Coarse-graining of collective dynamics models

Metric vs topological interactions

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Summary

1. Topological interactions
2. Smooth rank-based dynamics
3. Nearest neighbor
4. Conclusion

1. Topological interactions

Flocking dynamics

System with locally interacting agents

- Emergence of spatio-temporal coordination
- Patterns, structures, correlations, synchronization
- No leader





Interaction types

Mean-field (metric) interaction

Particle interact with all particles within a certain distance

Examples: Alignment (Vicsek . . .)

Consensus (Cucker-Smale, Motsch-Tadmor, . . .)

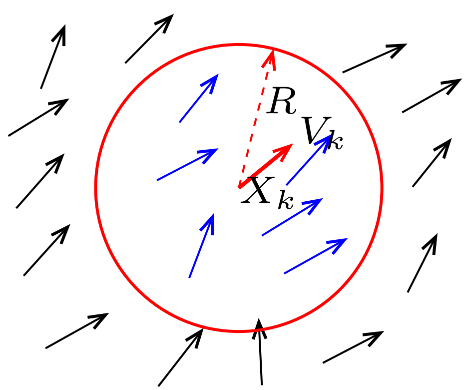
Attraction-repulsion (Bertozzi, Carrillo, . . .)

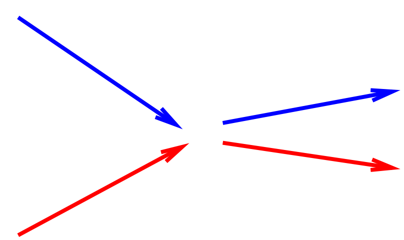
Binary (ternary, . . .)

Particle interacts with a partner at contact

Examples: Hard-sphere collisions (Boltzmann . . .)

Alignment (Bertin, Droz & Grégoire, . . .)





Particles interact with other particles

according to their rank

[Ballerini et al, PNAS 2008]

PNAS

Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study

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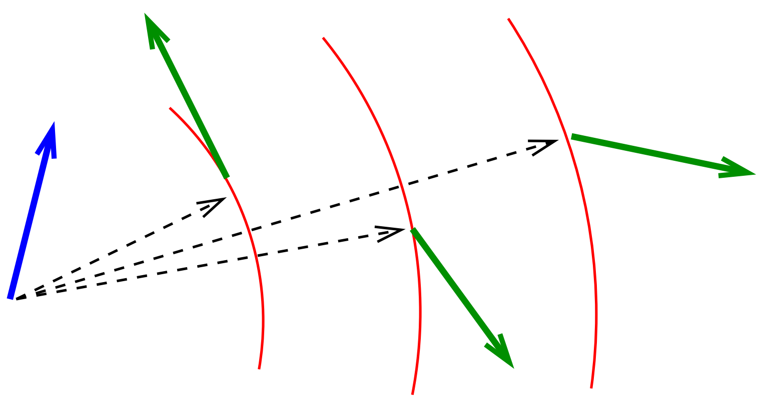
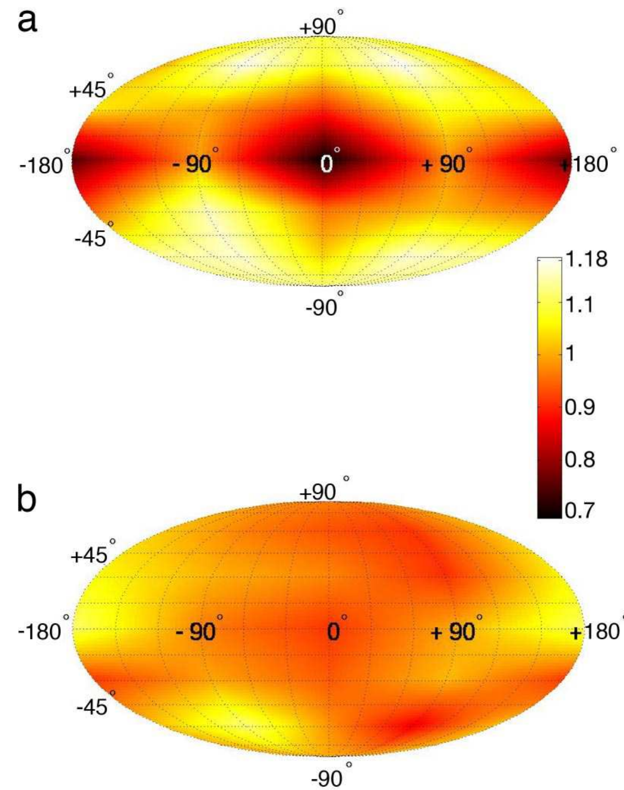




