Villani's program on constructive rate of convergence to the equilibrium : Part I - Coercivity estimates

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Outline of the talk

1. Introduction and main result
   - Villani's program
   - Boltzmann and Landau equation
   - Quantitative trend to the equilibrium
   - First step: quantitative coercivity estimates
2. Coercivity estimates for the Landau operator
   - Linearized Landau operator
   - Proof for the Maxwell molecules case $\gamma = 0$
   - Proof in the other cases ($\gamma \neq 0$)
3. Coercivity estimates for the Boltzmann operator
   - Linearized Boltzmann operator
   - Proof for $\gamma \in [0, \gamma^*)$, $\gamma^* > 0$
   - Proof for $\gamma \notin [0, \gamma^*)$

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Here is the program (Villani's Notes on 2001 IHP course, Section 8. Toward exponential convergence)

1. Find a constructive method for bounding below the spectral gap in $L^2(M^{-1})$, the space of self-adjointness, say for the Boltzmann operator with hard spheres.

▷ CIRM, April 2017 : coercivity estimates

3. Find a constructive argument to overcome the degeneracy in the space variable, to get an exponential decay for the linear semigroup associated with the linearized spatially inhomogeneous Boltzmann equation; something similar to hypo-ellipticity techniques.

▷ Trieste, May 2017 : hypocoercivity estimates

2. Find a constructive argument to go from a spectral gap in $L^2(M^{-1})$ to a spectral gap in L^1, with all the subtleties associated with spectral theory of non-self-adjoint operators in infinite dimension ...

4. Combine the whole things with a perturbative and linearization analysis to get the exponential decay for the nonlinear equation close to equilibrium.

▷ Granada, June 2017 : extension of spectral analysis and nonlinear problem

- Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995): non-constructive method for HS Boltzmann equation in the torus
- Desvillettes, Villani (2001 & 2005) if-theorem by entropy method
- Villani, 2001 IHP lectures on "Entropy production and convergence to equilibrium" (2008)
- Guo and Guo' school (issues 1,2,3,4)
  2002–2008: high energy (still non-constructive) method for various models
  2010–...: Villani's program for various models and geometries
- Mouhot and collaborators (issues 1,2,3,4)
  2005–2007: coercivity estimates with Baranger and Strain
  2006–2015: hypocoercivity estimates with Neumann, Dolbeault and Schmeiser
  2006–2013: $L^p(m)$ estimates with Gualdani and M.
- Carrapatoso, M., Landau equation for Coulomb potentials, 2017

The results presented in this talk are taken from

- M., *Semigroups in Banach spaces - factorization approach for spectral analysis and asymptotic estimates*, book in preparation

Consider the Boltzmann/Landau equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$
$$F(0, .) = F_0$$

on the density of the particle $F = F(t, x, v) \geq 0$, time $t \geq 0$, velocity $v \in \mathbb{R}^3$, position $x \in \Omega$

- $\Omega = \mathbb{T}^3$ (torus);
- $\Omega \subset \mathbb{R}^3$ + boundary conditions;
- $\Omega = \mathbb{R}^3$ + force field confinement (open problem?).

Q = nonlinear (quadratic) Boltzmann or Landau collisions operator
: conservation of mass, momentum and energy

We recall that $\varphi = 1, v, |v|^2$ are collision invariants, meaning

$$\int_{\mathbb{R}^3} Q(F,F)\varphi \, dv = 0, \quad \forall\, F.$$

$\Rightarrow$ laws of conservation

$$\int_{\mathbb{R}^6} F \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \int_{\mathbb{R}^6} F_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

We also have the H-theorem, namely

$$\int_{\mathbb{R}^3} Q(F,F) \log F \begin{cases} \leq 0 \\ = 0 \ \Rightarrow \ F = \text{Maxwellian} \end{cases}$$

From both pieces of information, we expect

$$F(t,x,v) \underset{t \to \infty}{\longrightarrow} M(v) := \frac{1}{(2\pi)^{3/2}} \, e^{-|v|^2/2}.$$

Existence, uniqueness and stability in small perturbation regime in large space and with constructive rate

Theorem 1. (Gualdani-M.-Mouhot; Carrapatoso-M.; Briant-Guo)

Take an "admissible" weight function m such that

$$\langle v \rangle^{2+3/2} \prec m \prec e^{|v|^2}.$$

There exist some Lebesgue or Sobolev space $\mathcal{E}$ associated with the weight m and some $\varepsilon_0 > 0$ such that if

$$\|F_0 - M\|_{\mathcal{E}(m)} < \varepsilon_0,$$

there exists a unique global solution F to the Boltzmann/Landau equation and

$$\|F(t) - M\|_{\mathcal{E}(m)} \leq \Theta_m(t),$$

with optimal rate

$$\Theta_m(t) \simeq e^{-\lambda t^\sigma} \text{ or } t^{-K}$$

with $\lambda > 0$, $\sigma \in (0,1]$, $K > 0$ depending on m and whether the interactions are "hard" or "soft".

