Mean field kinetic particles and the Vlasov-Fokker-Planck equation

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Mean field kinetic particles

- $(X_i(t), Y_i(t)) \ \in \mathbb{R}^{2d}$, position/velocity of the i^{th} particle (mass 1)
- $U : \mathbb{R}^d \to \mathbb{R}$ an external potential
- $W : \mathbb{R}^d \to \mathbb{R}$ an even interaction potential

For $i \in [\![1, N]\!]$,

$$\begin{aligned}
\mathrm{d}X_i &= Y_i \mathrm{d}t \\
\mathrm{d}Y_i &= -\nabla U(X_i)\mathrm{d}t - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_i - X_j)\mathrm{d}t - Y_i \mathrm{d}t + \sqrt{2}\mathrm{d}B_i
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Assuming $\pi_t^N \underset{N\to\infty}{\longrightarrow} m_t$,

$$\partial_t m_t + y \cdot \nabla_x m_t = \nabla_y \cdot \left(\nabla_y m_t + (\nabla U + \nabla W * m_t + y)\, m_t \right)$$

with $\nabla W * m_t(x) = \int \nabla W(x-u) m_t(u,v)\mathrm{d}u\mathrm{d}v$ (Vlasov-Fokker-Planck).

Non-linear process

For $i \in [\![1, N]\!]$,

$$\left\{ \begin{array}{rcl} \mathrm{d}\widetilde{X}_i & = & \widetilde{Y}_i \mathrm{d}t \\ \mathrm{d}\widetilde{Y}_i & = & -\nabla U \left(\widetilde{X}_i \right) \mathrm{d}t - (\nabla W * m_t) \left(\widetilde{X}_i \right) - \widetilde{Y}_i \mathrm{d}t + \sqrt{2} \mathrm{d}B_i \\ m_t & = & \mathcal{L} \left(\widetilde{X}_i(t), \widetilde{Y}_i(t) \right) \end{array} \right.$$

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We are interested in :

- The law m_t that solves the non-linear PDE,
- The non-independent $Z_i = (X_i, Y_i)$ with $Z = (Z_1, \dots, Z_N)$ Markov,
- The independent $\widetilde{Z}_i = \left(\widetilde{X}_i(t), \widetilde{Y}_i\right)$ with law m_t, $\widetilde{Z}$ non Markov.

Asymptotics

- $N \to \infty$: propagation of chaos

- $t \to \infty$: convergence to equilibrium

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  - $\pi_t^N = \frac{1}{N} \sum \delta_{Z_i}$ should converge to m_t,
  - Z_1 should behave like $\widetilde{Z}_1$,
  - $m_t^{(1,N)} = \mathcal{L}\left(Z_1(t)\right)$ should converge to m_t.

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Goal : quantitative estimates for the speed of these convergences.

Distances

Coupling of two laws :

$$\Pi(\mu,\nu) \quad = \quad \{(Q,R) \text{ r.v. such that } \mathcal{L}(Q)=\mu,\ \mathcal{L}(R)=\nu\}\,.$$

- Total variation distance :

$$\begin{aligned} d_{VT}(\mu,\nu) &= \inf_{\Pi(\mu,\nu)} \mathbb{P}\left(Q\neq R\right) \\ &= \frac{1}{2}\|\mu-\nu\|_1 \qquad \text{(if density)} \end{aligned}$$

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- Relative entropy (Kullback-Leibler divergence) :

$$\mathcal{H}\left(\mu \mid \nu\right) \;=\; \int \ln\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)\mathrm{d}\mu$$

Assumptions

A

- The external potential U is convex ($\nabla^2 U \geq c_1 > 0$) and $\nabla^2 W \geq -c_2$ with $c_2 < \frac{1}{2}c_1$. Moreover $\nabla^2 U$ and $\nabla^2 W$ are bounded.
- The law m_0 has a Lebesgue density, a finite 2nd moment and $\int m_0 \ln m_0 < \infty$.

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Remarks :

- Forbid the Coulomb interaction $W_c(x-y) = \pm\frac{1}{|x-y|}$, but allow $\xi * W_c$ with a smooth kernel, provided U is convex enough.
- $\ll W$ *small enough* $\gg$ not needed (contrary to [Villani 2007, Bolley-Guillin-Malrieu 2010, Hérau-Thomann 2015]).

