA)^{-1}$$

- The Boltzmann equation becomes

$$\partial_t g(w,t) - M(t)\,w\cdot\partial_w g(w,t) = Q(g,g)(w)$$

The matrix $M(t)$

Equidispersive solutions with the form: $f(x,v,t) = g(\underbrace{v-\xi(x,t)}_{w}, t)$

$$\xi(x,t) = M(t)x \qquad M(t) = \frac{A}{I+tA}$$

Asymptotics of $M(t) = (I+tA)^{-1}A$? Look at all the Jordan canonical forms of A

Physical interpretation of the possible cases:

- simple shear
- 3d dilatation (isotropic)
- 1d dilatation
- 2d dilatation
- mixed 1d dilatation and shear
- mixed 2d dilatation and shear
- mixed 3d dilatation and shear
- combined shear in orthogonal directions
- blow-up cases

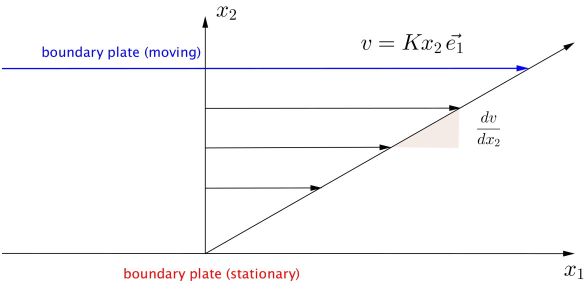
Paradigmatic cases

- Simple shear: $A = a \otimes n \quad a \cdot n = 0 \quad \mathrm{tr}(A) = 0$

$$\partial_t g - Aw \cdot \partial_w g = Q(g,g)(w) \qquad M(t) = A$$

Assume $a = Ke_1, \quad n = e_2$

$$\Rightarrow \qquad \partial_t g - Kw_2 \, \partial_{w_1} g = Q(g,g)(w)$$



Paradigmatic cases

- Mixed 1d dilatation and shear:
  $A = a \otimes n \qquad a \cdot n \neq 0$

$$\Rightarrow \qquad \partial_t g + \frac{a \otimes n}{1 + (a \cdot n)\, t} w \cdot \partial_w g = Q(g, g)(w)$$

Assume $\quad n = e_1, \quad a = k_1 e_1 + k_2 e_2$

$$\Rightarrow \qquad \partial_t g + \frac{k_1}{1 + k_1 t} w_1 \cdot \partial_{w_1} g + \frac{k_2}{1 + k_1 t} w_2 \cdot \partial_{w_1} g = Q(g, g)(w)$$

- Combined shear in orthogonal directions: $\qquad A = N \quad N^2 \neq 0 \quad N^3 = 0$

$$N = \begin{pmatrix} 0 & K_3 & K_2 \\ 0 & 0 & K_1 \\ 0 & 0 & 0 \end{pmatrix} \; K_1, K_3 \neq 0 \qquad M(t) = N - tN^2 \qquad \text{[orthogonal system]}$$

$$\Rightarrow \qquad \partial_t g - [K_3 w_2 + (K_2 - tK_1 K_3)\, w_3]\, \partial_{w_1} g - K_1 w_3 \partial_{w_2} g = Q(g, g)(w)$$

The homogeneity of the kernel $B(|v-v_*|,\omega)$

Collision operator : $Q(f,f)(v) = \int_{S^2}\int_{\mathbb{R}^3} B(v-v_*,\omega)\{f(v')f(v'_*) - f(v)f(v_*)\}d\omega\, dv_*$

- for inverse power law potential $\phi(r) = \dfrac{1}{r^{s-1}}, \quad s > 2$:

$$B(|v-v_*|,\omega) = \underbrace{b(\cos\theta)}_{\text{kinetic cross sec.}} \quad \underbrace{|v-v_*|^{\gamma}}_{\text{angular cross sec.}}, \qquad \gamma = \frac{s-5}{s-1}$$