- $(F_t)_{t\geq 0}$ solution to the inhomogeneous Boltzmann equation for hard spheres interactions in the torus with strong estimate

$$\sup_{t\geq 0}\left(\|F_t\|_{H^k}+\|F_t\|_{L^1(1+|v|^s)}\right)\leq C_{s,k}<\infty.$$

- Desvillettes, Villani proved [Invent. Math. 2005]: for any $s\geq s_0$, $k\geq k_0$

$$\forall\, t\geq 0 \qquad \int_{\Omega\times\mathbb{R}^3} F_t \log\frac{F_t}{M(v)}\,dvdx\leq C_{s,k}\,(1+t)^{-\tau_{s,k}}$$

with $C_{s,k}<\infty$, $\tau_{s,k}\to\infty$ when $s,k\to\infty$

Corollary. (Gualdani-M.-Mouhot; Briant-Guo)

$\exists\, s_1, k_1$ s.t. for any $a>\lambda_2$ exists C_a

$$\forall\, t\geq 0 \qquad \int_{\Omega\times\mathbb{R}^3} F_t \log\frac{F_t}{M(v)}\,dvdx\leq C_a\, e^{\frac{a}{2}\,t},$$

with $\lambda_2<0$ (2^{nd} eigenvalue of the linearized Boltzmann eq. in $L^2(M^{-1})$).

First step in Villani's program: quantitative coercivity estimates

We define the linearized Boltzmann / Landau operator in the space homogeneous framework

$$\mathcal{L}h := \frac{1}{2}\Big\{ Q(h,M) + Q(M,h) \Big\}$$

and the orthogonal projection π in $L^2(M^{-1})$ on the eigenspace

$$\mathrm{Span}\{(1, v, |v|^2)M\}.$$

Theorem 2. (..., Guo, Mouhot, Strain)

There exist two Hilbert spaces $\mathfrak{h} = L^2(M^{-1})$ and $\mathfrak{h}_*$ and constructive constants $\lambda, K > 0$ such that

$$(-\mathcal{L}h, g)_{\mathfrak{h}} = (-\mathcal{L}g, h)_{\mathfrak{h}} \le K \|g\|_{\mathfrak{h}_*} \|h\|_{\mathfrak{h}_*}$$

and

$$(-\mathcal{L}h, h)_{\mathfrak{h}} \ge \lambda \, \|\pi^\perp h\|^2_{\mathfrak{h}_*}, \quad \pi^\perp = I - \pi$$

Comments on Theorem 2

• Takes roots in Hilbert, Weyl, Carleman and Grad (non constructive) spectral analysis for the linearized Boltzmann operator

• Degond-Lemou (non constructive) spectral analysis for the linearized Landau operator

• Constructive by Wang Chang et al & Bobylev for Boltzmann operator ($\gamma = 0$) through Hilbert basis decomposition

• Constructive by Desvillettes-Villani for Landau operator ($\gamma = 0$) through log-Sobolev inequality and linearization of the entropy-dissipation of entropy inequality.

• Proved by Mouhot and collaborators (Baranger, Strain) in any cases $\gamma \in [-3, 1]$

• Our aim is to present a new and comprehensive proof :

- Integration by part for Landau operator when $\gamma = 0$
- Integration along the Ornstein-Uhlenbeck flow when $\gamma \sim 0$ (a trick already used by Toscani & Villani in a nonlinear context)
- strictly positive (but not sharp) estimates
- sharp (but not strictly positive) estimates

• Linearized Boltzmann operator (first)

[1] Wang Chang et al 70, Bobylev 88, $\gamma = 0$, L^2 estimate (direct Fourier analysis).

[2] Baranger-Mouhot 05, $\gamma > 0$, L^2 estimate (from [1] - intermediate collisions).

[3] Mouhot 06, $\gamma \in (-3, 1]$, L^2_γ estimate (from [1] for $\gamma < 0$ and [2] for $\gamma > 0$).

• Linearized Landau operator (next)

[4] Desvillettes-Villani 01, $\gamma = 0$, $H^1_{*,0}$ estimate (directly by linearization of nonlinear log-Sobolev inequality).

[5] Baranger-Mouhot 05, $\gamma \geq 0$, L^2 estimate (from [2] - grazing collisions).

[6] Mouhot 06, $\gamma \in (-3, 1]$, H^1_γ estimate (from [4,5] for $\gamma < 0$ and [5] for $\gamma > 0$).

[7] Mouhot-Strain 07, $\gamma \in (-3, 1]$, $H^1_{\gamma,*}$ estimate (from [6]).

Comments on Theorem 2 - scheme of our proof

• Linearized Landau operator (first)

(1) $\gamma = 0$, identity

(2) $\gamma > 0$, from (1) and splitting argument

(3) $\gamma < 0$, from (1) and splitting argument

• Linearized Boltzmann operator (next)

(4) $\gamma \in [0, \gamma^*]$, $\gamma^* > 0$, from (3) associated to $\gamma - 2$ by integration along the flow of the Ornstein-Uhlenbeck semigroup