Results

Theorem (M., 2016)

Under Assumption A, the exist $C, \chi > 0$ which depend neither on t, nor N, nor m_0, and there exists K that depends on m_0 but not on t, N, such that

- For the particle system, $m_\infty^{(N)}$ satisfies a log-Sobolev inequality with constant independent from N and

$$\mathcal{H}\left(m_t^{(N)} \mid m_\infty^{(N)}\right) \leq Ce^{-\chi t}\mathcal{H}\left(m_0^{(N)} \mid m_\infty^{(N)}\right).$$

- The Vlasov-Fokker-Planck PDE admits a unique equilibrium m_∞ and

$$\|m_t - m_\infty\|_1 \leq Ke^{-\chi t}, \qquad \mathcal{W}_2\left(m_t, m_\infty\right) \leq Ke^{-\chi t}.$$

Results

Theorem (M., 2016)

Under Assumption A, there exist $b, \alpha, > 0$ that depend neither on t, nor N, nor m_0, and there exists K that depends on m_0 but not on t, N, such that

- Uniform in time propagation of chaos :

$$W_2\left(m_t^{(1,N)}, m_t\right) \leq K \min\left(\frac{e^{bt}}{N}, \frac{1}{N^\alpha}\right)$$

  and

$$\|m_t^{(1,N)} - m_t\|_1 \leq \frac{K}{N^\alpha}.$$

- Numerical error bound (cf. Bolley-Guillin-Villani 2006) :

$$\mathbb{P}\left(\mathcal{W}_2\left(\pi_t^N, m_\infty\right) \geq \varepsilon\right) \leq \frac{K}{\varepsilon^2}\left(e^{-\chi t} + \frac{1}{N}\right)$$

Hypoercivity without interaction

$$\begin{cases} \mathrm{d}X &= Y\mathrm{d}t \\ \mathrm{d}Y &= -\nabla U(X)\mathrm{d}t - Y\mathrm{d}t + \sqrt{2}\mathrm{d}B \end{cases}$$

The entropy dissipation may vanish outside of equilibrium :

$$\partial_t \left(\mathcal{H}\left(m_t \mid m_\infty\right)\right) = -\int \left|\nabla_y \ln \frac{\mathrm{d}m_t}{\mathrm{d}m_\infty}\right|^2 \mathrm{d}m_t.$$

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Modified entropy (Hérau 2006, Villani 2007) : set $h_t = \frac{\mathrm{d}m_t}{\mathrm{d}m_\infty}$ and

$$\mathcal{N}(h) := \int h \ln h \mathrm{d}m_\infty + \int |P\nabla \ln h|^2 h \mathrm{d}m_\infty.$$

With a well-chosen P and a log-Sobolev inequality,

$$\partial_t \left(\mathcal{N} \left(h_t \right) \right) \leq -c \int \left| \nabla \ln h_t \right|^2 \mathrm{d}m_t \leq -c' \mathcal{N} \left(h_t \right)$$

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Large time for the particle system

- The system $Z = (X, Y) \in \mathbb{R}^{2dN}$ satisfies a Langevin SDE

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- $+$ modified entropy $+$ mean field,

$$\Rightarrow \quad \mathcal{H}\left(m_t^{(N)} \mid m_\infty^{(N)}\right) \leq Ce^{-\chi t}\mathcal{H}\left(m_0^{\otimes N} \mid m_\infty^{(N)}\right)$$

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with C, χ, K independent from t and N.

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  with C, χ, K independent from t and N.
- $+$ Talagrand Inequality independent from N,

$$\mathcal{W}_2^2\left(m_t^{(N)}, m_\infty^{(N)}\right) \leq KNe^{-\chi t}.$$

Crude propagation of chaos

The parallel coupling between

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and

$$\left\{ \begin{array}{rcl} \mathrm{d}\widetilde{X}_i & = & \widetilde{Y}_i \mathrm{d}t \\ \mathrm{d}\widetilde{Y}_i & = & -\nabla U\left(\widetilde{X}_i\right)\mathrm{d}t - (\nabla W * m_t)\left(\widetilde{X}_i\right) - \widetilde{Y}_i \mathrm{d}t + \sqrt{2}\mathrm{d}B_i. \end{array} \right.$$

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and

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with the same initial conditions yields

$$\mathbb{E}\left(\left|Z_1(t) - \widetilde{Z}_1(t)\right|^2\right) \leq \frac{Ke^{bt}}{N}.$$

