

A coupling approach to the (kinetic) Langevin equation

Andreas Eberle

University of Bonn

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Outline

Introduction

The coupling

The (semi-)metric

Main result

Introduction

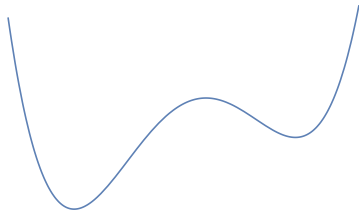
Langevin dynamics

$$dX_t = V_t dt$$

$$dV_t = -\nabla U(X_t) dt - \gamma V_t dt + \sqrt{2\gamma} dB_t$$

Invariant probability measure

$$\mu(dx dv) \propto e^{-U(x) - |v|^2/2} dx dv$$

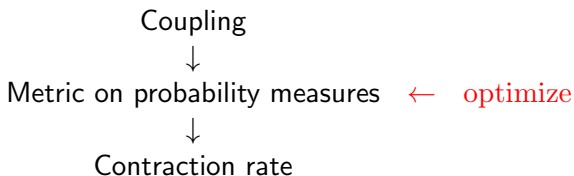


Analytic approaches

- Hérau/Nier (Witten Laplacian)
- Villani (Hypocoercivity), Dolbeault/Mouhot/Schmeiser

Probabilistic approach

- Coupling, Wasserstein distances, ...



Kantorovich (L^1 Wasserstein) distance based on ρ :

$$\mathcal{W}_\rho(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[\rho(X, Y)], \quad \mu, \nu \in \text{Prob}(\mathbb{R}^d).$$

Couplings for Langevin equation

$$dX'_t = V'_t dt$$

$$dV'_t = -\nabla U(X'_t) dt - \gamma V'_t dt + \sqrt{2\gamma} dB'_t$$

A. Synchronous coupling

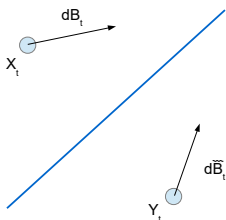
$$dB'_t = dB_t, \quad Z_t := X_t - X'_t, \quad W_t := V_t - V'_t$$

$$\Downarrow$$

$$dZ_t = W_t dt$$

$$dW_t = -(\nabla U(X_t) - \nabla U(X'_t)) dt - \gamma W_t dt$$

B. Reflection coupling



$$dB'_t = (I_d - 2e_t e_t^T) dB_t, \quad e_t = \frac{Q_t}{|Q_t|}$$

$$d(B_t - B'_t) = 2e_t e_t^T dB_t$$

Lévy's characterization $\Rightarrow B'$ Brownian motion.

C. Sticky coupling

$$dB'_t = (I_d - \mathbf{1}_{\{Q_t \neq 0, \alpha|Z_t| + |Q_t| < r_{max}\}} 2 e_t e_t^T) dB_t, \quad e_t = \frac{Q_t}{|Q_t|}$$

$$d(B_t - B'_t) = \mathbf{1}_{\{Q_t \neq 0, \alpha|Z_t| + |Q_t| < r_{max}\}} 2 e_t e_t^T dB_t$$

(Semi-)metric and contractivity

$$z := x - x', \quad q := x - x' + \gamma^{-1}(v - v'), \quad \alpha, \epsilon > 0.$$

$$\rho((x, v), (x', v')) = f(\alpha|z| + |q|) \cdot (1 + \epsilon H(x, v) + \epsilon H(x', v')),$$

$f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, concave; H Lyapunov function.

$$\mathcal{W}_\rho(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[\rho(X, Y)], \quad \mu, \nu \in \text{Prob}(\mathbb{R}^d).$$

Kantorovich (L^1 Wasserstein) distance based on semi-metric

Contractivity on average

$$\rho_t := \rho((X_t, V_t), (X'_t, V'_t)), \quad R_t := \alpha|Z_t| + |Q_t|.$$

- $Q_t = 0$: Contractivity by drift.
- $Q_t \neq 0, R_t < r_{max}$: Contractivity by noise and concavity of f .
- $R_t \geq r_{max}$: Contractivity by Lyapunov function.

Derivation of contraction rates

$$d\rho_t \stackrel{!}{\leq} -c\rho_t dt + dM_t$$

Then

$$\mathcal{W}_\rho(\mu_t, \nu_t) \leq E[\rho_t] \leq e^{-ct} E[\rho_0] = e^{-ct} \mathcal{W}_\rho(\mu_0, \nu_0).$$

Choose f, α, ϵ in order to maximize the contraction rate c .

Main result

Assumptions

$$(A1) \quad U(0) = 0 = \min U.$$

$$(A2) \quad |\nabla U(x) - \nabla U(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^d.$$

$$(A3) \quad \exists \beta, \mathcal{R} > 0: \quad x \cdot \nabla U(x) \geq \beta \cdot \left(\frac{|x|}{\mathcal{R}}\right)^2 \quad \forall |x| \geq \mathcal{R}.$$

Theorem (Contractivity)

Suppose that (A1), (A2) and (A3) hold. Then there exists an explicit concave function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and explicit constants $\alpha, \epsilon, c > 0$ such that

$$W_\rho(\mu p_t, \nu p_t) \leq \exp(-ct) W_\rho(\mu, \nu) \quad \text{for any } \mu, \nu \in \text{Prob}(\mathbb{R}^d).$$

For example, if $L\gamma^{-2} \leq 1/30$ and $\beta \geq LR^2/2$, then

$$c \geq \frac{\sqrt{\beta}}{76} R^{-1} \min \left((L\gamma^{-2})^2, \min \left(\sqrt{d}L\gamma^{-2}, \Lambda^{-1/2} \right) e^{-\Lambda} \right)$$

where $\Lambda := 4d + LR^2/4$.

References

- AE, A. Guillin, R. Zimmer: Couplings and quantitative contraction rates for Langevin dynamics, ArXiv March 2017.
- AE: A coupling approach to the kinetic Langevin equation on the torus, in preparation.
- AE, R. Zimmer: Sticky couplings of multidimensional diffusions with different drifts, ArXiv December 2016.