(5) $\gamma > \gamma^*$, from (4) and splitting argument

(6) $\gamma < 0$, from (4) and splitting argument

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The nonlinear Landau operator is defined by

$$Q_L(F,F) := \mathrm{div}\Big(\int_{\mathbb{R}^d} a(v-v_*)[F_*\,\nabla F - F\,\nabla_* F_*]\,dv_*\Big),$$

with the shorthand $F = F(v)$, $F_* = F(v_*)$. The matrix a is given by

$$a(z) = |z|^{2+\gamma}\,\Pi(z), \quad \Pi_{ij}(z) = \delta_{ij} - \hat{z}_i\hat{z}_j, \quad \forall\, z \in \mathbb{R}^d \backslash \{0\}$$

with

$$\hat{z} = \frac{z}{|z|} \quad \text{and} \quad \gamma \in [-3,1].$$

Observe that $\Pi(z)$ is the orthogonal projection on the plan $z^\perp$, implies $\Pi(z)z = 0$. Introducing the functions

$$b_i(z) = \partial_j a_{ij}(z) = -2\,|z|^\gamma\, z_i,$$
$$c(z) = \partial_{ij} a_{ij}(z) = -2(\gamma+3)\,|z|^\gamma \quad \text{if } \gamma > -3,$$
$$c(z) = \partial_{ij} a_{ij}(z) = -8\pi\delta_0 \quad \text{if } \gamma = -3,$$

we get

$$Q_L(F,F) = \nabla\cdot[a^F\nabla F - b^F F] = a^F_{ij}\partial_{ij}F - c^F F,$$

with $\alpha^F := \alpha * F$.

The linearized Landau operator on a variation $f := F - M$ writes

$$\mathcal{L}f := \mathrm{div}\Big(\int_{\mathbb{R}^d} a(v - v_*)[M_* \nabla f + f_* \nabla M - M \nabla_* f_* - f \nabla_* M_*]\, dv_*\Big),$$

or equivalently

$$\mathcal{L}f = \bar{a}_{ij}\partial_{ij} f - \bar{c}\, f + a^f_{ij}\partial_{ij} M - c^f M,$$

Observing that

$$\Pi(u)\left[M_* \nabla f + f_* \nabla M - M \nabla_* f_* - f \nabla_* M_*\right] = \Pi(u) M M_* \left[\nabla(f/M) - \nabla_*(f_*/M_*)\right],$$

we deduce

$$\int (\mathcal{L}f)\,\varphi = -\frac{1}{2}\int\!\!\int a\left[\nabla(f/M) - \nabla_*(f_*/M_*)\right][\nabla\varphi - \nabla_*\varphi_*]\, M M_* dv dv_*.$$

First consequence, we recover the same collisional invariants as for the nonlinear operator

$$\int (\mathcal{L}f)\varphi\, dv = 0, \quad \forall\, \varphi = 1, v_i, |v|^2.$$

Second consequence, with the choice $\varphi = g/M$, we obtain

$$\begin{aligned}(\mathcal{L}f, g)_{L^2(M^{-1})} &= \int (\mathcal{L}f)\, g\, M^{-1}\, dv \\ &= -\frac{1}{2} \iint a \left[\nabla(f/M) - \nabla_*(f_*/M_*)\right] \left[\nabla(g/M) - \nabla_*(g_*/M_*)\right] M M_* \, dv dv_*.\end{aligned}$$

Because of the symmetry of the RHS expression, we see that

$$(\mathcal{L}f, g)_{L^2(M^{-1})} = (f, \mathcal{L}g)_{L^2(M^{-1})},$$

and the linearized Landau operator $\mathcal{L}$ is a self-adjoint operator in $L^2(M^{-1})$. Finally, with the choice $g = f$ and the notation $h := f/M$, we get the positivity property of the associated Dirichlet form

$$\begin{aligned}D^L_\gamma(h) &:= (-\mathcal{L}f, f)_{L^2(M^{-1})} \\ &= \frac{1}{2} \iint a \left[\nabla h - \nabla_* h_*\right] \left[\nabla h - \nabla_* h_*\right] M M_* \, dv dv_* \geq 0.\end{aligned}$$

Our purpose is now to quantify the positivity property.

For $z \in \mathbb{R}^d \backslash \{0\}$, we define the projection $P = P_z$ on the straight line $\mathbb{R}z$ by

$$P_z \xi := \hat{z}\,(\hat{z} \cdot \xi), \quad \forall\, \xi \in \mathbb{R}^d, \quad \hat{z} := z/|z|.$$

In particular, $\Pi(z) = I - P_z$. We also define the anisotropic gradient

$$\widetilde{\nabla}_v f = P_v \nabla_v f + \langle v \rangle (I - P_v) \nabla_v f$$

and the related Sobolev norm

$$\|h\|^2_{*,\gamma} := \|\langle v \rangle^\gamma \widetilde{\nabla} h\|^2_{L^2(M)} + \|\langle v \rangle^{2+\gamma}\, h\|^2_{L^2(M)}.$$

We finally define

$$L^2_0(M) := \{h \in L^2(M);\ \langle h, \varphi \rangle_{L^2(M)} = 0,\ \forall\, \varphi = 1,\ v_j,\ |v|^2\}$$

$$\mathcal{S}_0 := \{h \in \mathcal{S}(\mathbb{R}^d);\ \langle h, \varphi \rangle_{L^2(M)} = 0,\ \forall\, \varphi = 1,\ v_j,\ |v|^2\}.$$

Lemma 1. (M.)

