The isotropic Landau equation

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Introduction

- One of the most interesting physical cases in gas and plasma physics is collisions of particles under the influence of Coulomb potential $1/r$.
- For this potential the Boltzmann equation in not a valid model anymore. The reason is that the **momentum exchanged among particles during a collision is divergent as $\theta \to 0$.**
- The physical explanation is that grazing collisions cannot be neglected when the potential is of Coulomb type.
- This problem was known by Landau, who in 1936 derived a kinetic equation that describes collisions of particles in plasma where grazing collisions are predominant. This equation was later named the **Landau equation**.

The Landau Equation

The time evolution of the particle density is described by

$$\partial_t f + x \cdot \nabla_v f + \nabla_x \phi \cdot \nabla_x f = Q(f, f)$$

with

$$Q(f, f) = \mathrm{div}_v \int_{\mathbb{R}^3} |v - y|^{\gamma+2} \Pi(v - y)[f(y)\nabla_v f(v) - f(v)\nabla_y f(y)]\, dy$$

and

$$\Pi(v) := Id - \frac{v \otimes v}{|v|^2}, \qquad -3 \leq \gamma \leq 1.$$

The Coulomb case corresponds to $\gamma = -3$.

Relevant literature (far from complete!)

The mathematical analysis heavily depends on the value of γ. We refer to hard potentials when $\gamma \geq 0$ and to soft potentials when $\gamma < 0$. The level of difficulty increases as $\gamma \to -3$.

- for $\gamma > 0$ global well-posedness was proven by Desvillettes and Villani in 2002.
- for $-2 < \gamma < 0$ Wu, Fournier-Guerin, Alexandre-Liao-Lin, Alexandre-Villani showed existence, uniqueness of solution and propagation of L^p estimates.

Literature (continued)

For $\gamma < -2$ much less is known:

- Arsenev-Peskov '77 showed **existence of weak solutions**, **uniqueness** by Fournier '10.
- Villani in '98 proved existence of the so called H-solutions.
- Guo in 2002 proved existence of smooth solutions when initial data are close to equilibrium.
- Alexander, Liao and Lin '13 gave a proof of existence of weak solutions in weighted L^2 -space under smallness assumption on initial data.
- Desvillettes '15 showed that the Villani's H-solutions are indeed weak-solutions.
- Carrapatoso, Desvillettes and He '16 proved time **convergence to equilibrium**.

More literature: higher regularity

$$\partial_t f + x \cdot \nabla_v f = Q(f, f)$$

with

$$Q(f,f) = \mathrm{div}_v \int_{\mathbb{R}^3} |v-y|^{\gamma+2}\Pi(v-y)[f(y)\nabla_v f(v) - f(v)\nabla_y f(y)]\, dy$$

Higher regularity $L^2(Q) \to C^\alpha(Q_{1/2})$

- Using De Giorgi-Nash and Moser's method: Golse, Imbert, Mouhot, Vasseur '16 (inhomogeneous equation, $\gamma = -3$)
- Using Krylov-Safonov method: Silvestre '15 (homogeneous): Global L^∞ - bounds for $\gamma > -2$
- Cameron, Silvestre and Nelson ('17) for $\gamma > -2$ (inhomogeneous).

Higher regularity

Let f be a solution to the homogeneous Landau equation with bounded local mass, energy and entropy. Then (Golse-Imbert-Mouhot-Vasseur '16) showed that

$$\|f\|_{C^\alpha(Q_{1/2})} \leq C(\|f\|_{L^\infty(Q_1)}^{1-\gamma/d} + \|f\|_{L^2(Q_1)}).$$

This result has been proven by applying Hoelder regularity theory for kinetic equations with rough coefficients to the solution to the Landau equation.
On the other hand, starting from the $L^2 \to L^\infty$ result on the linear equation by Golse-Imbert-Mouhot-Vasseur, with a rescaling argument and a change of variable Cameron-Silvestre-Snelson ('17) show that globally

$$f(x,v,t) \leq C(1+t^{-3/2})\frac{1}{1+|v|}.$$

The Coulomb Potential $\gamma = -3$

For $\gamma = -3$ **global well-posedness theory is still missing!**

- The issue of regularity (i.e. no finite time break down occurs) for all times has remained open.
- A blow-up configuration would become realistic if at some point the diffusion is not sufficient to prevent the instability caused by the **quadratic nonlinearity**.

