Bessel-like SPDEs

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Squared Bessel processes

Let $\delta \geq 0$, $y \geq 0$, and $(B_t)_{t \geq 0}$ a BM.

By Yamada-Watanabe’s Theorem, there exists a unique (strong) solution $(Y_t)_{t \geq 0}$ of

$$Y_t = y + \int_0^t 2\sqrt{|Y_s|} \, dB_s + \delta \, t,$$

and moreover $Y \geq 0$ so that $|Y| = Y$.

The transition semigroup is explicitly known and contains some Bessel functions and $(Y_t)_{t \geq 0}$ is called a Squared Bessel Process, see Pitman-Yor.

We define

$$X_t := \sqrt{Y_t}.$$ 

What equation does $X$ satisfy? The function $y \mapsto \sqrt{y}$ is not smooth and the Itô formula can not be applied (too) naively.
Bessel processes

For $\delta > 1$ we have

$$X_t = x + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, ds + B_t$$

where $x := \sqrt{y}$.

More precisely, by the Itô-Tanaka formula

$$X_t = x + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, ds + B_t + \frac{1}{2} L^0_t$$

where $(L^a_t)_{t \geq 0, a \geq 0}$ is defined by the occupation times formula

$$\int_0^t \varphi(X_s) \, ds = \int_0^\infty \varphi(a) \, L^a_t \, da$$

for all $\varphi \in C_b(\mathbb{R})$. Here $L^0 \equiv 0$ and $X$ is a semimartingale.
A very interesting SDE

- for $\delta > 1$ the drift $x \mapsto \frac{\delta - 1}{2x}$ is dissipative (i.e. decreasing) on $\mathbb{R}_+$, the SDE has pathwise uniqueness and the solution is Strong Feller
- $X$ is known as the Bessel process.
- As $\delta \downarrow 1$ the solution converges to the reflecting BM

$$X_t = X_0 + L_t + B_t$$

where $L$ is continuous monotone non-decreasing, $L_0 = 0$, $X$ is continuous non-negative, and $\int_0^{\infty} X_t \, dL_t = 0$.

- For $\delta \geq 2$ a.s. $X_t > 0$ for all $t > 0$.
- For $\delta \in ]1, 2[$ we have a.s. $X_t$ for some $t > 0$, but still $L_0^t = 0$.
- We can define diffusion local times $(\ell_t^a)_{t,a \geq 0}$ by

$$\int_0^t \varphi(X_s) \, ds = \int_0^{\infty} \varphi(a) \ell_t^a \, a^{\delta - 1} \, da.$$ 

Then $(\ell_t^0)_{t \geq 0}$ is the inverse of a $(1 - \frac{\delta}{2})$-stable subordinator.
It turns out that in this situation the process $X_t := \sqrt{Y_t}$ solves this SDE

$$X_t = X_0 + \frac{\delta - 1}{2} \int_0^\infty \frac{\ell^a_t - \ell^0_t}{a} a^{\delta-1} \, da + B_t,$$

where $(\ell^a_t)_{a,t \geq 0}$ is the family of diffusion local times. Formally this is equal to

$$X_t = X_0 + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, ds + \infty \ell^0_t + B_t,$$

This SDE has a very exotic drift: an increasing singular non-linearity (the opposite of dissipative) and a reflection at 0 but multiplied by an infinite constant. Indeed $X$ is not a semimartingale and $L^0_t = +\infty$.

The two infinite terms compensate each other in a renormalisation phenomenon.

To my knowledge, there is no pathwise uniqueness result for such SDE. One can prove the Strong Feller property (see Henri’s recent paper).
Bessel-like SPDEs with $\delta \geq 3$

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \xi + \eta \]

where $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$, $\xi$ is a space-time white noise, $\eta$ is a measure on $\mathbb{R}_+ \times [0, 1]$ s.t.

\[ \int_{\mathbb{R}_+ \times [0,1]} u \, d\eta = 0. \]

This is the Nualart-Pardoux equation, whose invariant measure is the 3-Bessel bridge.

The equation corresponding to the $\delta$-Bessel bridge for $\delta > 3$ is

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\delta - 1)(\delta - 3)}{8u^3} + \xi \]
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What about $\delta < 3$? This question has been open since 2001.
We concentrate on the drift

\[ \kappa(\delta) \int_0^t \frac{1}{u^3(s, x)} \, ds, \quad \kappa(\delta) := \frac{(\delta - 1)(\delta - 3)}{8}. \]

We introduce diffusion local times

\[ \int_0^t \varphi(u(s, x)) \, ds = \int_0^\infty \varphi(a) \ell_{t, x}^a a^{\delta - 1} \, da \]

and we can write

\[ \kappa(\delta) \int_0^t \frac{1}{u^3(s, x)} \, ds = \kappa(\delta) \int_0^\infty \frac{1}{a^3} \ell_{t, x}^a a^{\delta - 1} \, da, \]

which however diverges for \( \delta \leq 3 \) if \( \ell_{t, x}^0 > 0 \).
Bessel-like SPDEs with $1 < \delta < 3$