There holds

$$\frac{1}{2} D_0^L(h) = \|h\|_{**}^2 + \sum_{ij} T_{ij}(h)^2, \quad \forall h \in \mathcal{S}_0,$$

with

$$\|h\|_{**}^2 := \int \Big\{ (d-1)|\nabla h|^2 + |v|^2 \, |(I - P_v)\nabla h|^2 \Big\} M$$

and

$$T_{ij}(h) := \int h \, v_i \, v_j \, M \, dv.$$

In particular, thanks to the (strong) Poincaré inequality, there holds

$$\begin{aligned} \|h\|_{**}^2 &\geq \max\{\|\widetilde{\nabla} h\|_{L^2(M)}^2, \|\nabla h\|_{L^2(M)}^2, \|h\|_{L^2(M)}^2, \lambda_{SP} \|h\langle v\rangle\|_{L^2(M)}^2\} \\ &\geq \lambda \|h\|_{*,0}^2 \end{aligned}$$

for some constants $\lambda_{SP}, \lambda > 0$.

Observe $h \in L^2$ (resp $h \in \mathcal{S}$) implies $\pi^\perp h \in L^2_0$ (resp. $\pi^\perp h \in \mathcal{S}_0$)

We fix $h \in L^2_0(M)$ and we write

$$D^L_0(h) := \frac{1}{2} \int_{\mathbb{R}^{2d}} Y^T [|u|^2 I - u \otimes u] Y \, MM_* \, dvdv_*,$$

with the notations

$$Y := \nabla h - \nabla_* h_*, \quad u = v - v_*.$$

We observe that

$$Y^T [|u|^2 I - u \otimes u] Y = \sum_{i,j} [u_i Y_j - u_j Y_i|^2 = 2 \sum_{i,j} (u_i^2 Y_j^2 - u_i u_j Y_i Y_j).$$

Using a symmetry argument and the notation $h_i = \partial_i h$, $h_i^* = (\partial_i h)^*$, we have

$$\begin{aligned} A_{ij} &:= \int [(v_i - v_i^*)^2 (h_j - h_j^*)^2 - (v_j - v_j^*)(v_i - v_i^*)(h_i - h_i^*)(h_j - h_j^*)] MM_* \\ &= 2 \int [(v_i - v_i^*)^2 (h_j^2 - h_j h_j^*) - (v_i - v_i^*)(v_j - v_j^*)(h_i h_j - h_i h_j^*)] MM_* \\ &= B_{ij} + C_{ij}. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} \frac{1}{2} B_{ij} \quad &:= \quad \int [v_i^2 h_j^2 - 2 v_i v_i^* h_j^2 + v_i^{*2} h_j^2] M M_* \\ & \qquad - \int [v_i^2 h_j h_j^* - 2 v_i v_i^* h_j h_j^* + v_i^{*2} h_j h_j^*] M M_* \\ &= \quad \int [v_i^2 + 1] h_j^2 M + 2\, T_{ij}^2, \end{aligned}$$

where we have used that $\langle v M \rangle = 0$ and two integrations by parts in order to deduce

$$\int v_i v_i^* h_j h_j^* M M_* = \int h \partial_j (v_i M) \int h_* \partial_{*j} (v_i^* M_*) = T_{ij}^2.$$

On the other hand and with the same tricks, we have

$$\begin{aligned}\frac{1}{2} C_{ij} &:= -\int [v_j v_i h_i h_j - v_j v_i^* h_i h_j - v_j^* v_i h_i h_j + v_j^* v_i^* h_i h_j] M M_* \\ &\quad + \int [v_j v_i h_i h_j^* - v_j v_i^* h_i h_j^* - v_j^* v_i h_i h_j^* + v_j^* v_i^* h_i h_j^*] M M_* \\ &:= -\int [v_j v_i h_i h_j + \delta_{ij} h_i^2] M - T_{ij}^2 - T_{ii} T_{jj}.\end{aligned}$$

We deduce

$$\begin{aligned}\frac{1}{2} \sum_{ij} A_{ij} &= (d-1) \int |\nabla h|^2 M + \int \sum_{ij} (v_i^2 h_j^2 - v_j v_i h_i h_j)\, M \\ &\quad + \sum_{ij} T_{ij}^2 - \Big(\sum_i T_{ii} \Big)^2.\end{aligned}$$

We observe that the last term vanish because

$$\sum_i T_{ii} = \int |v|^2 hM = 0$$

and we compute

$$\begin{aligned}
\sum_{ij} (v_j^2 h_i^2 - v_j v_i h_i h_j) &= |v|^2 \sum_i \Big\{ h_i^2 - 2\hat{v}_i h_i \sum_j \hat{v}_j h_j + \hat{v}_i^2 \Big(\sum_j \hat{v}_j h_j \Big)^2 \Big\} \\
&= |v|^2 \sum_i \Big(h_i - \hat{v}_i \sum_j \hat{v}_j h_j \Big)^2 \\
&= |v|^2 |(I - P_v) \nabla h|^2.
\end{aligned}$$

We conclude by putting all the terms together.

Lemma 2.

