"On the Propagation of Chaos in Kinetic Theory "

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Plan of the lectures

1. Particle systems and BBKGY hierarchy (the paradigm of the Kinetic Theory)
2. Mean-field limit
3. Boltzmann-Grad limit
4. The weak-coupling limit and Landau equation
5. Size of chaos

The foundational problem in Kinetic Theory.

N identical particles. $N \approx 10^{20}$. Newton eq.ns (or Schrödinger). Statistical description. Look for a single equation for $f(x, v, t)$ the probability distribution of a single particle. Effective equation for f

N particles of unitary mass, in all the space $\mathbb{R}^3$. Configurations in the phase space

$$z_1 \dots z_N = Z_N = (X_N, V_N) = (x_1 \dots x_N, v_1 \dots v_N)$$

$z_i = (x_i, v_i)$ denotes position and velocity of the $i - th$ particle.
Two-body interaction $\varphi : \mathbb{R}^3 \to \mathbb{R}$, φ is spherically symmetric .
$m = 1$.

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = - \sum\limits_{\substack{j=1\dots N \\ j \neq i}} \nabla \varphi(x_i - x_j) \end{cases}$$

N is very large. Statistical description.

Probability measure $W^N(Z_N)dZ_N$ on $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$
Symmetry:

$$W^N(z_1 \dots z_i \dots z_j \dots z_N) = W^N(z_1 \dots z_j \dots z_i \dots z_N)$$

The time evolved measure is defined by

$$W^N(Z_N; t) = W^N(\Phi^{-t}(Z_N))$$

$\Phi^t(Z_N)$ the flow with initial datum Z_N.
An evolution equation for $W^N(Z_N; t)$, namely the Liouville equation.

$$\partial_t W^N(t) = \mathcal{L}_N W^N(t)$$

$$\mathcal{L}_N W^N(t) = -\sum_{i=1}^{N} [v_i \cdot \nabla_{x_i} + F_i \cdot \nabla_{v_i}]$$

$$-\sum_{i=1}^{N} (v_i \cdot \nabla_{x_i}) - \sum_{\substack{i,j=1 \\ i<j}}^{N} F(x_i - x_j) \cdot (\nabla_{v_i} - \nabla_{v_j})$$

$$F_i = -\sum_{j:j \neq i} \nabla \varphi(x_i - x_j)$$

N particle description vs one-particle description.
Define the j-particle marginals

$$f_j^N(Z_j; t) = \int dz_2 \dots dz_N W^N(Z_j, z_{j+1} \dots z_N; t). \qquad (1)$$

Evolution equation for $f_1(t)$ involves f_2.

$$\partial_t f_1^N(t) = -v \cdot \nabla_x f_1^N + (N-1) C_2 f_2^N, \qquad (2)$$

$$C_2 f_2^N(z_1) = (N-1) \int dz_2 \nabla_{x_1} \varphi(x_1 - x_2) \cdot \nabla_{v_1} f_2^N(z_1, z_2) \qquad (3)$$

BBKGY hierarchy.
Interpretation. Apparently not useful: we have in any case to solve Liouville.

However if

$$f_2^N(x_1, v_1, x_2, v_2) = f_1^N(x_1, v_1) f_1^N(x_2, v_2)$$

we get a single nonlinear eq.n. Not true in general. But, if for some reason

$$f_2^N(x_1, v_1, x_2, v_2) \approx f_1^N(x_1, v_1) f_1^N(x_2, v_2),$$

in some limiting situations...
Scaling limits and effective equations.
Propagation of chaos

$$\forall j \quad f_j^N(t) \to f(t)^{\otimes j}$$

as $N \to \infty$, where f solves the kinetic equation, provided that

$$f_j^N(0) \to f_0^{\otimes j} \text{ or } \quad f_j^N(0) \approx f_0^{\otimes j}.$$

Mean-Field limit and Vlasov equation.

The easiest case in which the program can be achieved is the mean-field limit. Force is small. Rescale $F \to \frac{1}{N} F$. Two particles behave almost independently. Then

$$C_2 f_2^N(z_1) \approx \frac{(N-1)}{N} \int dz_2 \nabla_{x_1} \varphi(x_i - x_2) \cdot \nabla_{v_1} (f_1^N)^{\otimes 2}(z_1, z_2)$$

$z_2 = (x_2, v_2)$ and $(f_1 = f)$

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0$$

$$E(x,t) = \int F(x-y)\rho(y,t)dy; \qquad \rho(x,t) = \int dv f(x,v)$$

Vlasov eq. (1938). Rigorous results (smooth potentials) : Neunzert, Baun-Hepp, Dobrushin.

Boltzmann equation

Boltzmann-Grad limit and B. equation.

Much more subtle. Now the force is $O(1)$. $N \to \infty$ $\varepsilon \to 0$ (ε the interaction range), $N\varepsilon^2 \to \lambda$ (Low-density or B-G limit).
Here the probability that two given particles collide is $O(\varepsilon^2)$. Given j particles, the probability that the j clusters of influence of the j particles are not disjoint is $O(\frac{j^2}{N})$.
A cluster of influence of a given particle is the set of particles which interact directly or indirectly with it.
Lanford (short times) , Illner-P. (expanding cloud of gas in the vacuum).
Recent results: Gallagher, Saint-Raymond, Texier, Bodineau, Saffirio, Simonella....