- $\gamma > 0$ Hard potentials $(s > 5)$
- $\gamma = 0$ Maxwellian molecules $(s = 5)$
- $\gamma < 0$ Soft potentials $(s < 5)$

- for a gas of hard spheres : $B(|v-v_*|,\omega) = |(v-v_*)\cdot\omega|, \quad K > 0 \qquad (\gamma = 1)$

General features

$$\partial_t g - M(t)\, w \cdot \partial_w g = Q(g,g)(w)$$

- the density $\rho(t) = \int g\, d^3 w =$ const or $\rho(t)$ decreases like a power law
- rescaling for the quadratic collision operator $\rho(t)[w]^{\gamma}[g]$
- rescaling for the hyperbolic term [if $M(t) \sim \eta(t)$] $\eta(t)[g]$
- different behaviours of the 'hyperbolic term" $M(t)\, w \cdot \partial_w g$ as $t \to \infty$
  - hyperbolic term >> collision term
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  - same order of magnitude for hyperbolic and collision term

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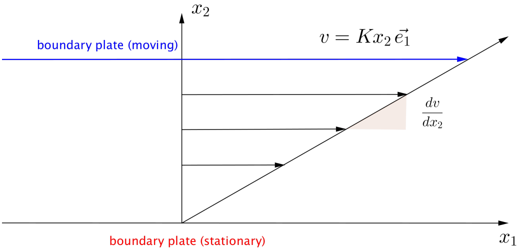
Paradigmatic examples

Simple Shear case

$$\partial_t g - Aw \cdot \partial_w g = Q(g,g)(w)$$

$$M(t) = A = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K \in \mathbb{R}$$



- critical homogeneity for the kernel B: $\gamma = 0$ (Maxwell molecules)
- supercritical case: homogeneity for the kernel B $\gamma > 0$ (hard potentials)
- subcritical case homogeneity for the kernel B: $\gamma < 0$ (soft potentials)

[Truesdell, Muncaster '80; Cercignani '89; Cercignani '01; Bobylev, Caraffini, Spiga '96]

Asymptotics: simple shear case

Goal: long time asymptotics of homoenergetic solutions ?

Critical case ($\gamma = 0$)	Subcritical case ($\gamma < 0$)	Supercritical case ($\gamma > 0$)
Self-similar solutions $\|w\|^2 \sim e^{bt}$, $b = b(K)$ (increasing temperature)	i) $-1 < \gamma < 0$? ii) $\gamma < -1$ frozen collisions	Hilbert expansion $\|w\|^2 \sim t^{\frac{1}{\gamma}}$

In the critical case $\gamma = 0$ when $t \to \infty$:

- the density ρ is constant
- the internal energy $e \sim e^{b\,t}$

Asymptotics: case of mixed 1d dilatation and shear

1d dilatation (K_1) and shear (K_2)

Critical homogeneity: $\gamma = 0$ (Maxwell molecules)

Critical case ($\gamma = 0$)	Subcritical case ($\gamma < 0$)	Supercritical case ($\gamma > 0$)
Self-similar solutions $\lvert w \rvert^2 \sim t^{\alpha(K_1,K_2)}$ $\alpha < 0$ or $\alpha > 0$	i) $-1 < \gamma < 0$? ii) $\gamma < -1$ frozen collisions	Hilbert expansion $\lvert w \rvert^2 \sim t^{\frac{1}{\gamma}}$

In the critical case $\gamma = 0$ when $t \to \infty$:

- the density $\rho \sim \frac{1}{t}$
- the internal energy $e \sim \frac{1}{t^\sigma}$

  [the average velocity increases ($\sigma < 1$) if $K_2^2 > K_1$ and decreases if $K_2^2 < K_1$]

Asymptotics: case of combined shear in orthogonal directions

Combined shear (K_1, K_2, K_3) $\quad K_1 K_3 \neq 0$

Critical homogeneity: $\gamma = 0$ $\quad$ (Maxwell molecules)