There exist $K_1, K_2 > 0$, such that

$$-(\mathcal{L}f, f)_{L^2(M^{-1})} \quad \geq \quad K_1 \|f/M\|^2_{*,\gamma} - K_2 \, \|f\|^2_{L^2}, \quad \forall\, f \in \mathcal{S}.$$

Idea of the proof:

$$Lh := M^{-1}\mathcal{L}(Mh) \simeq \bar{a}_{ij}\partial^2_{ij} h + ...$$

with leader term

$$\bar{a}_{ij}\xi_i\xi_j \approx \langle v\rangle^\gamma |P_v\xi|^2 + \langle v\rangle^{\gamma+2}|(I - P_v)\xi|^2, \quad -\partial_i \bar{a}_{ij}\, v_j \approx \langle v\rangle^{\gamma+2}.$$

Lemma 3.

There exist $K_3 > 0$, such that

$$D^L_\gamma(h) := -(\mathcal{L}f, f)_{L^2(M^{-1})} \quad \geq \quad K_3 \|f\|^2_{L^2}, \quad \forall f \in \mathcal{S}_0.$$

Both estimates together give

Theorem 2 holds for the Landau operator for any $\gamma \in [-3, 1]$ with

$$\|f\|_{\mathfrak{h}_*} := \|f/M\|^2_{*,\gamma}$$

We fix $h \in \mathcal{S}_0$ and for any $r \in (0,1)$, we write

$$\begin{aligned} D^L_\gamma(h) &\ge r^\gamma \int\!\!\int \mathbf{1}_{|u|\ge r}\, Y^T[|u|^2 I - u\otimes u] Y\, MM_*\, dvdv_* \\ &= r^\gamma D^L_0(h) - \varepsilon_r(h), \end{aligned}$$

with

$$\begin{aligned} \varepsilon_r(h) &:= \frac{r^\gamma}{2} \int_{\mathbb{R}^{2d}} \mathbf{1}_{|u|\le r} Y^T[|u|^2 I - u\otimes u] Y\, MM_*\, dvdv_* \\ &\le 2\, r^{\gamma+2} \int_{\mathbb{R}^{2d}} |\nabla h|^2\, MM_*\, dvdv_* \\ &= 2\, r^{\gamma+2} \, \|\nabla h\|^2_{L^2(M)} \end{aligned}$$

Using the estimate for the Maxwell molecules case $\gamma = 0$, we have in particular

$$D^L_0(h) \ge 2(d-1)\|\nabla h\|^2_{L^2(M)}.$$

Gathering the above three inequalities, we deduce

$$D^L_\gamma(h) \geq 2\|\nabla h\|^2_{L^2(M)}((d-1)\, r^\gamma - r^{\gamma+2}) \geq K\|\nabla h\|^2_{L^2(M)},$$

with $K > 0$ and $r > 0$ small enough.

Using finally Poincaré inequality, we obtain our first inequality

$$D^L_\gamma(h) \geq K'\|h\|^2_{L^2(M)}.$$

We also recall that from Lemma 2, we have

$$D^L_\gamma(h) \geq C_1\,\|h\|^2_{*,\gamma} - C_2\,\|h\|^2_{L^2}.$$

The two last inequalities together, we deduce that

$$D^L_\gamma(h) \geq \lambda C_1\,\|h\|^2_{*,\gamma} + [(1-\lambda)K - \lambda C_2]\,\|h\|^2_{L^2},$$

from what we conclude by choosing $\lambda > 0$ small enough.

Proof of Lemma 3 in the case $\gamma < 0$

We fix $h \in \mathcal{S}_0$ and we write

$$D^L_\gamma(h) = \iint |u|^{\gamma+2}\, \Delta_h M M_*\, dv dv_*,$$

with the notation

$$\Delta_h = \Delta_h(v, v_*) = |\Pi(u)\, (\nabla_v h - \nabla_{v_*} h_*)|^2.$$

Introducing the change of variables

$$x = \frac{1}{\sqrt{2}}(v - v_*), \quad y = \frac{1}{\sqrt{2}}(v + v_*),$$

and using $|x|^\gamma M(x) \gtrsim M(\eta\, x)$ and $M(y) \gtrsim M(\eta\, y)$ for any $\eta > 1$, we have

$$\begin{aligned} D^L_\gamma(h) &= C_1 \iint |x|^{\gamma+2}\, \Delta_h M(x) M(y)\, dx dy \\ &\geq C_{2,\eta} \iint |x|^2\, \Delta_h\, M(\eta\, x) M(\eta\, y)\, dx dy \\ &= C_{3,\eta} \iint |u|^2\, \Delta_h(v/\eta, v_*/\eta)\, M M_*\, dv dv_*, \end{aligned}$$

for some constants $C_1, C_{i,\eta} \in (0, \infty)$.