Large-time in $\mathcal{W}_2$ for the non-linear process

At fixed t and for all N,

$$\mathcal{W}_2\left(m_t, m_\infty\right)$$

$$\leq \quad \mathcal{W}_2\left(m_t, m_t^{(1,N)}\right) + \mathcal{W}_2\left(m_t^{(1,N)}, m_\infty^{(1,N)}\right) + \mathcal{W}_2\left(m_\infty^{(1,N)}, m_\infty\right)$$

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$$\leq \frac{Ke^{bt}}{N} + Ke^{-\chi t} + \frac{K}{N}$$

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\underset{N\to\infty}{\longrightarrow} \quad & Ke^{-\chi t}.
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For $t \geq \varepsilon \ln N$, coupling through the equilibria :

$$\begin{aligned}
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&\leq Ke^{-\chi t} + \frac{K}{N} + Ke^{-\chi t}
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For $t \leq \varepsilon \ln N$, parallel coupling :

$$\mathcal{W}_2^2\left(m_t^{(1,N)}, m_t\right) \leq \frac{Ke^{bt}}{N} \leq \frac{K}{N^{1-b\varepsilon}}.$$

For $t \geq \varepsilon \ln N$, coupling through the equilibria :

$$\begin{aligned}
&\mathcal{W}_2\left(m_t^{(1,N)}, m_t\right) \\
&\leq \mathcal{W}_2\left(m_t^{(1,N)}, m_\infty^{(1,N)}\right) + \mathcal{W}_2\left(m_\infty^{(1,N)}, m_\infty\right) + \mathcal{W}_2\left(m_\infty, m_t\right) \\
&\leq Ke^{-\chi t} + \frac{K}{N} + Ke^{-\chi t} \\
&\leq \frac{3K}{N^{\varepsilon\chi}}
\end{aligned}$$

Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \varepsilon \ln N$, parallel coupling :

$$\mathcal{W}_2^2\left(m_t^{(1,N)}, m_t\right) \leq \frac{Ke^{bt}}{N} \leq \frac{K}{N^{1-b\varepsilon}}.$$

For $t \geq \varepsilon \ln N$, coupling through the equilibria :

$$\begin{aligned}
&\mathcal{W}_2\left(m_t^{(1,N)}, m_t\right) \\
&\leq \mathcal{W}_2\left(m_t^{(1,N)}, m_\infty^{(1,N)}\right) + \mathcal{W}_2\left(m_\infty^{(1,N)}, m_\infty\right) + \mathcal{W}_2\left(m_\infty, m_t\right) \\
&\leq Ke^{-\chi t} + \frac{K}{N} + Ke^{-\chi t} \\
&\leq \frac{3K}{N^{\varepsilon\chi}}
\end{aligned}$$

Conclusion, for all time, $\mathcal{W}_2\left(m_t^{(1,N)}, m_t\right) \leq \frac{K}{N^\alpha}$.

Total variation

Based on Malrieu's 2001 guideline,

$$\partial_t \left(\mathcal{H}\left(m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \;\leq\; K N \mathcal{W}_2 \left(m_t^{(1,N)}, m_t \right) \;\leq\; K N^{1-\alpha}$$

Total variation

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$$\partial_t \left(\mathcal{H}\left(m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \ \leq \ KN\mathcal{W}_2\left(m_t^{(1,N)}, m_t \right) \ \leq \ KN^{1-\alpha}$$

hence

$$\| m_t^{(1,N)} - m_t \|_1^2 \ \leq \ \frac{1}{N}\mathcal{H}\left(m_t^{(N)} \mid m_t^{\otimes N} \right) \ \leq \ \frac{Kt}{N^\alpha}$$

Total variation

Based on Malrieu's 2001 guideline,

$$\partial_t \left(\mathcal{H} \left(m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \leq KN\mathcal{W}_2 \left(m_t^{(1,N)}, m_t \right) \leq KN^{1-\alpha}$$

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Not uniform, but sufficient :

$$\begin{aligned} & \| m_t - m_\infty \|_1 \\ \leq \quad & \| m_t - m_t^{(1,N)} \|_1 + \| m_t^{(1,N)} - m_\infty^{(1,N)} \|_1 + \| m_\infty^{(1,N)} - m_\infty \|_1 \end{aligned}$$