Critical case ($\gamma = 0$)	Subcritical case ($\gamma < 0$)	Supercritical case ($\gamma > 0$)
Non Maxwellian distr. $\|w\|^2 \sim t^{\sigma} \exp\left(a t^{\frac{5}{3}}\right)$ $a > 0$	i) $-1 < \gamma < 0$? ii) $\gamma < -1$ frozen collisions	Hilbert expansion $\|w\|^2 \sim t^{\frac{1}{\gamma}}$

In the critical case $\gamma = 0$ when $t \to \infty$:

- the density ρ is constant
- the internal energy $e \sim t^{\sigma} e^{a t^{\frac{5}{3}}}$ $\qquad a = \frac{3}{5} \left(K_1 K_3\right)^{\frac{2}{3}}$

Simple Shear Case

Simple Shear for Maxwell molecules ($\gamma = 0$)

Assumption : $B(\omega, |v - v_*|) = B(\cos(\theta))$ s.t. $\int_{S^2} d\omega\, B(\omega, |v - v_*|) < \infty$

(Grad's angular cut-off)

$$\partial_t \int_{\mathbb{R}^3} \varphi(w)\, g(dw,t) = -\int_{\mathbb{R}^3} \partial_w \cdot (Aw\varphi)\, g(dw,t) + \\ + \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} d\omega g(dw,t)\, g(dw_*,t)\, B(\omega) \left[\varphi(w') + \varphi(w'_*) - \varphi(w) - \varphi(w_*)\right]$$

Evolution eq. for $M_{j,k}(t) = \int_{\mathbb{R}^3} w_j w_k g(w,t)\, dw$? Choose $\varphi = w_j w_k$

$$\partial_t M_{j,k} = -K\left[\delta_{1,j} M_{2,k} + \delta_{1,k} M_{2,j}\right] - 6b\left(M_{j,k} - m\delta_{j,k}\right)$$

$$M_{j,k}(0) = N_{j,k} \qquad m = \frac{1}{3} tr\left(M_{j,k}\right)$$

$\Leftrightarrow$

$$M_{j,k} = N_{j,k} e^{6b\sigma t}$$

(exponential growth)

$\Rightarrow$ the set of moment equations is closed for Maxwell molecules

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Self-similar solutions

Look for self-similar solutions with the form $g(w,t) = e^{-3\beta t} G\left(\frac{w}{e^{\beta t}}\right)$

- The rate of growth of the tensor $M_{j,k}$ determines β

$$N_{j,k} e^{6b\sigma t} = M_{j,k} = e^{-3\beta t} \int G\left(\frac{w}{e^{\beta t}}\right) w_j w_k d^3 w = e^{2\beta t} \int G(\xi)\, \xi_j \xi_k d^3 \xi$$

$$\beta = 3b\sigma$$

- Evolution eq. for $M_{j,k}$ for $\mathbf{K} = \frac{k}{6b}$ and $\mathbf{t} = 6bt$

$$\partial_{\mathbf{t}} M_{j,k} = -\mathbf{K}\left[\delta_{1,j} M_{2,k} + \delta_{1,k} M_{2,j}\right] + m\delta_{j,k} - (1+\sigma) M_{j,k} \qquad (\diamond)$$

$$M_{j,k} = e^{\alpha \mathbf{t}} N_{j,k} \quad \Rightarrow \quad \alpha N_{j,k} = -\mathbf{K}\left[\delta_{1,j} N_{2,k} + \delta_{1,k} N_{2,j}\right] + n\delta_{j,k} - (1+\sigma) N_{j,k}$$

[*eigenvalue pb.*]

$(\bar{N}_{j,k})$: eigenvector associated to the zero eigenvalue

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Existence of self-similar profiles

$\mathcal{M}_+ \left(\mathbb{R}^3_c\right)$: Radon measures in $\mathbb{R}^3_c$ with the weak$-*$ topology of measures.