Landau eq.n

Landau (1936) introduced a new kinetic equation to deal with dense gases, but with weak intermolecular forces. Grazing collision limit from Boltzmann. $Q_B \to Q_L$

$$Q_L(f,f)(v) = \int dv_1 \nabla_v \, a(\nabla_v - \nabla_{v_1}) f f_1,$$

($f = f(v)$ and $f_1 = f(v_1)$) where $a = a(v - v_1)$ is the matrix

$$a_{i,j}(V) = \frac{B}{|V|}(\delta_{i,j} - \hat{V}_i \hat{V}_j).$$

where $\hat{V}$ and $\hat{P}$ is the versor of V and B is a constant depending on the interaction. Nature of the diffusion. Remark on the terminology in the math literature.

Weak-coupling limit and Landau equation.

Is the Landau eq.n arising from particle systems under suitable scaling limits? (Bogolyubov) Scale

$$x \to \varepsilon x, \quad t \to \varepsilon t, \quad \varphi \to \sqrt{\varepsilon}\varphi, \quad N = O(\varepsilon^{-3}).$$

In the new (macroscopic) variables

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1...N \\ j \neq i}} \nabla\varphi(\frac{x_i - x_j}{\varepsilon}) \end{cases}$$

why we expect a diffusion in velocity in the limit ?
Force $\frac{1}{\sqrt{\varepsilon}}$, $\Delta v \approx \sqrt{\varepsilon}$ (the time is $O(\varepsilon)$). Number of particles collided per unit time $= O(\varepsilon^{-1})$. Then

$$\sum |\Delta v| \approx \frac{1}{\sqrt{\varepsilon}}$$

But $\sum \Delta v = 0$ and $\sum |\Delta v|^2 = O(1)$ which is typical of the diffusion.

Validity problem: no result even for short times

The only consistency result (validity at time zero) Bobylev, P. Saffirio (2012).

Size of chaos

Estimate $j = j(N)$ s.t.

$$\lim_{N \to \infty} \sup_{j \leq j(N)} \| f_j^N(t) - f^{\otimes j}(t) \| = 0$$

Heuristics. Starting point is the hierarchy:

$$\partial_t f_j^N + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k} f_j + T_j^N f_j^N = C_{j+1}^N f_{j+1}^N$$

The corresponding kinetic equation is recovered by the formal limit

$$\partial_t f_j + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k} f_j = C_{j+1} f_{j+1}$$

admitting factorized solutions.

B-G limit:

$$T_j \approx j^2 \varepsilon^2 = \frac{j^2}{N} \quad j << N^{1/2}$$

W-C limit ($N = \varepsilon^{-3}$):

$$T_j = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{k=1\dots N \\ k \neq i}} \nabla \varphi(\frac{x_i - x_k}{\varepsilon})$$

$$T_j \approx \frac{j^2}{\sqrt{\varepsilon}} \varepsilon^2 = \frac{j^2}{\sqrt{N}}, \quad j << N^{1/4}$$

M-F limit

$$T_j \approx \frac{j^2}{N}$$

........but lack of regularity. Conjecture $j << N^{1/3}$
For the Kac's model it seems optimal $j << N^{1/2}$

Mischler and Mouhot (2012) for the Kac's model. Estimates uniform in time. Vasserstein distance and relative entropy.
P and Simonella (2016) for the B-G limit (short times). In this case the best estimates recently available are

$$\sup_x \| f_j^N(t) - f^{\otimes j}(t) \| \leq C^j \varepsilon^{1-\eta}$$

η arbitrarily small. Actually a log. So that $j(N) = O(|\log N|)$.
Try to improve to a power law.

Kinetic error

Define $J \subset \mathbb{N}$, $|J|$ its cardinality:

$$E_J(t) = \sum_{K \subset J} (-1)^{|K|} f^N_{J/K}(t) f^{\otimes K}(t); \;\; f^N_J(t) = \sum_{K \subset J} f^{\otimes K}(t) E_{J/K}(t)$$

the kinetic error. Replace $f(t)$ by $f^N_1(t)$ the correlation error $E^{corr}_J(t)$.
$E^{corr}_J(t)$ is an efficient way to measure the deviation from the factorization. $E_J(t)$ measure also the convergence to the kinetic equation. Hard to estimate in the B-G limit. P and Simonella 2016

$$\|E_J(t)\|_{L^1(x,v)} \leq \varepsilon^{\gamma j} \;\text{ if }\; j = |J| \leq \varepsilon^{-\alpha}$$

In the Grand-canonoical formalism: i.e. $N = \varepsilon^{-2}$ is the average number of particles. Less correlations.

Then $(|K| = k,\ |J| = j)$:

$$\|f_j^N(t) - f^{\otimes j}(t)\|_{L^1(x,v)} \leq \sum_{K \subset J, |K| \geq 1} \|f^{\otimes (j-k)}(t) E_K(t)\|_{L^1(x,v)}$$

$$\leq \sum_{k=1}^{j} \binom{j}{k} \varepsilon^{\gamma k}$$

$$\leq \sum_{k=1}^{\infty} \frac{(\varepsilon^{\gamma} j)^k}{k!}$$

$$\leq (e^{\varepsilon^{\gamma'}} - 1) \leq C\varepsilon^{\gamma'}$$

Size of chaos for the Kac's (canonical) model (and other models) (Paul,P. Simonella in progress).
Use E_J and similar arguments to prove

$$\|f_j^N(t) - f^{\otimes j}(t)\|_{L^1(x,v)} \leq C(t)\frac{g(j)}{N}$$

for g superlinear slowly increasing e.g. $g(j) = j^{1+\eta}$, η arbitrarily small.
Somehow optimal because the recollision operator is $O(\frac{j^2}{N})$.