On the Boltzmann equation without cutoff

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Kinetic theory

- System described by the evolution of the density of particles $f = f(t, x, v) \geq 0$, $t \in \mathbb{R}^+$ the time, $x \in \Omega$ the position and $v \in \mathbb{R}^3$ the velocity.

$f(t, x, v)\, dx\, dv$ = quantity of particules in the volume element $dx\, dv$ centered in $(x, v) \in \Omega \times \mathbb{R}^3$.

Kinetic theory

– No external force or interaction: free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0.$$

– If interaction between particles or with a background medium, equation of kind

$$\partial_t f + v \cdot \nabla_x f = \underbrace{C(f)}_{\text{collision term}}.$$

– Maxwell (1867), Boltzmann (1872): Boltzmann collision operator for neutral particles (gaz).

The Boltzmann equation

Boltzmann equation in the torus

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3$$

$$Q(g, f)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} \underbrace{B(v - v_*, \sigma)}_{\text{collision kernel}} \Big(\underbrace{f(v') g(v'_*)}_{\text{“appearing”}} - \underbrace{f(v) g(v_*)}_{\text{“disappearing”}} \Big) \, dv_* \, d\sigma$$

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v, v_* and v', v'_* are the velocities of a pair of particles before and after collision.

- Conservation of momentum and energy:

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

- Parametrization of (v', v'_*):

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$

The collision kernel

- Physical motivation: particles interacting according to a repulsive potential of the form $\phi(r) = r^{-(p-1)}$, $p \in (2, +\infty)$. We only deal with the case $p > 5 \leftrightarrow$ Hard potentials.
- The collision kernel $B(v - v_*, \sigma)$ satisfies

$$B(v - v_*, \sigma) = C|v - v_*|^\gamma \, b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

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- b is not integrable on $\mathbb{S}^2$:

$$\sin\theta \, b(\cos\theta) \approx \theta^{-1-2s}, \quad s = \frac{1}{p-1}, \quad \forall\, \theta \in (0, \pi/2].$$

  Hard potentials $\leftrightarrow s \in (0, 1/4)$.
- The kinetic factor $|v - v_*|^\gamma$ satisfies $\gamma = \frac{p-5}{p-1}$.
  Hard potentials $\leftrightarrow \gamma > 0$.

Weak form and consequences

For $\phi = \phi(v)$ a test function,

$$\int_{\mathbb{R}^3} Q(f,f)\,\phi\,dv = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,f\,f_*\left(\phi' + \phi'_* - \phi - \phi_*\right)d\sigma\,dv_*\,dv.$$

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– Conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^3} Q(f,f)(v)\begin{pmatrix}1\\ v_i\\ |v|^2\end{pmatrix} dv = 0$$

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– Entropy inequality (H-theorem):

$$D(f) := -\int_{\mathbb{R}^3} Q(f,f)(v)\,\log f(v)\,dv \geq 0$$

and

$$D(f) = 0 \Longleftrightarrow f = \mu = \text{Maxwellian (Gaussian in } v\text{)}$$

$$Q(\mu,\mu) = 0$$

A priori estimates

We fix $\mu = (2\pi)^{-3/2} e^{-|v|^2/2}$.
In what follows, we shall consider initial data f_0 with same mass, momentum, energy as μ.

A priori estimates: if f_t is solution of the Boltzmann equation associated with f_0 s.t.

$$\int \left(1 + |v|^2 + |\log f_0|\right) f_0 \, dx \, dv < \infty$$

then $f \in L^\infty_t \left(L^1_2 \cap L \log L\right)$:

$$\sup_{t \geq 0} \int \left(1 + |v|^2 + |\log f_t|\right) f_t \, dx \, dv + \int_0^\infty D(f_s) \, ds < \infty.$$

Problem:
Does $f_t \xrightarrow[t \to \infty]{} \mu$? If yes, what is the rate of convergence?

1) Introduction

2) The homogeneous case

3) The non homogeneous case

Main result in the homogeneous case

$$\partial_t f = Q(f, f), \quad (t, v) \in \mathbb{R}^+ \times \mathbb{R}^3$$

Theorem (T. '14)

Consider $f_0 \geq 0$ of finite entropy. Then if f_t is a "smooth" solution associated to the initial datum f_0, $\exists\, \lambda > 0,\ C > 0$ s.t.

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_{L^1} \leq C\, e^{-\lambda t}.$$

- ↪ Cauchy theory: Arkeryd '81, Goudon '97, Villani '98, Fournier-Mouhot '09, Desvillettes-Mouhot '09...
- ↪ Regularization properties of the equation: Elmroth '83, Desvillettes '93, Wennberg '97, Alexandre-Desvillettes-Villani-Wennberg '00, Chen-He '11...

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- ↪ Improvement of the (better than any) polynomial rate of Villani '03 (see also Carlen-Carvalho '94, Toscani-Villani '99...).
- ↪ Improvement of Mouhot '06 result (cutoff case).

Strategy of the proof

$$\partial_t f = Q(f, f)$$

- Linearization around the equilibrium: $f = \mu + h$

$$\partial_t h = \underbrace{Q(\mu, h) + Q(h, \mu)}_{\Lambda h = \text{linear part}} \quad \Big(+ \quad \underbrace{Q(h, h)}_{\text{Nonlinear part: negligible?}} \quad \Big).$$

- Study of the linearized operator Λ (semigroup estimates):
  ↪ enlargement argument to link the linear and nonlinear theories.

- Proof of bilinear estimates on the collision operator Q.