Kinetic models for topological interactions

Kinetic models

Largely investigated for mean-field metric
or binary interactions

Almost inexistent for topological interactions

1st work is by J. Haskovec [Physica D 2013]

for Cucker-Smale type interactions

Also work by Y. Brenier

for a competition model

Our work: Boltzmann approach

Adapting 'Choose the Leader' model from
[Carlen, D., Wennberg, M3AS 2013],
[Carlen, Chatelin, D., Wennberg, Physica D 2013]

2. Smooth rank-based dynamics

A. Blanchet & P. D., J. Stat. Phys **163** (2016) 41-60

Rank

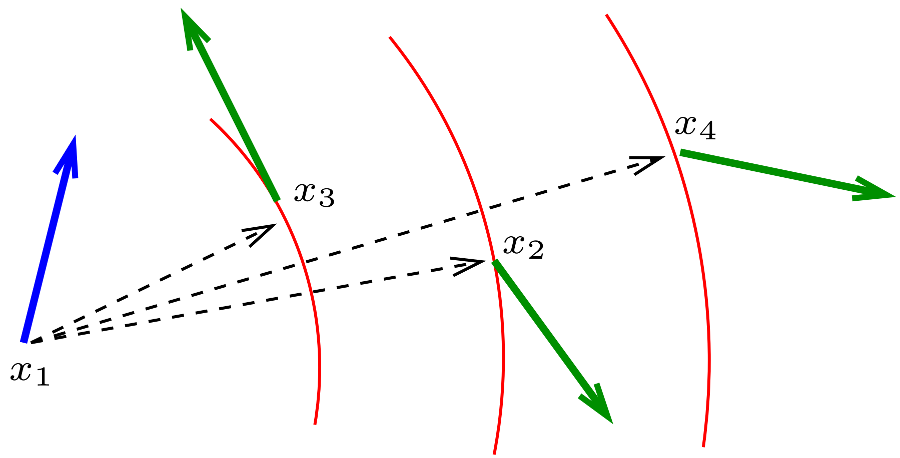
N particles $(x_i(t), v_i(t))_{i=1,\ldots,N}$, $(x_i, v_i) \in \mathbb{R}^{2n}$

At given time t, Rank $R^N(i,j)$ of j w.r.t. i:

Sort $(|x_k - x_i|)_{k=1,\ldots,n,\, k\neq i}$ by increasing order

$R^N(i,j)$ is rank of $|x_j - x_i|$ in the list

$R^N(i,j) \in \{1, \ldots, N-1\}$



$R(1,3) = 1$ $R(1,2) = 2$ $R(1,4) = 3$

Interaction probabilities

Normalized rank: $r^N(i,j) = \dfrac{R^N(i,j)}{N-1} \in \bigcup_{k=1}^{N} \{\dfrac{k}{N-1}\}$

Given: $K : [0,1] \to [0,\infty)$ s.t. $\int_0^1 K(r)\,dr = 1$

define $K^N(r) = \dfrac{K(r)}{\sum_{k=1}^{N-1} K(\frac{k}{N-1})}$

Interaction probabilities:

π_{ij}^N probability of i interacting with j: $\pi_{ij}^N = K^N(r^N(i,j))$

Note: $\displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{N-1} \pi_{ij}^N = K^N(r^N(i,j)) = 1$