The homogeneous Landau equation (with $\gamma = -3$) exhibits a **quadratic nonlinearity**:

$$\partial_t f = \mathrm{div}_v \int_{\mathbb{R}^3} \frac{1}{|v-y|} \Pi(v-y)(f(y)\nabla_v f(v) - f(v)\nabla_y f(y))\, dy$$

$$= \mathrm{div}_v \left(A[f]\nabla f - f\nabla a[f]\right) = Tr(A[f]D^2 f) + f^2$$

with

$$A[f] = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{1}{|v-y|} \Pi(v-y) f(y)\, dy, \qquad a[f] = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|v-y|}\, dy$$

$$\partial_t f = Tr(A[f]D^2 f) + f^2$$

Difficulties:

- Quadratic non-linearity,
- Degeneracy and unboundness of the diffusion coefficient

$$\frac{C(\|f\|_{L^1})}{(1+v)^3} < A[f] <?$$

- Non-locality of the diffusion coefficients $A[f]$ prevents comparison principle:

$$f(v,t) \quad < \quad g(v,t), \quad \text{and} \quad f(v_0,t_0) \quad = \quad g(v_0,t_0)$$

$$\Rightarrow \Delta f \leq \Delta g \quad \text{and} \quad A[f] \leq A[g]$$

$$\nRightarrow A[f]\Delta f \leq A[g]\Delta g$$

The isotropic Landau equation

Consider a modification of the Landau equation; **isotropic Landau equation**

$$\partial_t f = div\,(TrA[f]\nabla f - f\nabla a[f]) = div\,(a[f]\nabla f - f\nabla a[f])$$

- Previously Gressmann-Krieger-Strain in '12 showed that any solution to

$$\partial_t f = div\,(a[f]\nabla f - f\nabla a[f]) - \alpha f^2 \qquad \alpha \gg 0$$

stays bounded for all time.

Theorem

(G., Guillen, '16) The isotropic Landau equation

$$\begin{cases} \partial_t f & = div(a[f]\nabla f - f\nabla a[f]) = a[f]\Delta f + f^2 \\ f|_{t=0} & = f_{in}, \end{cases}$$

*with radially symmetric and decreasing (but not small!) initial condition f_{in} has **bounded smooth solutions for all times** $t > 0$.*

New idea: Find **partial barriers**. Since we deal with radially symmetric decreasing functions, we direct our attention to a neighborhood of $v = 0$.

Barrier to get higher integrability

Lemma: Let $f(v,t)$ be a solution to

$$\partial_t f = \mathrm{div}(a[g]\nabla f - f\nabla a[g]) = Q(g,f)$$

Assume

$$\frac{|v|^2 g(v,t)}{a[g](v,t)} \le \alpha(1-\alpha), \quad 0 < \alpha < 1,$$

then

$$f(v,t) \le \frac{C}{|v|^\alpha}.$$

Proof: Rewrite the equation in spherical coordinates and get

$$Q(g,|v|^{-\alpha}) = |v|^{-\alpha}(g - \alpha(1-\alpha)a[g]|v|^{-2}) < 0$$

Maximum principle implies

$$f \le \frac{C}{|v|^\alpha} \quad v \in B_R.$$

From $L^2 \to L^\infty$

Lemma: Any solution $f(v,t)$ to

$$\partial_t f = \mathrm{div}(a[g]\nabla f - f\nabla a[g])$$

such that

$$f \le \frac{C}{|v|^\alpha}, \quad v \in B_R,$$

satisfies

$$\sup_{t>0,\mathbb{R}^3} f(v,t) < +\infty.$$

Proof: The barrier $f \le \frac{C}{|v|^\alpha}$ implies

- $f \in L^2$
- $a[f],\ \nabla a[f] \in L^\infty$
- Use Stampacchia's theorem to get the bound: given $\lambda < B < \Lambda$, any weak solutions to

$$\partial_t u \le \mathrm{div}(B\nabla u + ub)$$

satisfies

$$\|u\|_{L^\infty(Q_{1/2})} \le C(\lambda,\Lambda)(\|b\|_{L^\infty(Q)} + \|u\|_{L^2(Q)}).$$

The sufficient condition for preventing blow-up

All what remains to see is whether the condition

$$\frac{|v|^2 f(v,t)}{a[f](v.t)} \leq \alpha(1-\alpha) \quad 0 < \alpha < 1,$$

is true. We first observe that

$$\frac{|v|^2 f(v,t)}{a[f](v,t)} \leq c\frac{1}{|v|}\int_{B(0,|v|)} f(y,t)\, dy.$$

$\Longrightarrow$ **Sufficient condition** for our assumption to hold is

$$\int_{B(0,|v|)} f(y,t)\, dy =: M(|v|,t) \leq |v|^{1+\beta}.$$

A barrier argument argument for the mass function

The function $M(r,t) := \int_{B_r} f(y,t)\, dy$ satisfies the following non-linear equation

$$\partial_t M = a[f]\partial_{rr} M + \frac{2}{r}\left(\frac{M}{8\pi r} - a[f]\right)\partial_r M$$

A simple barrier argument shows that

$$M(r,t) \leq r^m$$

for $m \leq 2$.