Theorem

For any $\zeta \geq 0$ and any $\mathbf{K} \leq \kappa_0$ there exists at least one weak solution $G \in \mathcal{M}_+ \left(\mathbb{R}^3_c\right)$ of

$$-\beta \partial_w \cdot (wG) - \partial_{w_1} (K w_2 G) = Q(G, G)(w) \qquad (\star)$$

such that

$$\int_{\mathbb{R}^3} G(dw) = 1 \ , \quad \int_{\mathbb{R}^3} w_j G(dw) = 0, \int_{\mathbb{R}^3} w_j w_k G(dw) = \zeta \bar{N}_{j,k} \quad \text{for} \quad j, k \in \{1, 2, 3\} \, .$$

Strategy:

- change of variables to transform the self-similar solution into the steady state of a new system
- well-posedness for the corresponding evolution problem (evolution semigroup $T(t)$)
- $T(t)$ has an invariant convex set of functions and a fixed point argument

  $\Rightarrow$ existence of solution for $(\star)$

Tools for the evolution problem

Mild solutions

$$G(w,t) = [S[G](t)]\, G_0(w) + \int_0^t S[G](t-s) Q^{(+)}(G,G)(w,s)\, ds$$

where $[S[G](t)]\, h_0(w)$ solves, for any $h_0 \in \mathcal{M}_+(\mathbb{R}^3_c)$ and any $G \in C([0,\infty] : \mathcal{M}_+(\mathbb{R}^3_c))$,

$$\partial_t h - \beta \partial_w \cdot (wh) - \partial_{w_1}(K w_2 h) = -\mathbb{A}[G] h(w)$$
$$h(\cdot, 0) = h_0$$

with
$$\mathbb{A}[f] f(v) = f \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega B(\omega) f_* = Q^{(-)}(f,f)(v)$$

The (nonlinear) evolution semigroup

$T : [0,\infty) \times \mathcal{M}_+(\mathbb{R}^3_c) \to \mathcal{M}_+(\mathbb{R}^3_c)$ is defined by $T(t)\, G_0 = G(\cdot, t)$, $G(\cdot, t)$ mild solution

Tools for the evolution problem

Weak solutions

$G \in C\left([0,\infty];\mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ with $\int G(dw,t) < \infty$ is a weak solution if for any $\varphi \in C\left(\mathbb{R}^3_c\right)$ and for any $T > 0$

$$\int_{\mathbb{R}^3_c} \varphi(w)\, G(dw,T) - \int_{\mathbb{R}^3_c} \varphi(w)\, G_0(dw)$$

$$= -\int_0^T dt \int_{\mathbb{R}^3_c} \left[\beta w \cdot \partial_w \varphi + \partial_{w_1} \cdot (w_2 \varphi)\right] G(dw,t)$$

$$+ \frac{1}{2}\int_0^T dt \int_{\mathbb{R}^3_c}\int_{\mathbb{R}^3_c}\int_{S^2} d\omega\, G(dw,t)\, G(dw_*,t)\, B(\omega)\left[\varphi(w') + \varphi(w'_*) - \varphi(w) - \varphi(w_*)\right]$$

- the weak formulation implies conservation of mass choosing $\varphi = 1$

Well-posedness for the evolution problem

Evolution pb. for the time-dependent self-similar profile $g(w,t) = e^{-3\beta t} G\left(\frac{w}{e^{\beta t}}, t\right)$?

$$\partial_t G - \beta \partial_\xi \cdot (\xi G) - \partial_{\xi_1}(K\xi_2 G) = Q(G,G)(\xi,t) \qquad t > 0$$
$$G(\cdot,0) = G_0$$

Proposition

Let $g_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ such that $\int_{\mathbb{R}^3} G_0(dw) = 1, \int_{\mathbb{R}^3} |w|^s G_0(dw) < \infty$ for some $s > 2$.

Then, there exists a unique mild solution $G \in C\left([0,\infty] : \mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ of

$$\partial_t G - \beta \partial_w \cdot (wG) - \partial_{w_1}(K w_2 G) = Q(G,G)(w,t) \qquad t > 0$$
$$G(\cdot,0) = G_0(\cdot)$$

[Cercignani '89]

- Invariant set & Schauder fixed point th. $\Rightarrow$ existence of self-similar sol.

