Flocking for stochastic variations of the Cucker-Smale model

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based on a joint work with
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A flight of birds in Linum (Brandeburg, Germany)

Collective motions of population without central direction

A few examples :

- flocks of birds or insects (common cranes, marching locusts)
- schools of fish, pods of dolphins
- populations of bacteria
- herds of cattle (when trying to escape a predator)
- human groups (with respect to a focal or repulsion point)

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How can these phenomena be explained ?

- it is safer : the "safety in numbers" concept (dissuasion and anonymity)
- finding food is easier : the appeal of an increased foraging efficiency
- chances of mating are higher
- traveling is less tiring (higher locomotion (aerodynamic, hydrodynamic) efficiency)

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- Collective motion of a population (groups of animals on the move, including wolf packs or elephant herds, bacteria aggregations).
- Emergence of a new language in (primitive) societies.
- Control mechanisms providing basic coordination and consensus algorithms for systems of mobile autonomous agents (robotic vehicles, mobile sensors, satellites).
- Applications in military, environmental controls.
- Belief in a price system in an active market (for instance, the correlation structure of credit worthiness index is modelled through flocking mechanics, since firms do not exist in isolation, but can influence one another).

Collective behaviours and literature

Collective motion of population :

- Equations descriptive of fish schools and other animal aggregations (Breder '54)
- Modeling Social Animal Aggregations (Grunbaum-Okubo '86)
- Tendency-distance models of social cohesion in animal groups (Warburton-Lazarus '91)

Other fields :

- Modeling language evolution (Cucker-Smale-Zhou '04)
- The Computational Nature of Language Learning and Evolution (Niyogi '06)
- Distributed anonymous mobile robots : formation of geometric patterns (Suzuki-Yamashita '09)
- A mathematical model for multi-name credit based on community flocking (Ha-Kim-Lee, '15)

Flocking and mathematics

In a mathematical sense, *flocking* means the following long-time behaviour :

- *velocity alignment*
- *stability of the group structure.*

Consider N particles evolving in $\mathbb{R}^d$. Position (resp. speed) of i-th particle at time t is denoted by $x_i(t)$ (resp. $v_i(t)$).

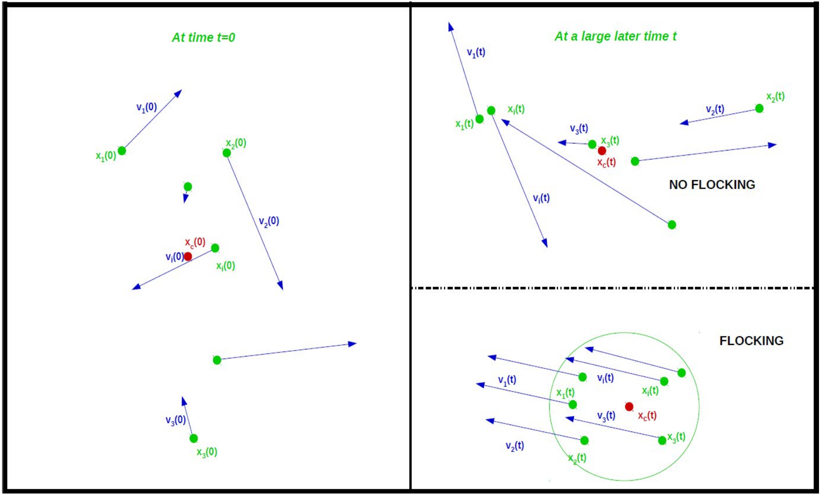
The center of mass is given by : $x_c(t) := \frac{1}{N}\sum_{j=1}^{N} x_j(t)$ (resp. $v_c(t) := \frac{1}{N}\sum_{j=1}^{N} v_j(t)$).

Definition

Flocking happens for a set of N particles if :

- For all $i \in \{1, \ldots N\}$, $\lim_{t\to\infty} |v_i(t) - v_c(t)|^2 = 0$;
- For all $i \in \{1, \ldots N\}$, $\sup_{0\leq t<\infty} \left|x_i(t) - x_c(t)\right|^2 < \infty$.

