Convergence rates for a particle approximation of conservation laws

Julien Reygner

CERMICS – École des Ponts ParisTech

Joint works with Benjamin Jourdain and Régis Monneau.

Outline of the talk

Introduction of **deterministic particle systems** to approximate the solution to

- **scalar conservation laws** (Kružkov's **entropy solution**),
- **diagonal hyperbolic systems** (Bianchini-Bressan's **viscosity solution**),

in one space dimension, and:

- computation of (optimal) **rates of convergence** of these approximations,
- discussion of **numerical schemes**.

Outline

Sticky Particle Dynamics and scalar conservation laws

Multitype SPD and diagonal hyperbolic systems

Scalar conservation laws

Scalar Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \Lambda(u) = 0, & t \geq 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases}$$

- $\Lambda : \mathbb{R} \to \mathbb{R}$ is the **flux** function, assumed to be C^1,
- in general, weak solutions possess **shocks** (discontinuities) and are **not unique**.

Definition: entropy solution

A function $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\partial_t E(u) + \partial_x F(u) \leq 0$$

in the distributional sense, for pairs of **entropy-entropy flux** functions (E, F) such that

- E is convex,
- $F' = \Lambda' E'$,

is an **entropy solution**.

Actually $E(u) = |u - c|$, $c \in \mathbb{R}$ is sufficient.

Kružkov's Theorem

Kružkov's Theorem

If $u_0 \in L^\infty(\mathbb{R})$ then **there exists a unique entropy solution** to the Cauchy problem.

Some useful properties:

- If u_0 is monotonic, then for all $t \geq 0$, $u(t, \cdot)$ has the same monotonicity and

$$\inf_{x \in \mathbb{R}} u(t,x) = \inf_{x \in \mathbb{R}} u_0(x), \qquad \sup_{x \in \mathbb{R}} u(t,x) = \sup_{x \in \mathbb{R}} u_0(x).$$

- L^1-contraction: for all $t \geq 0$,

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

Assumption

Up to rescaling of the flux function, we now assume that **u_0 is the Cumulative Distribution Function (CDF) of a probability measure m on $\mathbb{R}$**.

- For all $t \geq 0$, $u(t, \cdot)$ **remains a CDF**.
- The L^1-contraction reads as a **W_1-contraction**.
  Bolley-Brenier-Loeper, Jourdain-R.: actually W_p-contraction for any $p \in [1, +\infty]$.

Sticky Particle Dynamics

Deterministic evolution of n particles with positions $x_1(t) \leq \cdots \leq x_n(t)$ on the line:

- the k-th particle has an **initial velocity**

$$\lambda_k = n\left(\Lambda\left(\frac{k}{n}\right) - \Lambda\left(\frac{k-1}{n}\right)\right) = \frac{1}{1/n}\int_{v=(k-1)/n}^{k/n} \Lambda'(v)\mathrm{d}v,$$

- particles travel at constant velocity, and **aggregate into clusters at collisions**,
- the **post-collisional velocity** of a cluster is the **average of the initial velocities** of the particles.

If each particle is assigned a weight $1/n$, this dynamics **preserves mass and momentum, but dissipates kinetic energy**.

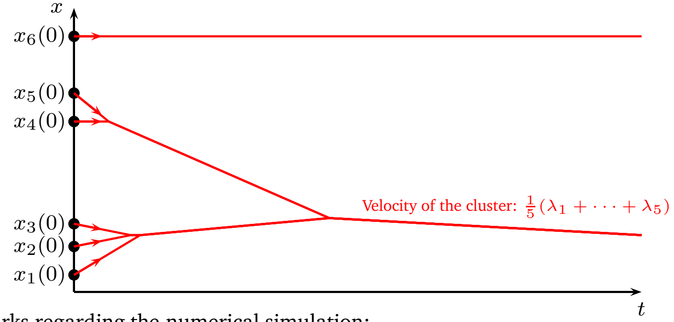
- Relevant model in astrophysics and gas dynamics.
- See Zeldovitch, Bouchut, Grenier, E-Rykov-Sinai...

An example of the SPD

An example of the SPD

An example of the SPD

An example of the SPD



Remarks regarding the numerical simulation:

- Event-driven system, exactly simulable.
- In fact, Hamiltonian formulation allows to compute the configuration at time t directly by convexity arguments; namely:

$$\underbrace{k \mapsto \frac{1}{n}\sum_{\ell=1}^{k} x_\ell(t)}_{\text{Sticky Particle Dynamics}} \quad \text{is the convex hull of} \quad \underbrace{k \mapsto \frac{1}{n}\sum_{\ell=1}^{k} x_\ell(0) + t\lambda_\ell}_{\text{Free Transport}} .$$

Relation with the scalar conservation law

Let $u_n(t,x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{x_k(t) \leq x\}}$, and $\phi : \mathbb{R} \to \mathbb{R}$ smooth with compact support.