[Gamba,Panferov,Villani '04, Escobedo,Mischler,Rodriguez-Ricard'05; Niethammer,V.'13; Niethammer,V,Throm'16; Kierkels,V'15]

Moments bounds

Proposition

Let be $s = 4$ and $\int_{\mathbb{R}^3} |w|^s \, G_0(dw) < \infty$ and

$$\int_{\mathbb{R}^3} w_j G_0(dw) = 0, \qquad \int_{\mathbb{R}^3} w_j w_k G_0(dw) = \zeta \bar{N}_{j,k}, \quad j,k \in \{1,2,3\}$$

where $(\bar{N}_{j,k})$ is the eigenvector associated to the zero eigenvalue and $\zeta > 0$

$$\Rightarrow \quad \int_{\mathbb{R}^3} w_j T(t) G_0(dw) = 0 \ , \ \int_{\mathbb{R}^3} w_j w_k T(t)(G_0)(dw) = \zeta \bar{N}_{j,k} \ \ \forall\, t \geq 0.$$

Moreover $\exists \, \kappa_0 > 0$ sufficiently small s.t. if $\mathbf{K} \leq \kappa_0 \ \exists \ C_* = C_*(\zeta) > 0$ s.t. if

$$\int_{\mathbb{R}^3} |w|^s \, G_0(dw) \leq C_* \qquad \Rightarrow \qquad \int_{\mathbb{R}^3} |w|^s \, T(t)(G_0)(dw) \leq C_* \quad \forall\, t \geq 0.$$

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Tools for controlling the moments

1. Choosing $\varphi(w) = w_j$, $\varphi(w) = w_k w_j$ in the weak formulation
   $\Rightarrow$ conservation of the pressure tensor $M_{j,k}$
   [$M_{j,k}(t) = M_{j,k}(0) = \bar{N}_{j,k}$ for any $t \geq 0$]

2. To prove the invariance of the set $\left\{ \int_{\mathbb{R}^3} |w|^s G(dw) \leq C_* \right\}$
   choose $\varphi(w) = |w|^s$ (for $s = 4$) in the weak formulation &

Pozvner estimates (for $s = 4$)

$$|w'|^4 + |w'_*|^4 - |w|^4 - |w_*|^4 \leq -\kappa(\theta)|w|^4 + C\left[|w|^3|w_*| + |w_*|^3|w|\right]$$

with $\kappa(\theta) > 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

Geometrical interpretation: Boltz. coll. dyn. for particles with large velocities

$\Downarrow$

$M_4(t) \leq C_*$ [the fourth moment is globally bounded!]

Simple shear for hard potentials ($\gamma > 0$)

$$\partial_t g - K\, w_2 \partial_{w_1} g = Q(g,g)\,(w) \qquad B(|v-v_*|,\omega) = b(\cos\theta)|v-v_*|^{\gamma} \quad \gamma > 0$$

Long time asymptotics? $g\,(w,t) \simeq C_0\,(\beta\,(t))^{\frac{3}{2}} \exp\left(-\beta\,(t)\,|w|^2\right),\ \beta\,(t) \to 0$

[Maxwellian distribution with increasing temperature]

- Hilbert expansion:

$$g(w,t) \sim C_0\big(\beta(t)\big)^{\frac{3}{2}} \exp\left(-\beta(t)\,|w|^2\right)\left[1 + h_1\,(w,t) + h_2\,(w,t)\,...\right]$$

$$\int |w|^2\, g\,(dw,t) = C_0\,(\beta\,(t))^{\frac{3}{2}} \int \exp\left(-\beta\,(t)\,|w|^2\right)|w|^2\, dw \qquad \text{[definition of } \beta(t)\text{]}$$

$$\int \exp\left(-\beta\,(t)\,|w|^2\right) h_k\,(w,t)\, dw = 0 \quad , \quad k = 1,2,3,.. \quad \text{[normalization]}$$