Observing that

$$\Delta_h(v/\eta, v_*/\eta) = \Delta_{h_{\eta^{-1}}}(v, v_*)$$

with $h_\eta(w) := h(w/\eta)$, we get

$$D^L_\gamma(h) \geq C_{3,\eta} D^L_0(h_{\eta^{-1}}).$$

Introducing the function $\phi(v) := a_\eta + b_\eta \cdot v + c_\eta |v|^2$, where

$$(a_\eta, b_\eta, c_\eta) := \eta^{2+d} \int_{\mathbb{R}^d} h \left(\frac{d+2}{2\eta^2} - |v|^2, v, \frac{\eta^2}{2d}|v|^2 - \frac{1}{2} \right) M_\eta \, dv,$$

we have

$$h_{\eta^{-1}} - \phi_{\eta^{-1}} \in L^2_0(M).$$

As a consequence of the positivity of the Dirichlet form in the case $\gamma = 0$, we get

$$\begin{aligned} D^L_\gamma(h) &\geq C_{3,\eta} \| h_{\eta^{-1}} - \phi_{\eta^{-1}} \|^2_{L^2(M)} \\ &\geq C_{4,\eta} \left(\|h\|_{L^2(M_\eta)} - \|\phi\|_{L^2(M_\eta)} \right)^2 \\ &\geq C_{5,\eta} \left\{ \|h\|_{L^2(M_\eta)} - K \left(a^2_\eta + |b_\eta|^2 + c^2_\eta \right) \right\}, \end{aligned}$$

for a numerical constant $K \in (0, \infty)$ in the range $\eta \in (1, \sqrt{2})$.

Using the vanishing moment conditions on h, we easily estimate

$$a_\eta^2 + |b_\eta|^2 + c_\eta^2 \quad \lesssim \quad \varepsilon(\eta) \|h\|^2_{L^2(M_\eta)},$$

with $\varepsilon(\eta) \to 0$ when $\eta \to 1$
We may then fix $\eta \in (1, \sqrt{2}]$ small enough, such that

$$D^L_\gamma(h) \quad \geq \quad C_{6,\eta} \, \|h\|^2_{L^2(M_{\sqrt{2}})} = C_{7,\eta} \, \|h\|^2_{L^2(M^2)}.$$

On the other hand, from Lemma 2, for any $h \in \mathcal{S}(\mathbb{R}^3)$, we have

$$D^L_\gamma(h) \geq K_1 \, \|h\|^2_{*,\gamma} - K_2 \, \|h\|^2_{L^2(M^2)}.$$

Putting together the above two estimates, we easily end the proof of Lemma 3.

Outline of the talk

The nonlinear collision Boltzmann operator Q_B is defined by

$$Q_B(F,F) := \int_{\mathbb{R}^3} \int_{S^2} \Gamma(v - v_*)\, b(\cos\theta)\, (F' F'_* - F F_*)\, d\sigma dv_*,$$

and we use the shorthands $F = F(v)$, $F' = F(v')$, $F_* = F(v_*)$ and $F'_* = F(v'_*)$. Moreover, v' and v'_* are parametrized by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\,\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\,\sigma, \qquad \sigma \in \mathbb{S}^2.$$

Finally, $\theta \in [0, \pi]$ is the *deviation angle* between $v' - v'_*$ and $v - v_*$ defined by

$$\cos\theta = \sigma \cdot \hat{u}, \quad u = v - v_*, \quad \hat{u} = \frac{u}{|u|},$$

and $\Gamma\, b$ is the *collision kernel* determined by the physical context of the problem. We consider

$$\Gamma(z) = |z|^\gamma,\ \gamma \in (-3, 1], \quad b \in L^1 \text{ (Grad's cut-off)}.$$

The linearized Boltzmann operator on a variation $f := F - M$ writes

$$\mathcal{L}f := \int_{\mathbb{R}^3} \int_{S^2} \Gamma b \, (f' M'_* + M' f'_* - f \, M_* - M \, f_*) \, d\sigma dv_*.$$

Observing that $M' M'_* = M M_*$, denoting $h := f/M$ and using changes of variables

$$\int (\mathcal{L}f) \, \varphi = -\frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \Gamma b \, (h' + h'_* - h - h_*)(\varphi' + \varphi'_* - \varphi - \varphi_*) \, M M_* d\sigma dv dv_*,$$

for any nice function $\varphi : \mathbb{R}^3 \to \mathbb{R}$.
As for the linearized Landau equation, we deduce that the collisional invariants are the mass, momentum and energy, that the operator is self-adjoint in $L^2(M^{-1})$ and the non-negativity of the Dirichlet form

$$\begin{aligned} D^B_\gamma(h) &:= -(\mathcal{L}f, f)_{L^2(M^{-1})} \\ &= \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \Gamma b \, (h' + h'_* - h - h_*)^2 \, M M_* d\sigma dv dv_* \geq 0, \end{aligned}$$

which is nothing but the linearized version of the H-Theorem for the Boltzmann equation.

We aim now to establish an optimal lower bound on $D^B_\gamma(h)$ in the space $L^2_0(M)$.

Theorem 2 holds for the Boltzmann operator for any $\gamma \in (-3, 1]$ with

$$\|f\|_{\mathfrak{h}_*} := \|f/M\|^2_{L^2(M\langle v\rangle^\gamma)} = \|f\|^2_{L^2(M^{-1}\langle v\rangle^\gamma)}$$

It is a consequence of the sharp (but not positive) estimate

Lemma 4.

There exist $K_1, K_2 \in (0, \infty)$ such that

$$D^B_\gamma(h) \geq K_1 \, \|h\|^2_{L^2(\langle v\rangle^\gamma M)} - K_2 \, \|h\|^2_{L^2(M^2)}, \quad \forall h \in \mathcal{S}(\mathbb{R}^3),$$

together with the next strictly positive (but not sharp) estimates

Lemma 5.