N-particle dynamics

Free flights $\dot{x}_i = v_i, \quad \dot{v}_i = 0$

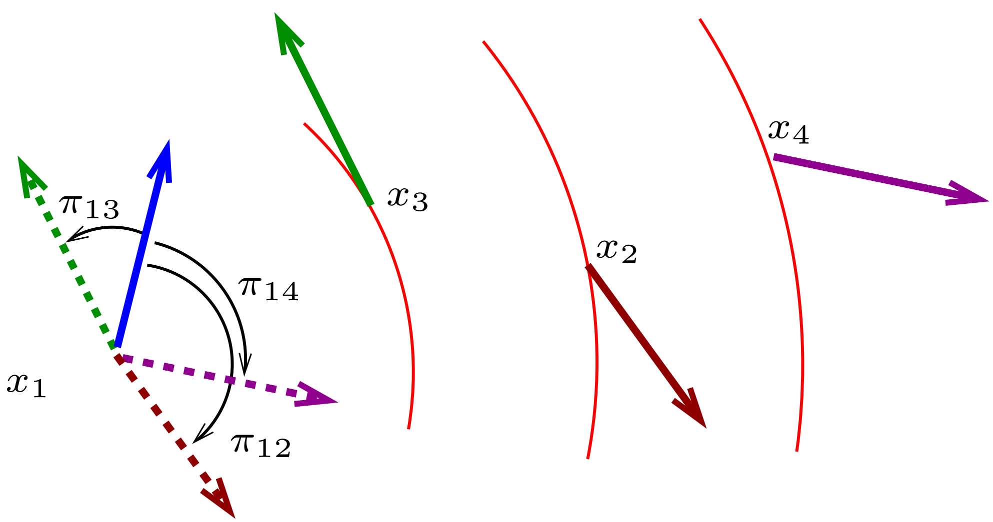
Collisions: at Poisson random times with rate N:

Pick $i \in \{1, \ldots, N\}$ with uniform probability $1/N$

Pick $j \in \{1, \ldots, N\}$, $j \neq i$ with probability π_{ij}^N

Perform: (x_i, x_j) remains unchanged

(v_i, v_j) changed into (v_j, v_j)



Properties

Rank is a function of positions $r^N(i,j) = r^N(i,j)(x_1,\ldots,x_N)$
and so is π_{ij}^N: $\pi_{ij}^N = K^N(r^N(i,j)(x_1,\ldots,x_N))$

Rank is permutation-invariant
$r^N(\sigma(i)\,\sigma(j))(x_{\sigma(1)},\ldots,x_{\sigma(N)}) = r^N(i,j)(x_1,\ldots,x_N)$

Goal

Derive a Boltzmann operator
under Propagation of Chaos assumption

Note: BBGKY does not help
because all particles interact with all particles

Previous work using similar ideas:
[D. & Ringhofer, SIAM Appl. Math. 2007] (min interaction)

Master equation

Notations: $\vec{x} = (x_1, \dots, x_N)$, $\vec{v} = (v_1, \dots, v_N)$

$Z_j = (x_j, v_j)$, $\vec{Z} = (Z_1, \dots, Z_N)$, $d\vec{Z} = dZ_1 \dots dZ_N$

$f^N(\vec{Z}, t)$: N-particle distribution function

Find eq. for f^N (master equation)

Follow strategy of [Carlen, D., Wennberg, M3AS 2013]

Take $\Phi^N(\vec{Z})$ a test function

Drop drift term for simplicity

$$\frac{d}{dt} \int f_N \, \Phi^N \, d\vec{Z} =$$

$$N \int \Big[\frac{1}{N} \sum_{i \neq j} \pi^N_{ij}(\vec{x}) \, \Phi^N(Z_1, \dots, x_i, v_j, \dots x_j, v_j, \dots Z_N) - \Phi^N(\vec{Z}) \Big] f^N(\vec{Z}, t) \, d\vec{Z}$$