Then, in analogy with Bessel processes, we write a renormalised version of the drift

$$\kappa(\delta) \int_0^\infty \frac{1}{a^3} (\ell^a_{t,x} - \ell^0_{t,x}) a^{\delta-1} \, da,$$

which may work as long as $\delta > 2$. 
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which may work as long as \( \delta > 2 \). For \( \delta \in ]1, 2] \) one expects

\[ \kappa(\delta) \int_0^\infty \frac{1}{a^3} \left( \ell^a_{t,x} - \ell^0_{t,x} - a \frac{\partial}{\partial a} \ell^a_{t,x} \bigg|_{a=0} \right) a^{\delta-1} \, da, \]
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which may work as long as $\delta > 2$. For $\delta \in [1, 2]$ one expects

$$\kappa(\delta) \int_0^\infty \frac{1}{a^3} \left( \ell_{t,x}^a - \ell_{t,x}^0 - a \left. \frac{\partial}{\partial a} \ell_{t,x}^a \right|_{a=0} \right) a^{\delta-1} \, da,$$

however it turns out that

$$\left. \frac{\partial}{\partial a} \ell_{t,x}^a \right|_{a=0} = 0$$

so that the same expression is valid for $\delta \in [1, 3[$.
The most important and interesting case is $\delta = 1$, which, together with $\delta = 3$, is a critical case.

As $\delta \downarrow 1$, the previous expression can be seen to converge to

$$\frac{-1}{8} \frac{\partial^2}{\partial a^2} \ell_{t,x}^a \bigg|_{a=0}.$$

We write therefore the SPDE for $\delta = 1$:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{8} \frac{\partial}{\partial t} \frac{\partial^2}{\partial a^2} \ell_{t,x}^a \bigg|_{a=0} + \xi.$$

Motivated by scaling limits of dynamical critical pinning models.
For $\delta \in ]0, 1[$ one expects

$$
\kappa(\delta) \int_0^\infty \frac{1}{a^3} \left( \ell^a_{t,x} - \ell^0_{t,x} - \frac{a^2}{2} \frac{\partial^2}{\partial a^2} \ell^a_{t,x} \bigg|_{a=0} \right) a^{\delta-1} \, da
$$

with a Taylor expansion of order 2 of $a \mapsto \ell^a_{t,x}$.
For $\alpha > 0$ we define the measure on $\mathbb{R}_+$

$$
\mu_\alpha(dx) := \frac{x^{\alpha-1}}{\Gamma(\alpha)} \mathbb{1}(x>0) \, dx.
$$

For $\alpha \leq 0$ we define the Schwartz distribution on $\mathbb{R}_+$

$$
\mu_\alpha(\varphi) := (-1)^{-\alpha} \varphi(-\alpha)(0)
$$

if $-\alpha \in \mathbb{N}$, and

$$
\mu_\alpha(\varphi) := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} \left( \varphi(x) - \sum_{0 \leq i \leq -\alpha} \frac{x^i}{i!} \varphi^{(i)}(0) \right) \, dx
$$

otherwise.
Then we can write the above family of SPDEs in a unified way for all \( \delta > 0 \)

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \Gamma(\delta - 3) \kappa(\delta) \mu_{\delta-3}(\mathcal{L}_{t,x}^{(\cdot)}) + \xi
\]
Most of the above is conjectural.

We do have integration by parts formulae on the law of \( \delta \)-Bessel processes for \( \delta < 3 \), see Henri’s talk, which give the form of the equation.

By Dirichlet forms methods, at least in the cases \( \delta = 1, 2 \) we can construct (stationary) solutions to the SPDE.

Major open questions

- pathwise uniqueness ???
- local times for SPDEs ???
- the Strong Feller property ??
  (Henri proved it for Bessel processes uniformly in \( \delta \))
- the associated Dirichlet forms ? (for \( \delta \neq 1, 2 \))
Recall that for Bessel processes and $\delta < 1$, we have pathwise uniqueness by setting $Y_t := X_t^2$ and applying

- the Itô formula in order to compute the SDE solved by $Y$,
- the Yamada-Watanabe theorem, since the diffusion coefficient of this SDE is only $\frac{1}{2}$-Hölder.
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For space-time white noise driven SPDEs, Itô calculus is notoriously difficult. Carlo Bellingeri is investigating this with regularity structures.
Since the drift is proportional to $u^{-3}$, it seems reasonable to set $v := u^4$ and study pathwise uniqueness for $v$. The equation is

$$\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} - \frac{3}{2} : \left( \frac{\partial \sqrt{v}}{\partial x} \right)^2 : + \frac{(\delta - 1)(\delta - 3)}{2} + 4v^{\frac{3}{4}} \xi$$

This equation is impossible to solve today, but the exponent $\frac{3}{4}$ has been shown by Mueller-Mytnik-Perkins to be critical for pathwise uniqueness, as for Yamada-Watanabe in one-dimensional diffusions.
Theorem (Dalang, Mueller, Z. (2006))

Let $\delta \geq 3$ and $k \in \mathbb{N}$ such that

$$k > \frac{4}{\delta - 2}.$$

Then $\mathbb{P}(\exists t > 0, x_1, \ldots, x_k \in [0, 1[ : u(t, x_i) = 0) = 0$. 

Now we can conjecture that the same formula holds for all $\delta \geq 2$!!!

Correctly, $\delta = 2$ is the critical case for hitting infinitely many times in space.

What about $\delta < 2$, in particular $\delta = 1$?
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