Total variation

Based on Malrieu's 2001 guideline,

$$\partial_t \left(\mathcal{H} \left(m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \ \leq \ K N \mathcal{W}_2 \left(m_t^{(1,N)}, m_t \right) \ \leq \ K N^{1-\alpha}$$

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Not uniform, but sufficient :

$$\begin{aligned} & \| m_t - m_\infty \|_1 \\ \leq \ & \| m_t - m_t^{(1,N)} \|_1 + \| m_t^{(1,N)} - m_\infty^{(1,N)} \|_1 + \| m_\infty^{(1,N)} - m_\infty \|_1 \\ \leq \ & \frac{\sqrt{Kt}}{N^{\frac{\alpha}{2}}} + \sqrt{KN} e^{-\frac{1}{2}\chi t} + \frac{K}{\sqrt{N}} \end{aligned}$$

Total variation

Based on Malrieu's 2001 guideline,

$$\partial_t \left(\mathcal{H} \left(m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \ \leq \ K N \mathcal{W}_2 \left(m_t^{(1,N)}, m_t \right) \ \leq \ K N^{1-\alpha}$$

hence

$$\| m_t^{(1,N)} - m_t \|_1^2 \ \leq \ \frac{1}{N} \mathcal{H} \left(m_t^{(N)} \mid m_t^{\otimes N} \right) \ \leq \ \frac{Kt}{N^\alpha}$$

Not uniform, but sufficient : for N of order $e^{\frac{\chi t}{\alpha+1}}$.

$$\begin{aligned} & \| m_t - m_\infty \|_1 \\ \leq & \ \ \| m_t - m_t^{(1,N)} \|_1 + \| m_t^{(1,N)} - m_\infty^{(1,N)} \|_1 + \| m_\infty^{(1,N)} - m_\infty \|_1 \\ \leq & \ \ \frac{\sqrt{Kt}}{N^{\frac{\alpha}{2}}} + \sqrt{KN} e^{-\frac{1}{2}\chi t} + \frac{K}{\sqrt{N}} \ \leq \ K e^{-\chi' t}. \end{aligned}$$

Total variation

Based on Malrieu's 2001 guideline,

$$\partial_t \left(\mathcal{H}\left(m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \leq KN\mathcal{W}_2\left(m_t^{(1,N)}, m_t \right) \leq KN^{1-\alpha}$$

hence

$$\|m_t^{(1,N)} - m_t\|_1^2 \leq \frac{1}{N}\mathcal{H}\left(m_t^{(N)} \mid m_t^{\otimes N} \right) \leq \frac{Kt}{N^\alpha}$$

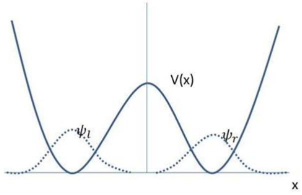
Not uniform, but sufficient : for N of order $e^{\frac{\chi t}{\alpha+1}}$.

$$\begin{aligned} &\|m_t - m_\infty\|_1 \\ \leq\ & \|m_t - m_t^{(1,N)}\|_1 + \|m_t^{(1,N)} - m_\infty^{(1,N)}\|_1 + \|m_\infty^{(1,N)} - m_\infty\|_1 \\ \leq\ & \frac{\sqrt{Kt}}{N^{\frac{\alpha}{2}}} + \sqrt{KN}e^{-\frac{1}{2}\chi t} + \frac{K}{\sqrt{N}} \leq Ke^{-\chi' t}. \end{aligned}$$

($\Rightarrow$ uniform in time propagation of chaos in the total variation sense. . .)

Without convexity

If U has several minima and the interaction is attractive, in the small noise regime, the non-linear PDE has several distinct equilibria, but there is unicity for a large enough noise



- If uniqueness, uniform estimates, with respect to t or N ?
- Without uniqueness, replace THE invariant measure by quasi-stationary ones ? Are there two regimes

$$t \ll e^{aN} \Rightarrow \mathcal{W}_2\left(m_t^{(1,N)}, m_t\right) \leq \frac{K}{N}$$
$$t \gg e^{aN} \Rightarrow \mathcal{W}_2\left(m_t^{(1,N)}, m_t\right) \geq K$$

  and convergence of the QSD towards the equilibria of the PDE ?
- toy model (Curie-Weiss).

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