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x\in\mathbb{R}} u_n(t,x)\phi(x)\mathrm{d}x &= \frac{1}{n}\sum_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x=x_k(t)}^{+\infty}\phi(x)\mathrm{d}x\\ &= -\frac{1}{n}\sum_{k=1}^{n}\dot{x}_k(t)\phi(x_k(t)),\end{aligned}$$

and grouping the terms by consecutive clusters,

$$\begin{aligned}-\frac{1}{n}\sum_{k=1}^{n}\dot{x}_k(t)\phi(x_k(t)) &= -\int_{v=0}^{1}\Lambda'(v)\phi(u_n^{-1}(t,v))\mathrm{d}v\\ &= -\int_{v=0}^{1}\Lambda'(v)\int_{x=-\infty}^{u_n^{-1}(t,v)}\partial_x\phi(x)\mathrm{d}x\mathrm{d}v\\ &= \int_{x\in\mathbb{R}}\Lambda(u_n(t,x))\partial_x\phi(x)\mathrm{d}x,\end{aligned}$$

so that $u_n(t,x)$ **is a weak solution** to the conservation law.

Approximation of the entropy solution by the SPD

We have seen that **the empirical CDF of the SPD** is a **weak solution** to the conservation law.

- Weak solution to the Cauchy problem with **'discretised' initial datum**

$$u_{n,0} = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{x_k(0) \leq x\}}.$$

- There are always shocks (of size $1/n$ at least) in the profile, which in general **prevents this solution from being entropic**.

But when $n \to +\infty$...

Convergence of SPD - Brenier-Grenier, Jourdain-R.

Assume that $u_{0,n}(x)$ converges $\mathrm{d}x$-a.e. to $u_0(x)$. Then, for all $t \geq 0$, $u_n(t,x)$ **converges** $\mathrm{d}x$**-a.e. to the entropy solution** $u(t,x)$ of the Cauchy problem.

Looks like a **propagation of chaos** result!

Rates of convergence

We may now look for **rates of convergence**. Defining the function $\bar{u}_n(t,x)$ as the **entropy solution** of the Cauchy problem with initial datum $u_{n,0}$, we get

$$\|u(t,\cdot)-u_n(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u(t,\cdot)-\bar{u}_n(t,\cdot)\|_{L^1(\mathbb{R})} + \|\bar{u}_n(t,\cdot)-u_n(t,\cdot)\|_{L^1(\mathbb{R})}.$$

- By L^1-contraction, $\|u(t,\cdot)-\bar{u}_n(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0-u_{n,0}\|_{L^1(\mathbb{R})}$, so that we are looking for the W_1-optimal approximation of a probability measure m on $\mathbb{R}$ by a measure of the form $\frac{1}{n}\sum_{k=1}^n \delta_{x_k}$.
- The quantity $\|\bar{u}_n(t,\cdot)-u_n(t,\cdot)\|_{L^1(\mathbb{R})}$ measures the **'entropicity defect'** introduced by the evolution of the particle system.

Remark: for all these quantities to be finite, m should have a finite first order moment.

Optimal discretisation of the initial condition

We fix a probability measure m with CDF u_0 and finite first-order moment.

- The choice $x_k = u_0^{-1}(\frac{2k-1}{2n})$ is a minimiser of $W_1(m, \frac{1}{n}\sum_{k=1}^n \delta_{x_k})$.
- If m has compact support, then $W_1(m, \frac{1}{n}\sum_{k=1}^n \delta_{x_k}) = \mathrm{O}(1/n)$.
- By comparison with $\mathbb{E}[W_1(m, \frac{1}{n}\sum_{k=1}^n \delta_{X_k})]$ with X_k iid according to m,

$$W_1\left(m, \frac{1}{n}\sum_{k=1}^n \delta_{x_k}\right) \leq \frac{1}{\sqrt{n}} \int_{x\in\mathbb{R}} \sqrt{u_0(x)(1-u_0(x))}\mathrm{d}x,$$

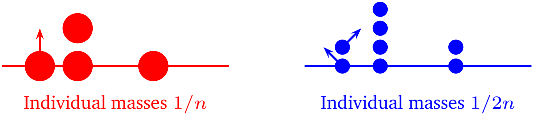
  (see e.g. Bobkov-Ledoux).