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- $\beta_t = -a\,\beta^{\frac{\gamma}{2}+1}$ (Green-Kubo formula) $\Rightarrow$ $\beta(t) \sim \frac{C}{t^{\frac{2}{\gamma}}}$ as $t \to \infty$

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Some comments on the entropy

For self-similar solutions $g(w,t) = \frac{1}{a(t)} G\left(\frac{w}{\lambda(t)}\right)$:

- $\rho = \frac{\lambda^3}{a} \int G(\xi)\, d^3\xi$
- $e = \frac{\lambda^2}{2} \frac{\int |\xi|^2 G d^3\xi}{\int G(\xi)\, d^3\xi}$
- $H = -\log\left(\frac{e^{\frac{3}{2}}}{\rho}\right) + H_G$ with

$$H_G = \frac{\int G \log(G)\, d^3\xi}{\int G(\xi)\, d^3\xi} + \log\left[\left(\frac{1}{2}\right)^{\frac{3}{2}} \frac{\left(\int |\xi|^2 G d^3\xi\right)^{\frac{3}{2}}}{\left(\int G(\xi)\, d^3\xi\right)^{\frac{5}{2}}}\right]$$

H_G is different from the constant for the Maxwellian H_M !

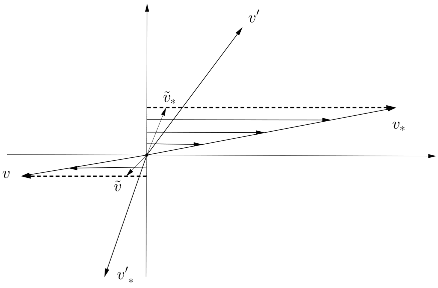
Some comments on the entropy

In nonselfsimilar solutions the formula of the entropy might differ much from the one for equilibrium distributions.

- Example: 2d dilatation $\rho \sim \frac{\rho_0}{t^2}$, $e \sim \frac{\rho_0}{t^2}$ as $\to \infty$
- However $H \to const$ as $t \to \infty$
- Then H is very different from $-\log\left(\frac{e^{\frac{3}{2}}}{\rho}\right) + H_0$
- The reason for this is that the geometry of the particle distribution is extremely nonisotropic.

Problems dominated by hyperbolic terms with non-negligible collision effects

Shear combined with collision yields a huge increase of average velocity.



Problems dominated by hyperbolic terms with non-negligible collision effects

- I some cases (for instance 2d dilatation with $\gamma < -2$), there are regimes in which the collisions are frozen.
- Simplified shear model

$$
\begin{aligned}
f &= f(t,\rho,\zeta) \\
\partial_t f + \partial_\zeta f &= -\varepsilon(t) f \quad , \quad \zeta > 1, \quad \rho > 0, \quad t > 0 \\
f(t,\rho,1) &= \varepsilon(t) \int_1^\infty f\left(t, \frac{\rho}{\zeta}, 1\right) \frac{d\zeta}{\zeta} \\
f(0,\rho,1) &= f_0(\rho)\,\delta(\zeta - 1) \\
\varepsilon(t) &= \int_0^\infty d\rho \int_1^\infty d\zeta \frac{f(t,,\zeta,\rho)}{\rho^\alpha \zeta^\alpha} \quad , \quad 0 < \alpha < 1
\end{aligned}
$$

- Asymptotics $\varepsilon \sim \dfrac{1-\alpha}{t^2}$ as $\rightarrow \infty$

Concluding Remarks

- Homoenergetic flows provide an interesting tool to study the properties of Boltzmann gases in the presence of strong shears, dilatations or compressions
- The long time asymptotics of the solutions depends on the balance between the "deformation terms' (hyperbolic)' and the homogeneity of the collision terms.
- Some of the solutions describing the long time asymptotics are non-Maxwellian self-similar solutions. In other cases can be aproximated by Maxwellians with changing temperature and/or density.
- There are many interesting open questions in cases in which the hyperbolic terms are asymptotically much larger than the collisions, but these ones play a crucial role in the dynamics.