Flocking and mathematics



The Cucker-Smale model : a particle kinetic system for flocking

For all $i \in \{1, \ldots N\}$

$$\begin{cases} x_i'(t) & = & v_i(t) \\ v_i'(t) & = & -\lambda \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \psi(x_j(t), x_i(t))(v_i(t) - v_j(t)) \end{cases}$$

where λ is a positive number and ψ a positive, symmetric function called *communication rate*.

Theorem (F. Cucker, S. Smale '06)

When the communication rate is defined by $\psi(x,y) = \bar{\psi}(\|x-y\|^2)$ with $\bar{\psi}(u) = \dfrac{C_1}{(C_2+u)^r}$, there is always flocking if $r < 0.5$; otherwise, conditional flocking, for a subset of the initial configuration.

The Cucker-Smale model : a mean-field particle kinetic system for flocking

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where λ is a positive number and ψ a positive, symmetric function called *communication rate*.

- There are interactions between all the particles, and not only the nearest neighbours.
- The closer two particles are, the higher the communication rate between them. Other choices can be championed.
- Only one **deterministic** interaction is taken into account, through ψ. What about the impact of the **random** environment (wind, flow)? Or free will ?

Stochastic flocking for a stochastic dynamics

==> Interest of adding random noise.

==> Two main problems :

- How to define stochastic flocking ?
- Which form for the random perturbation ?

==> And then, what happens to the dynamics ? Disturbance of a certain equilibrium ? Or instead “improvement" of the group properties ?

The problem of stochastic flocking : different definitions

Recall that in a deterministic framework, there is flocking if

$$\lim_{t\to\infty} |v_i(t) - v_c(t)|^2 = 0 \quad \text{and} \quad \sup_{0\leqslant t<\infty} \left|x_i(t) - x_c(t)\right|^2 < \infty$$

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How to define flocking in a random setting ?

- *mean-flocking* : the weakest form of flocking (?). Deterministic flocking is satisfied by the $\mathbb{E}[x_i]$ and $\mathbb{E}[v_i]$:

$$\lim_{t\to\infty} |\mathbb{E}[v_i(t)] - \mathbb{E}[v_c(t)]|^2 = 0 \quad \text{and} \quad \sup_{0\leqslant t<\infty} \left|\mathbb{E}[x_i(t)] - \mathbb{E}[x_c(t)]\right|^2 < \infty$$

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- *weak-flocking* : it more or less corresponds to the convergence in probability towards the center of mass. There is weak flocking with rate $\varepsilon(R)$ if for all $R > 0$ and all $i \in \{1, ..., N\}$,

$$\limsup_{t\to+\infty} \mathbb{P}(|v_i(t) - v_c(t)| > R) \leq \varepsilon(R).$$

The problem of stochastic flocking : different definitions

- *almost-sure-flocking* : the definition for deterministic flocking holds almost surely :

$$\lim_{t\to\infty} |v_i(t) - v_c(t)| = 0 \ a.s. \quad \text{and} \quad \sup_{0\leqslant t<\infty} \left|x_i(t) - x_c(t)\right| < \infty \ a.s.$$

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- $L^{p,q}$*-flocking* : convergence in L^p of the velocities towards the center of mass, and boundedness in L^q of the difference between the positions and their center of mass :

$$\lim_{t\to\infty} \mathbb{E}[|v_i(t) - v_c(t)|^p] = 0 \quad \text{and} \quad \sup_{0\leqslant t<\infty} \mathbb{E}[\left|x_i(t) - x_c(t)\right|^q] < \infty$$

If $q = 1$, we simply say that there is L^p-flocking.
In practice, $p, q = 1, 2$.

Which stochastic touch ?

==> A diffusion coefficient with a Brownian motion :

$$\sigma(t)\, dW_t$$

with σ constant, or depending only on v_i, or on x_i, or all of them, and so on.