There exist $\gamma^* \in (0,1)$ and $\lambda > 0$ such that for any $\gamma \in [0,\gamma^*]$

$$D^B_\gamma(h) \ge \lambda \, \|h\|^2_{L^2(\langle v\rangle^{\gamma-2} M)}, \quad \forall h \in \mathcal{S}_0.$$

For the proof, we mainly follow Villani's paper "Cercignani's conjecture is sometimes true and always almost true" (03) (as suggested to us by Mouhot).

We take $b_0 = 1$, $\gamma \in [0,1)$ to be fixed later and $h \in \mathcal{S}_0$.
Thanks to the Jensen inequality, we have

$$4D^B_\gamma(h) \ge \int\!\!\int |u|^\gamma \, q^2 \, MM_* \, dvdv_* := \bar{D}_\gamma(h),$$

with

$$q := H - G, \quad H = h + h_*, \quad G = \frac{1}{|S^2|} \int_{S^2} (h' + h'_*) \, d\sigma.$$

We define the Ornstein-Uhlenbeck operator

$$\mathcal{C} h := \Delta_w h - w \cdot \nabla_w h,$$

either on $w = v \in \mathbb{R}^d$ or $w = (v, v_*) \in \mathbb{R}^{2d}$ and the corresponding semigroup U_t.
We recall that $U_t h = \mathcal{O}(e^{at})$ in $L^2(\langle v \rangle M)$ as $t \to \infty$ with $a < 0$.
As a consequence $U_t q = \mathcal{O}(e^{at})$ in the product space $L^2(\langle v \rangle M \langle v_* \rangle M_*)$ and

$$\bar{D}_\gamma(U_t h) = \mathcal{O}(e^{2at}) \text{ as } t \to \infty.$$

With the notation $\mathcal{A} = \nabla := (\nabla_v, \nabla_{v_*})$, we have $\mathcal{C} = -\mathcal{A}^*\mathcal{A}$ on $L^2(MM_*)$ and we compute

$$2q\,\mathcal{C} q = -2|\nabla q|^2 + \mathcal{C} q^2,$$

from what we deduce

$$\begin{aligned} -\frac{d}{dt}\bar{D}_\gamma(U_t h) &= -2\iint |u|^\gamma (U_t q)\,\mathcal{C}(U_t q)\, MM_* \\ &= 2\iint |u|^\gamma |\nabla(U_t q)|^2\, MM_* + \iint \mathcal{A}|u|^\gamma \cdot \mathcal{A}\big((U_t q)^2\big)\, MM_*. \end{aligned}$$

We introduce the linear operator from $\mathbb{R}^{2d}$ to $\mathcal{B}(\mathbb{R}^{2d}, \mathbb{R}^d)$ defined by

$$\mathcal{P} : (A, B) \mapsto \Pi_{v-v_*}(A - B),$$

where A and B stand for the component in $\mathbb{R}^d_v$ and $\mathbb{R}^d_{v_*}$.
We estimate

$$\begin{aligned} 2|\nabla(U_t q)|^2 &\geq |\mathcal{P}\nabla(U_t H) - \mathcal{P}\nabla(U_t G)|^2 \\ &= |\mathcal{P}\nabla(U_t H)|^2 = |\Pi_{v-v_*}(\nabla U_t h - \nabla_* U_t h_*)|^2, \end{aligned}$$

where we have used $\|\mathcal{P}\|_{L^\infty(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}, \mathbb{R}^d))} \leq \sqrt{2}$ and the fact that G only depends on $|v - v_*|$ thanks to the parallelogram identity, so does $U_t G$.
Using the coercivity estimate for the Dirichlet form $D^L_{\gamma-2}$, we get

$$\begin{aligned} 2 \iint |u|^\gamma |\nabla(U_t q)|^2 \, MM_* \, dvdv_* &\geq \iint |u|^\gamma |\Pi_u(\nabla U_t h - \nabla_* U_t h_*)|^2 \, MM_* \, dvdv_* \\ &= D^L_{\gamma-2}(U_t h) \\ &\geq \lambda_L \int |\nabla_v (U_t h)|^2 \langle v \rangle^{\gamma-2} M \, dv, \end{aligned}$$

for a constant λ_L which is uniform with respect to $\gamma \in [0, 1]$.

On the other hand, we have

$$\left|\mathcal{A}|u|^\gamma \cdot \mathcal{A}\big((U_t q)^2\big)\right| \lesssim \gamma\,|u|^{\gamma-1}\,|U_t q|^2\,|\nabla U_t q|).$$

Observing that for any $h_t^\dagger, h_t^\ddagger \in \{U_t h, U_t h_*, U_t h', U_t h'_*\}$, we have

$$\left|\iint |u|^{\gamma-1}\,|h_t^\dagger|\,|\nabla h_t^\ddagger| M M_*\, dv dv_*\right| \lesssim \Big(\iint |u|^\gamma\,|h_t^\dagger|^2\, M M_*\, dv dv_*\Big)^{1/2} \Big(\iint |u|^{\gamma-2}\,|\nabla h_t^\ddagger|^2\, M M_*\, dv dv_*\Big)^{1/2},$$

we deduce

$$\left|\iint \mathcal{A}|u|^\gamma \cdot \mathcal{A}\big((U_t q)^2\big) M M_*\, dv dv_*\right| \lesssim \gamma\,\|U_t h\|_{L^2(\langle v\rangle^\gamma M)}\|\nabla U_t h\|_{L^2(\langle v\rangle^{\gamma-2} M)}.$$