Thank you for your attention !!!

Conjectures

Simple shear (K)

Critical homogeneity: $\gamma = 0$ (Maxwellian molecules)

Asymptotics ($\gamma = 0$)	Asymptotics ($\gamma < 0$)	Asymptotics ($\gamma > 0$)	ρ	e
Self-similar sol. $\lvert w\rvert^2 \sim e^{bt}$, $b = b(K)$	$i)$ $-1 < \gamma < 0$???? $ii)$ $\gamma < -1$ frozen collisions	Hilbert expansion $\lvert w\rvert^2 \sim t^{\frac{1}{\gamma}}$	const.	$e^{b\,t}$

Pure 3d dilatation (isotropic and anisotropic)

Critical homogeneity: $\gamma = -2$ (very soft potential)

Asymptotics ($\gamma = -2$)	Asymptotics ($\gamma < -2$)	Asymptotics ($\gamma > -2$)	ρ	e
Maxwellian distr. ($t \to \infty$)	Maxwellian distr. ($t \to \infty$)	Frozen collisions	$\frac{1}{t^3}$	$\frac{1}{t^2}$

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1d dilatation (a)

Critical homogeneity: $\gamma = 0$ (Maxwellian molecules)

Asymptotics ($\gamma = 0$)	Asymptotics ($\gamma < 0$)	Asymptotics ($\gamma > 0$)	ρ	e
self-similar sol. $\lvert w\rvert^2 \sim t^{\alpha(a)}$ $\alpha > 0$ (large a) $\alpha < 0$ (small a)	$i)$ $-1 < \gamma < 0$???? $ii)$ $\gamma < -1$ frozen collisions	Hilbert expansion	$\frac{1}{t}$	$\frac{1}{t^\sigma}$

2d dilatation or cylindrical dilatation (a_1, a_2)

Critical homogeneity: $\gamma = \gamma_{crit}$ γ_{crit} eigenvalue ($\gamma_{crit} = \infty$?)

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Mixed 1d dilatation (K_1) and shear (K_2)

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self-similar sol. $\|w\|^2 \sim t^{\alpha(K_1,K_2)}$ $\alpha < 0$ or $\alpha > 0$	$i)$ $-1 < \gamma < 0$???? $ii)$ $\gamma < -1$ frozen collisions	Hilbert expansion	$\frac{1}{t}$	$\frac{1}{t^\sigma}$

Combined shear (K_1, K_2, K_3) $K_1 K_3 \neq 0$

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Asymptotics ($\gamma = 0$)	Asymptotics ($\gamma < 0$)	Asymptotics ($\gamma > 0$)	ρ	e
Non Maxwellian $\|w\|^2 \sim t^\sigma \exp(at^{\frac{5}{3}})$ $a > 0$	$i)$ $-1 < \gamma < 0$???? $ii)$ $\gamma < -1$ frozen collisions	Hilbert expansion	const.	$t^\sigma e^{at^{\frac{5}{3}}}$

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The matrix $M(t)$

$A = D + N \quad D \geq 0$ diagonal, N nilpotent (Jordan canonical form)

$$M(t) = \frac{D}{(1+Dt)} + \frac{N}{(1+Dt)^2} - \frac{tN^2}{(1+Dt)^3}$$

Long time asymptotics of $M(t)$?

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- $D \neq 0$ & the zero eigenvalue has degeneracy two ($\Rightarrow N$ disappears)

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- $D \neq 0$ & $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_3 \neq 0$

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  - $N^2 = 0, \ N \neq 0$

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$$\rightsquigarrow \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = M(t)\, x = \begin{pmatrix} K_3 x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ K_1 x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} (K_2 - tK_1 K_3)\, x_3 \\ 0 \\ 0 \end{pmatrix}$$

(velocity as a combination of three simple shears!)

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