Enlargement argument

$$\underbrace{\mathsf{E} := L^2(\mu^{-1/2})}_{\text{small Hilbert space}} \quad \subset \quad \underbrace{\mathcal{E} := L^1(\langle v \rangle^k)}_{\text{large Banach space}}$$

$\downarrow$ Baranger-Mouhot ’05

$\downarrow$ Decay of the semigroup?

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small Hilbert space ↓ Baranger-Mouhot '05

large Banach space ↓ Decay of the semigroup?

(1) There exists $\lambda > 0$ such that for any $h_0 \in E$, we have:

$$\forall\, t \geq 0, \qquad \|S_\Lambda(t)(I - \Pi_0)h_0\|_E \leq e^{-\lambda t}\|(I - \Pi_0)h_0\|_E.$$

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small Hilbert space $\downarrow$ Baranger-Mouhot '05

large Banach space $\downarrow$ Decay of the semigroup?

(2) Exhibit a splitting of Λ:

$$\Lambda = \underbrace{\mathcal{A}}_{\text{“}\mathcal{B}\text{-regular”}} + \underbrace{\mathcal{B}}_{a-\text{(hypo)dissipative}}, \qquad a < 0$$

(3) Notion of regularity: $\mathcal{A}S_{\mathcal{B}}(t)$ has some regularizing properties ($\mathcal{E} \rightsquigarrow \mathrm{E}$) with an exponential rate e^{at}.

The conclusion: decay of the semigroup $S_\Lambda(t)$ in $\mathcal{E}$

Theorem (Gualdani-Mischler-Mouhot '13)

For any $h_0 \in \mathcal{E}$ and any $a' > \max(a, -\lambda)$, there exists a constant $C \geq 1$ such that:

$$\forall\, t \geq 0, \quad \left\| S_\Lambda(t)(I - \Pi_0)\, h_0 \right\|_{\mathcal{E}} \leq C\, e^{a' t} \, \| (I - \Pi_0)\, h_0 \|_{\mathcal{E}} \,.$$

Key element of the proof: Duhamel formula

$$S_\Lambda(t) = S_\mathcal{B}(t) + \int_0^t S_\Lambda(t-s) \mathcal{A} S_\mathcal{B}(s)\, ds.$$

Proof of exponential decay to equilibrium

- Exponential decay of $S_\Lambda(t)$ in $L^2(\mu^{-1/2})$ (already known from Baranger-Mouhot '05).
- Enlargement argument $\Rightarrow$ exponential decay of $S_\Lambda(t)$ in the larger space $L^1(\langle v \rangle^k)$ with $k > 2$.

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- There exists a neighborhood of μ in which the linear part of the equation is dominant.
- Thanks to Villani's result, we know that a solution of the equation is going to reach this stability neighborhood.
- Then, we use our result of exponential convergence in this neighborhood.

Main result in the non homogeneous case

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3.$$

Theorem (Hérau-Tonon-T. '17)

If f_0 is close enough to the equilibrium μ, then there exists a unique global solution $f \in L^\infty_t(X)$ to the Boltzmann equation.
Moreover, for any $0 < \lambda < \lambda_\star$ there exists $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_X \leq C\, e^{-\lambda t}\, \|f_0 - \mu\|_X.$$

- ↪ X is a Sobolev space of type $H^3_x L^2_v(\langle v \rangle^k)$ with k large enough.
- ↪ $\lambda_\star > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator.

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↪ Perturbative Cauchy theory in spaces of type $H^{\ell,m}_{x,v}(\mu^{-1/2})$: Gressman-Strain '11, Alexandre-Morimoto-Ukai-Xu-Yang '11

↪ Improvement in the weight.

Strategy of the proof - I

Steps of the proof:

- Study of the linearized problem around equilibrium (based on Mouhot-Neumann '06 result) // homogeneous case.

- Nonlinear estimates in weighted non homogeneous spaces.

- Construction of solutions thanks to *a priori* estimates: if $f_t = \mu + h_t$,

$$\frac{1}{2}\frac{d}{dt}|||h_t|||_X^2 \leq -K_0|||h_t|||_X^2 - \left(K_1 - C|||h_t|||_X\right)\|h_t\|_Y^2$$

where $|||\cdot|||_X \sim \|\cdot\|_X$ and $Y \subset X$.

Strategy of the proof - II

Main difficulties:

- Adapt the computations in L^1 from the homogeneous case to the L^2 framework in the non homogeneous case.
- Regularization properties in x? Hypoellipticity of the equation? Lyapunov functional (see Hérau '07 and Villani '09 for the kinetic Fokker-Planck equation) for the linearized equation (see the Boltzmann operator as a pseudo-differential operator).
- Estimates on the nonlinear term in a non homogeneous framework with polynomial weight.

Theorem (Hérau-Tonon-T. '17)

Let $r \in \mathbb{N}$. We have for k large enough and $k' > k$ large enough:

$$\|S_\Lambda(t)h_0\|_{H^{r,s}_{x,v}(\langle v\rangle^k)} \le \frac{C_r}{t^{1/2}}\|h_0\|_{H^{r,0}_{x,v}(\langle v\rangle^{k'})}, \quad \forall\, t \in (0,1],$$

and

$$\|S_\Lambda(t)h_0\|_{H^{r+s,s}_{x,v}(\langle v\rangle^k)} \le \frac{C_r}{t^{1/2+s}}\|h_0\|_{H^{r,0}_{x,v}(\langle v\rangle^{k'})}, \quad \forall\, t \in (0,1].$$

↪ Key point to develop our perturbative Cauchy theory.

↪ In the spirit of Alexandre-Hérau-Li '15.

Thanks for your attention!