If $f^N|_{t=0}$ is permutation invariant

then $f^N(t)$ is permutation invariant

Marginals and propagation of chaos

Marginal: $f_N^k(Z_1, \ldots, Z_k, t) = \int f^N(\vec{Z}, t)\, dZ_{k+1} \ldots dZ_N$

To compute eq. for f_N^1, use $\Phi^N(\vec{Z}) = \phi(Z_1)$

Propagation of chaos

Assume $f^N(\vec{Z}, t) = \prod_{\ell=1}^{N} f_N^1(Z_\ell, t) +$ negligible terms as $N \to \infty$

Define $\rho_N^1(x, t) = \int f_N^1(x, v, t)\, dv$

Eq. for first marginal

$$\frac{d}{dt}\int f_1^N(Z_1)\,\phi(Z_1)\,dZ_1 = \frac{1}{S^N(K)}\int \big(\phi(x_1,v_2)-\phi(x_1,v_1)\big)\, f_1^N(Z_1)\, f_1^N(Z_2)$$

$$\Big(\int K\big(r^N(1,2)(\vec{x})\big)\prod_{\ell=3}^{N}\rho_N^1(x_\ell)\,dx_\ell\Big)\,dZ_1\,dZ_2$$

where

$$S^N(K) = \frac{1}{N-1}\sum_{k=1}^{N-1} K\big(\frac{k}{N-1}\big) \approx \int_0^1 K(r)\,dr = 1$$

Need to estimate the behavior as $N \to \infty$ of:

$$\int K\big(r^N(1,2)(\vec{x})\big)\prod_{\ell=3}^{N}\rho_N^1(x_\ell)\,dx_\ell$$

$$\int K\big(r^N(1,2)(\vec{x})\big) \prod_{\ell=3}^{N} \rho_N^1(x_\ell)\, dx_\ell = \mathbb{E}\Big\{K\big(r^N(1,2)(x_1,x_2,x_3,\ldots,x_N)\big)\Big\}$$

when $x_3,\ldots,x_N$ are iid random variables following the law ρ_N^1

$$= \sum_{j=0}^{N-2} K\big(\frac{j+1}{N-1}\big) \binom{N-2}{j} \text{Prob}\Big(\text{Card}\{\, k \in \{3,\ldots N\} \text{ s.t. } |x_k - x_1| \leq |x_2 - x_1|\} = j\Big)$$

$$= \sum_{j=0}^{N-2} K\big(\frac{j+1}{N-1}\big) \binom{N-2}{j}\, m^j\, (1-m)^{(N-2)-j}$$

where $m = M_{\rho_N^1}(x_1, |x_2 - x_1|) = \int_{|x-x_1| \leq |x_2 - x_1|} \rho_N^1(x)\, dx$

$\approx$ Bernstein polynomial approximation of $K(m)$

$= K(m) + \mathcal{O}(\frac{1}{N})$

In the limit $N \to \infty$, if $f_N^1(Z_1) \to f(Z_1)$, then

$$\frac{d}{dt} \int f(Z_1,t)\, \phi(Z_1)\, dZ_1 =$$

$$\int \big(\phi(x_1,v_2) - \phi(x_1,v_1)\big)\, f(Z_1)\, f(Z_2)\, K\big(M_\rho(x_1,|x_2-x_1|)\big) dZ_1\, dZ_2$$

In strong form (adding back the drift term):

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

$$Q(f)(x,v) = \rho(x) \int f(x',v)\, K\big(M_\rho(x,|x'-x|)\big)\, dx' - f(x,v)$$

with $M_\rho(x,s) = \int_{|x'-x|\le s} \rho(x')\, dx'$

$$\int Q(f)\,dv = \rho(x)\,\Big(\int \rho(x')\,K\big(M_\rho(x,|x'-x|)\big)\,dx' - 1\Big)$$

$$= \rho(x)\,\big(\int K(m)\,dm - 1\big) = 0$$

So, the continuity eq. is satified

$$\partial_t \rho + \nabla_x \cdot j = 0, \qquad j(x,t) = \int_{\mathbb{R}^n} f(x,v,t)\,v\,dv$$