In general, the order of magnitude of $W_1(m, \frac{1}{n}\sum_{k=1}^n \delta_{x_k})$ is quite sensitive to the tail behaviour of m.

Entropicity defect of the SPD

We now address the evolution of $\|\bar{u}_n(t,\cdot) - u_n(t,\cdot)\|_{L^1(\mathbb{R})}$.
Assumption: the velocity field Λ' is Lipschitz continuous, with constant L.

First, compare SPD with n and $2n$ particles, same initial CDF $u_{n,0}$.



The difference of the velocities in each particle is of order $1/n$, the computation yields

$$\|u_n^{(0)}(t,\cdot) - u_n^{(1)}(t,\cdot)\|_{L^1(\mathbb{R})} \leq \frac{Lt}{2n}$$

where $u_n^{(m)}(t,x)$ corresponds to SPD with $2^m n$ particles and initial CDF $u_{n,0}$, whence

$$\begin{aligned}\|\bar{u}_n(t,\cdot) - u_n(t,\cdot)\|_{L^1(\mathbb{R})} &\leq \sum_{m=0}^{+\infty} \|u_n^{(m+1)}(t,\cdot) - u_n^{(m)}(t,\cdot)\|_{L^1(\mathbb{R})} \\ &\leq \sum_{m=0}^{+\infty} \frac{Lt}{2^{m+1}n} = \frac{Lt}{n}.\end{aligned}$$

Conclusion

The SPD provides a particle scheme:

- which is **easily implementable**,
- which error at time t is of order $(1+t)/n$,

if m has compact support and the derivative of the flux is Lipschitz continuous.

Outline

Sticky Particle Dynamics and scalar conservation laws

Multitype SPD and diagonal hyperbolic systems

Diagonal hyperbolic systems

We now consider **hyperbolic systems**, which after **diagonalisation** write

$$\forall \gamma \in \{1, \ldots, d\}, \qquad \begin{cases} \partial_t u^\gamma + \lambda^\gamma(\mathbf{u})\partial_x u^\gamma = 0, \\ u^\gamma(0,x) = u_0^\gamma(x), \end{cases}$$

with:

- $\mathbf{u} = (u^1, \ldots, u^d) : [0, +\infty) \times \mathbb{R} \to \mathbb{R}^d$,
- $\lambda^1, \ldots, \lambda^d : \mathbb{R} \to \mathbb{R}$ characteristic fields,
- $u_0^1, \ldots, u_0^d$ are assumed to be the CDFs of $m^1, \ldots, m^d$.

Generalisation of entropy solution to this case: Bianchini-Bressan's theory of **viscosity solution** (not detailed here).

Main assumptions:

(USH) For all $\gamma \in \{1, \ldots, d-1\}$, $\inf \lambda^\gamma > \sup \lambda^{\gamma+1}$.

(LC) For all $\gamma \in \{1, \ldots, d\}$, λ^γ is Lipschitz continuous.

Multitype SPD

System of $d \times n$ particles, with positions

$$\mathbf{x}(t) = \left(x_k^\gamma(t)\right)_{1 \leq \gamma \leq d, 1 \leq k \leq n}$$

such that:

- for all γ, the system of particles of **type γ** satisfies

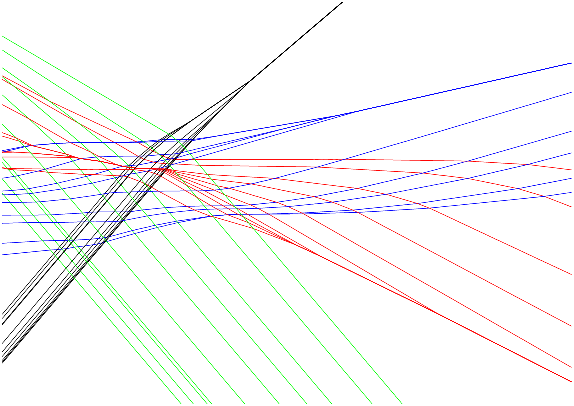
  $$x_1^\gamma(t) \leq \cdots \leq x_n^\gamma(t)$$

  and follows the SPD with 'initial' velocities

  $$\lambda_k^\gamma \simeq \lambda^\gamma \left(u_n^1(t, x_k^\gamma(t)), \ldots, u_n^d(t, x_k^\gamma(t))\right)$$

  where $u_n^{\gamma'}(t, \cdot)$ is the **empirical CDF** of the system of type γ';
- after collisions between clusters of **different types**, the velocities are updated and the **clusters drift away from each other** according to the order prescribed by **Assumption USH**.