==> Three main optics :

- a noise common to all the particles : winds, currents, other environmental effects, ...
- a noise specific to each individual : free will, right to self-determination, craziness, ...
- a noisy communication rate : "interference" in the transmission, deformation of the information (parasitic noise, fog,...)

Stochastic Cucker-Smale models with a constant communication rate

Recall that, in the general case, for all $i \in \{1, ..., N\}$,

$$\begin{cases} dx_i(t) &= v_i(t)\,dt \\ dv_i(t) &= -\lambda \frac{1}{N} \sum_{j=1}^{N} \psi(x_j(t), x_i(t))(v_i(t) - v_j(t))\,dt + \sigma_i(t)\,dW_i(t) \end{cases}$$

If $\psi = 1$, it becomes

$$\begin{cases} dx_i(t) &= v_i(t)\,dt \\ dv_i(t) &= -\lambda\,(v_i(t) - v_c(t))\,dt + \sigma_i(t)\,dW_i(t) \end{cases}$$

as $\sum_{i=1}^{N} (v_i - v_j) = N\,v_i - \sum_{i=1}^{N} v_j = N(v_i - v_c)$.

Flocking properties of different stochastic Cucker-Smale models with a constant communication rates

- Independent noise for each particle and a constant diffusion coefficient [Ha, Lee, Levy '09] (*individuals with a noisy mind*) :

$$dv_i(t) = -\lambda\,(v_i(t) - v_c(t))\,dt + \sqrt{D}\,dw_i(t).$$

Weak flocking, with a rate of convergence given by a χ^2-tail.

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- Linear independent noise for each individual [Erban, Haszkovec, Sun '16] (*noisy communication rate*):

$$dv_i(t) = -\lambda\,(v_i(t) - v_c(t))\,dt + \sigma\,(v_i(t) - v_c(t))\,dw_i(t).$$

$\alpha = (1 - 1/N)\,\sigma^2 - 2\lambda$ determines the behaviour of the system : **almost-sure** and $L^{2,2}$-**flocking** if $\alpha < 0$ but else no L^2-flocking , and the norm in L^2 of v_i is going to infinity if $\alpha > 0$.

Flocking properties of different stochastic Cucker-Smale models with a constant communication rates

- With a common noise [Ahn, Ha '10] (*noisy environment*), for some constant v_e :

$$dv_i(t) = -\lambda\,(v_i(t) - v_c(t))\,dt + D\,(v_i(t) - v_e)\,dw(t).$$

Always **almost-sure flocking** and L^1-**flocking**, and, $L^{2,2}$-**flocking** if and only if $2\lambda > D^2$.

==> Importance of the choices of noise and of flocking definition...

With a noisy environment and a general communication rate

Consider the same random noise $w(t)$ affecting all particles,

$$dv_i(t) = -\frac{\lambda}{N}\sum_{j=1}^{N}\psi_{ij}(t)(v_i(t)-v_j(t))\,dt+\sigma(v_i(t))\,dw(t),$$

with $\psi_{i,j}=\psi_{i,j}(v(.),x(.))$ locally Lipschitz, non-negative and symmetric (for instance the Cucker-Smale rate), and σ globally K-Lipschitz continuous.

Theorem (Cattiaux, Delebecque, P. '16)

(i) If $2\lambda \inf_{i,j,x,v}\psi_{i,j}(v,x) > 4K^2d^2$, there is almost-sure and $L^{2,2}$-flocking.

(ii) If $\sigma(v_i)$ is linear in v_i, then, the system *always flocks almost surely*.

(iii) Assuming that $\psi_{i,j}=\bar{\psi}(|x_i-x_j|^2)$, there is *conditional flocking* (that is for a subset of initial conditions) with a *positive probability*.

A few comments

- The most demanding type of flocking appears to be the L^2-flocking.
- Almost-sure flocking can happen even in scenarios where deterministic flocking does not for the corresponding dynamics.
- In other cases, in particular with independent noises, the introduction of noise can lead to a scattering of the particles.

Thank you for your attention !