Why do energy estimates not work?

Recall the Landau equation for any potentials $-3 \leq \gamma \leq 0$:

$$\partial_t f = \mathrm{div}_v \left(A[f]\nabla f - f\nabla a[f] \right)$$

with

$$\Delta a[f] = -h, \quad h = \int_{\mathbb{R}^3} \frac{f(y)}{|v-y|^{-\gamma}}\, dy, \quad \gamma > -3$$

and

$$h = f \quad \text{if} \;\; \gamma = -3.$$

Multiply by $f^{p-1}$and integrate by parts; one obtains

$$\frac{p}{(p-1)}\partial_t \int f^p \, dv = -4 \int \langle A[f]\nabla f^{p/2}, \nabla f^{p/2} \rangle \, dv + p \int f^p h \, dv$$

Energy estimates (continued)

Goal is to control the higher order term with the coercive term:

$$\int f^p h \, dv \leq C \int \langle A[f] \nabla f^{p/2}, \nabla f^{p/2} \rangle \, dv$$

with C small enough. But this is impossible!
.... unless one can prove that (Gualdani-Guillen '17)

$$\int f^p h \, dv \leq \varepsilon \int \langle A[f] \nabla f^{p/2}, \nabla f^{p/2} \rangle \, dv + C_\varepsilon \int f^p \, dv$$

with ε small as one wishes. This brings us back to the theory of weighted Sobolev's and Poincare's inequalities (Chanillo-Sewer-Wheeden '80).

Weighted Poincare's and Sobolev inequality

Let $1 < p \leq q < +\infty$ and $w(x)$and $v(x)$ two measurable functions. If

$$|Q_r|^{2/3} \frac{\int_{Q_r} v(x)\, dx}{\int_{Q_r} w(x)\, dx} \leq c, \quad \forall Q_r \subset Q$$

then

$$\int_Q |f - f(Q)|^q v(x)\, dx \leq c\left(\int_Q |\nabla f|^p w(x)\, dx\right)^{q/p}$$

A weighted Poincare's for Landau

If one can show that there exists a modulus of continuity $\eta(r)$ such that

$$|Q_r|^{2/3} \frac{\int_{Q_r} h(v)\, dv}{\int_{Q_r} a^*(v)\, dv} \le \eta(r), \quad \forall Q_r \subset Q$$

where a^* is the smallest eigenvalue of the Landau diffusion matrix

$$A[f](v) = \int |v-y|^{\gamma+2} \Pi(v-y) f(y)\, dy$$

then any smooth function satisfies the ε-**Poincare inequality**

$$\int_Q f^p h\, dv \le \varepsilon \int_Q a^* |\nabla f^{p/2}|^2\, dv + C_\varepsilon \int_Q f^p\, dv$$

Weighted Poincare's for moderately soft potentials

When $\gamma > -2$ one can show that

$$|Q_r|^{2/3} \frac{\int_{Q_r} h(v)\, dv}{\int_{Q_r} a^*(v)\, dv} \leq c|Q_r|^{2+\gamma}, \quad \forall Q_r \subset Q$$

For $\gamma \leq -2$ so far we are only able to show that

$$|Q_r|^{2/3} \frac{\int_{Q_r} h(v)\, dv}{\int_{Q_r} a^*(v)\, dv} \leq C, \quad \forall Q_r \subset Q$$

Long time behavior

Theorem

(G. Guillen, '17) Any solution to the Landau equation with $\gamma > -2$ *stays bounded for all times and*

$$\|f\|_{L^\infty} \leq C + \frac{1}{t^{\frac{d}{2}}}.$$

Idea of the Proof

- Energy Estimates with cut-off test function $\chi_Q f^{q-1}$

$$\frac{q}{q-1}\partial_t \int_Q f^q \, dv \leq -4 \int_Q a^* |\nabla f^{q/2}|^2 \, dv + \int_Q f^q h \, dv + l.o.t.$$

$$\leq -(4-\varepsilon) \int_Q a^* |\nabla f^{q/2}|^2 \, dv + \frac{1}{\varepsilon} \int_Q f^q \, dv + l.o.t$$

- Moser's iteration: let $p_k = (q/2)^k$and show that

$$\int_{T/2}^{T} \int_Q a[f] f^{p_k} \, dv \leq C \quad \text{for all } k > 0.$$

Thank you!