Uniformly in $0 \le \gamma \le \gamma^*$, $\gamma^* \in (0,1)$ small enough, we obtain

$$\begin{aligned} -\frac{d}{dt}\bar{D}_\gamma(U_t h) &\ge \frac{\lambda_L}{2}\|\nabla U_t h\|^2_{L^2(M\langle v\rangle^{\gamma-2})} - \gamma^* \, C\|U_t h\|^2_{L^2(M\langle v\rangle^{\gamma})} \\ &\ge \frac{\lambda_L}{4}\|\nabla U_t h\|^2_{L^2(M\langle v\rangle^{\gamma-2})}, \end{aligned}$$

by using the strong Poincaré inequality for the probability measure $cM\langle v\rangle^{\gamma-2}$.

We recall here that for the Ornstein-Uhlenbeck semigroup, there holds

$$\begin{aligned} -\frac{d}{dt}\|U_t h\|^2_{L^2(M\langle v\rangle^{\gamma-2})} &\lesssim K\|\nabla U_t h\|^2_{L^2(M\langle v\rangle^{\gamma-2})} + \|U_t h\|^2_{L^2(M\langle v\rangle^{\gamma})} \\ &\le K\|\nabla U_t h\|^2_{L^2(M\langle v\rangle^{\gamma-2})}, \end{aligned}$$

by using again the strong Poincaré inequality for the measure $M\langle v\rangle^{\gamma-2}$ and the constraint $\langle U_t h M\rangle = 0$. The two last differential inequalities yields

$$-\frac{d}{dt}\bar{D}_\gamma(U_t h) \ge -\frac{\lambda_L}{4K}\frac{d}{dt}\|U_t h\|^2_{L^2(M\langle v\rangle^{\gamma-2})}.$$

We conclude by integrating in time that differential inequation.

Lemma 6. For any $\gamma \in (\gamma^*, 1]$, there exists $\lambda > 0$ such that

$$D^B_\gamma(h) \geq \lambda \, \|h\|^2_{L^2(M)}, \quad \forall h \in \mathcal{S}_0.$$

We proceed as for the Landau operator. Denoting

$$\Delta_h := \int_{S^2} [h + h_* - h' - h'_*]^2 \, b \, d\sigma,$$

for any $r \in (0,1)$, we write

$$\begin{aligned} D^B_\gamma(h) &\geq r^{\gamma-\gamma^*} \int \mathbf{1}_{|u| \geq r} |u|^{\gamma^*} \, \Delta_h \, M M_* \, dv dv_* \\ &= r^{\gamma-\gamma^*} D^B_{\gamma^*}(h) - \varepsilon_r(h), \end{aligned}$$

with

$$\varepsilon_r(h) := r^{\gamma-\gamma^*} \int \mathbf{1}_{|u| \leq r} |u|^{\gamma^*} \, \Delta_h \, M M_* \, dv dv_* \leq r^\gamma \, C \, \|h\|^2_{L^2(M)}.$$

Using Theorem 2 for $D^B_{\gamma^*}(h)$, we deduce

$$D^B_\gamma(h) \geq r^{\gamma-\gamma^*} \, (\lambda - C \, r^{\gamma^*}) \, \|h\|^2_{L^2(M)}.$$

Lemma 7. For any $\gamma \in (-3, 0)$, there exists $\lambda > 0$ such that

$$D^B_\gamma(h) \ge \lambda \, \|h\|^2_{L^2(M^2)}, \quad \forall h \in \mathcal{S}_0.$$

For any $\eta > 1$, there exist some constants C_1, $C_{i,\eta} \in (0, \infty)$, such that

$$\begin{aligned} D^B_\gamma(h) &= C_1 \iint |x|^\gamma \, \Delta_h M(x) M(y) \, dxdy \\ &\ge C_{3,\eta} \iint \Delta^\eta_h \, MM_* \, dvdv_* = C_{3,\eta} \, D^B_0(h_{\eta^{-1}}), \end{aligned}$$

where

$$x = \frac{1}{\sqrt{2}}(v - v_*), \quad y = \frac{1}{\sqrt{2}}(v + v_*), \quad h_\eta(w) := h(w/\eta)$$

$$\Delta^\eta_h := \int_{S^2} \big\{ h(v'(v/\eta, v_*/\eta, \sigma)) + h(v'_*(v/\eta, v_*/\eta, \sigma)) - h(v/\eta) - h(v_*/\eta) \big\}^2 \, d\sigma.$$

From the positivity estimate of the Dirichlet form D^B_0, we have

$$D^B_\gamma(h) \ge C_{4,\eta} \, \|h_{\eta^{-1}} - \phi_{\eta^{-1}}\|^2_{L^2(M)},$$

where ϕ is defined as for the Landau operator and we conclude in the same way.