A realisation of the MSPD



System with $d = 4$ types and $n = 10$ particles per type.

Approximation result

Once again, $\mathbf{u}_n = (u_n^1, \dots, u_n^d)$ is a **weak solution** with **discretised initial data** $\mathbf{u}_{n,0}$.

Convergence of MSPD — Jourdain, R.

Under Assumptions USH and LC, if $\mathbf{u}_{n,0}(x)$ converges to $\mathbf{u}_0(x)$, $\mathrm{d}x$-a.e., then for all $t \geq 0$, $\mathbf{u}_n(t,x)$ converges to $\mathbf{u}(t,x)$, $\mathrm{d}x$-a.e., where $\mathbf{u}$ is the unique Bianchini-Bressan solution to the initial Cauchy problem.

Further properties:

- The Bianchini-Bressan solution is a semigroup.
- For all $p \in [1, +\infty]$, there exists $C_p \in [1, +\infty)$ such that

$$\sum_{\gamma=1}^d W_p\left(u^\gamma(t,\cdot), v^\gamma(t,\cdot)\right) \leq C_p \sum_{\gamma=1}^d W_p\left(u_0^\gamma, v_0^\gamma\right). \quad (1)$$

The proof essentially relies on similar stability estimates at the level of the MSPD.

Rate of convergence

Thanks to the W_1 stability estimate, the same decomposition of the error holds:

$$\|\mathbf{u}(t,\cdot) - \mathbf{u}_n(t,\cdot)\|_{L^1(\mathbb{R})^d} \leq C_1 \times \text{initial discretisation} + \text{entropicity defect},$$

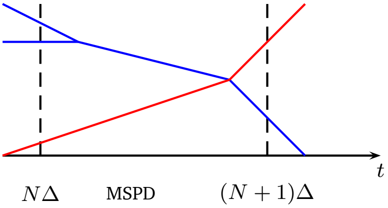
where $\|\cdot\|_{L^1(\mathbb{R})^d}$ is the sum of the W_1 errors over the coordinates u^γ.

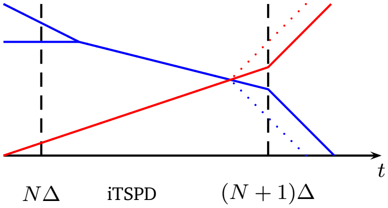
- The analysis of the discretisation of the initial data is unchanged, and the best rate is $1/n$ **for compactly supported data**.
- The analysis of the entropicity defect induced by the evolution of the particle system is more complicated and relies on an auxiliary particle system: the **iterated Typewise Sticky Particle Dynamics**.

Iterated Typewise Sticky Particle Dynamics

Fix a time step $\Delta > 0$. Define $\tilde{\mathbf{x}}(t) = (\tilde{x}_k^\gamma(t))_{1 \leq \gamma \leq d, 1 \leq k \leq n}$ as follows:

- on the time interval $[N\Delta, (N+1)\Delta)$, let each system evolve according to the SPD with initial velocities given by the configuration $\tilde{\mathbf{x}}(N\Delta)$;
- update the velocities according to the configuration obtained at time $(N+1)\Delta$.





Easily implementable!

Error estimate

There exists $C > 0$ such that $\sup_{t \geq 0} \|\mathbf{u}_n(t,\cdot) - \tilde{\mathbf{u}}_n(t,\cdot)\|_{L^1(\mathbb{R})^d} \leq C\Delta$.

Global entropicity defect estimates

Using the comparison between n and $2n$ particles on intervals $[N\Delta, (N+1)\Delta]$ again, we get

$$\|\bar{\mathbf{u}}_n(t, \cdot) - \tilde{\mathbf{u}}_n(t, \cdot)\|_{L^1(\mathbb{R})^d} \leq td\frac{L}{n} + C\Delta,$$

which is the **useful bound** since in practice we simulate the iTSPD.

Letting $\Delta \downarrow 0$ yields

$$\|\bar{\mathbf{u}}_n(t, \cdot) - \mathbf{u}_n(t, \cdot)\|_{L^1(\mathbb{R})^d} \leq td\frac{L}{n},$$

which is the **theoretical bound** on the convergence of MSPD.

Conclusion

The (MSPD and) iTSPD provide numerical schemes to approximate the diagonal hyperbolic system, with error of order

$$\frac{1+t}{n} + \Delta$$

which allows to select n and Δ in order to reach a global error ϵ at time t with an optimal number of elementary computations.

Main references: Jourdain, R. — JHDE 2016, DCDS 2016.