3. Nearest neighbor

A. Blanchet & P. D., arXiv:1703.05131

At each collision event:

A particle is selected randomly with uniform probability

When selected, particle i follows its nearest neighbor

i.e. the probability π_{ij}^N of i interacting with j is

$$\pi_{ij}^N = \delta\Big(R^N(i,j) - 1\Big)$$

here: rank is un-normalized: $R^N(i,j) \in \{1, \ldots, N-1\}$, so:
$\pi_{ij}^N = 0$ except if $j =$ nearest neighbor in which case $\pi_{ij}^N = 1$

Rate of collision events: $\lambda(N)\, N$ with $\lambda(N) \to \infty$ TBD

Master eq. unchanged: - multiply by $\lambda(N)$

- replace $K^N(r^N(i,j))$ by $\delta(R^N(i,j) - 1)$

Eq. for first marginal

$$\frac{d}{dt}\int f_1^N(Z_1)\,\phi(Z_1)\,dZ_1 = \int \big(\phi(x_1,v_2)-\phi(x_1,v_1)\big)\, f_1^N(Z_1)\, f_1^N(Z_2)$$

$$\Big(\lambda(N)\,(N-1)\int \delta\big(R^N(1,2)(\vec{x})-1\big)\prod_{\ell=3}^N \rho_N^1(x_\ell)\,dx_\ell\Big)\,dZ_1\,dZ_2$$

Need to estimate the behavior as $N\to\infty$ of:

$$\lambda(N)\,(N-1)\int \delta\big(R^N(1,2)(\vec{x})-1\big)\prod_{\ell=3}^N \rho_N^1(x_\ell)\,dx_\ell$$

Drop scripts N and 1 for clarity i.e. $f_1^N \to f$, $\rho_1^N \to \rho$

$$\int \delta\big(R^N(1,2)(\vec{x}) - 1\big) \prod_{\ell=3}^{N} \rho(x_\ell)\, dx_\ell = \mathbb{E}\Big\{ \delta\big(R^N(1,2)(\vec{x}) - 1\big) \Big\}$$

when $x_3, \ldots, x_N$ are iid random variables following the law ρ

$$= \text{Prob}\Big(\text{Card}\big\{ k \in \{3, \ldots N\} \text{ s.t. } |x_k - x_1| \leq |x_2 - x_1| \big\} = 0 \Big)$$

$$= (1-m)^{(N-2)}$$

where $m = M_\rho(x_1, |x_2 - x_1|) = \displaystyle\int_{|x-x_1| \leq |x_2 - x_1|} \rho(x)\, dx$

So: $\lambda(N)\,(N-1) \displaystyle\int \delta\big(R^N(1,2)(\vec{x}) - 1\big) \prod_{\ell=3}^{N} \rho(x_\ell)\, dx_\ell$

$$= \lambda(N)\,(N-1)\big(1 - M_\rho(x_1, |x_2 - x_1|)\big)^{(N-2)}$$

Estimate of first marginal

$$\frac{d}{dt}\int f(Z_1)\,\phi(Z_1)\,dZ_1 = \int \big(\phi(x_1,v_2)-\phi(x_1,v_1)\big)\,f(x_1,v_1)\,f(x_2,v_2)$$
$$\lambda(N)\,(N-1)\big(1-M_\rho(x_1,|x_2-x_1|)\big)^{(N-2)}\,dx_1\,dx_2\,dv_1\,dv_2$$

Use polar coordinates $x_2 = x_1 + r\omega$, $r \in [0,\infty)$, $\omega \in \mathbb{S}^{n-1}$

$$\frac{d}{dt}\int f(Z_1)\,\phi(Z_1)\,dZ_1 = \int \big(\phi(x_1,v_2)-\phi(x_1,v_1)\big)\,f(x_1,v_1)\,F_f^N(x_1,v_2)\,dx_1\,dv_1\,dv_2$$

$$F_f^N(x_1,v_2) = \lambda(N)\,(N-1)\int f(x_1+r\omega,v_2)\,\big(1-M_\rho(x_1,r)\big)^{(N-2)}\,r^{n-1}\,dr\,d\omega$$

Change of variables $m = M_\rho(x_1, r) \Leftrightarrow r = R_\rho(x_1, m)$, $m \in [0, 1)$

$$F_f^N(x_1, v_2) = \lambda(N)\,(N-1) \int_0^1 G_f(x_1, v_2, m)\,(1-m)^{N-2}\,dm$$

$$G_f(x_1, v_2, m) = \frac{\int f(x_1 + R_\rho(x_1, m)\omega, v_2)\,d\omega}{\int \rho(x_1 + R_\rho(x_1, m)\omega)\,d\omega}$$

As $m \to 0$

$$G_f(x_1, v_2, m) = \frac{f(x_1, v_2)}{\rho(x_1)} + \frac{1}{2n}\left(\frac{m}{\rho(x_1)}\right)^{\frac{2}{n-1}} D(\rho, f)(x_1, v_2) + o\big(m^{\frac{2}{n-1}}\big)$$

$$D(\rho, f)(x_1, v_2) = \frac{1}{\rho(x_1)}\Big(\Delta_x f(x_1, v_2) - \frac{f(x_1, v_2)}{\rho(x_1)}\Delta_x \rho(x_1)\Big)$$

Denote:

$$C_{N,n} = (N-1)\frac{n^{\frac{2}{n}-1}}{2}\int_0^1 m^{\frac{2}{n}}\,(1-m)^{N-2}\,dm$$

$C_{N,n} \to 0$ as $N \to \infty$. For instance: $C_{N,2} = \frac{1}{2N}$

Choose: $\lambda(N) = 1/C_{N,n}$

larger frequency: $\lambda(N) \to \infty$ as $N \to \infty$

Then, as $N \to \infty$:

$$F_f^N(x_1,v_2) - \lambda(N)\frac{f(x_1,v_2)}{\rho(x_1)} \longrightarrow \left(\frac{1}{\rho(x_1)}\right)^{\frac{2}{n-1}} D(\rho,f)(x_1,v_2)$$

Probability $p_{N,n}(dm) \sim m^{\frac{2}{n-1}}\,(1-m)^{N-2}\,dm \to \delta_0$ as $N \to \infty$

Kinetic equation (nearest neighbor)

In the limit $N \to \infty$, if $f_N^1(Z_1) \to f(Z_1)$, then

$$\frac{d}{dt'} \int f(x_1, v_1, t')\, \phi(x_1, v_1)\, dx_1\, dv_1 =$$

$$\int \big(\phi(x_1, v_2) - \phi(x_1, v_1)\big)\, f(x_1, v_1)\, \Delta_x f(x_1, v_2)\, \frac{1}{\rho(x_1)^{\frac{n+1}{n-1}}}\, dx_1\, dx_2\, dv_2$$

In strong form (adding back the drift term):

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

$$Q(f)(x, v) = \frac{1}{\rho(x)^{\frac{2}{n-1}}} \Big(\Delta_x f(x, v) - \frac{f(x, v)}{\rho(x)} \Delta_x \rho(x) \Big)$$

$$\int Q(f)\, dv = 0 \qquad \text{(obvious)}$$

So, the continuity eq. is satisfied

$$\partial_t \rho + \nabla_x \cdot j = 0, \qquad j(x,t) = \int_{\mathbb{R}^n} f(x,v,t)\, v\, dv$$

4. Conclusion

Conclusion and perspectives

Smooth rank-based topological interaction

- Derivation of spatially non-local Boltzmann model
- Highly nonlinear new collision operator

Nearest-neighbor interaction

- gives rise to spatial nonlinear diffusion
- Still preserves continuity eq.

Perspectives

- Well-posedness of the resulting eqs.
- Proof of propagation of chaos and convergence
- Hydrodynamic limits
- Extension to more complex interaction rules