Bessel-like SPDEs

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Squared Bessel processes

Let $\delta \geq 0$, $y \geq 0$, and $(B_t)_{t\geq 0}$ a BM.

By Yamada-Watanabe's Theorem, there exists a unique (strong) solution $(Y_t)_{t\geq 0}$ of

$$Y_t = y + \int_0^t 2\sqrt{|Y_s|}\,\mathrm{d}B_s + \delta\, t,$$

and moreover $Y \geq 0$ so that $|Y| = Y$.

The transition semigroup is explicitly known and contains some Bessel functions and $(Y_t)_{t\geq 0}$ is called a Squared Bessel Process, see Pitman-Yor.

We define

$$X_t := \sqrt{Y_t}.$$

What equation does X satisfy? The function $y \mapsto \sqrt{y}$ is not smooth and the Itô formula can not be applied (too) naively.

Bessel processes

For $\delta > 1$ we have

$$X_t = x + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, \mathrm{d}s + B_t$$

where $x := \sqrt{y}$.

More precisely, by the Itô-Tanaka formula

$$X_t = x + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, \mathrm{d}s + B_t + \frac{1}{2} L_t^0$$

where $(L_t^a)_{t \geq 0, a \geq 0}$ is defined by the occupation times formula

$$\int_0^t \varphi(X_s) \, \mathrm{d}s = \int_0^\infty \varphi(a) \, L_t^a \, \mathrm{d}a$$

for all $\varphi \in C_b(\mathbb{R})$. Here $L^0 \equiv 0$ and X is a semimartingale.

A very interesting SDE

- for $\delta > 1$ the drift $x \mapsto \frac{\delta-1}{2x}$ is dissipative (i.e. decreasing) on $\mathbb{R}_+$, the SDE has pathwise uniqueness and the solution is Strong Feller
- X is known as the Bessel process.
- As $\delta \downarrow 1$ the solution converges to the reflecting BM

$$X_t = X_0 + L_t + B_t$$

where L is continuous monotone non-decreasing, $L_0 = 0$, X is continuous non-negative, and $\int_0^\infty X_t \, \mathrm{d}L_t = 0$.
- For $\delta \geq 2$ a.s. $X_t > 0$ for all $t > 0$.
- For $\delta \in]1,2[$ we have a.s. X_t for some $t > 0$, but still $L_t^0 = 0$.
- We can define diffusion local times $(\ell_t^a)_{t,a\geq 0}$ by

$$\int_0^t \varphi(X_s) \, \mathrm{d}s = \int_0^\infty \varphi(a) \, \ell_t^a \, a^{\delta-1} \, \mathrm{d}a.$$

Then $(\ell_t^0)_{t\geq 0}$ is the inverse of a $(1 - \frac{\delta}{2})$-stable subordinator.

$\delta < 1$

It turns out that in this situation the process $X_t := \sqrt{Y_t}$ solves this SDE

$$X_t = X_0 + \frac{\delta - 1}{2} \int_0^\infty \frac{\ell_t^a - \ell_t^0}{a} \, a^{\delta-1} \, \mathrm{d}a + B_t,$$

where $(\ell_t^a)_{a,t\geq 0}$ is the family of diffusion local times. Formally this is equal to

$$X_t = X_0 + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, \mathrm{d}s + \infty \ell_t^0 + B_t,$$

This SDE has a very exotic drift: an increasing singular non-linearity (the opposite of dissipative) and a reflection at 0 but multiplied by an infinite constant. Indeed X is not a semimartingale and $L_t^0 = +\infty$.

The two infinite terms compensate each other in a renormalisation phenomenon.

To my knowledge, there is no pathwise uniqueness result for such SDE. One can prove the Strong Feller property (see Henri's recent paper).

Bessel-like SPDEs with $\delta \geq 3$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \xi + \eta$$

where $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$, ξ is a space-time white noise, η is a measure on $\mathbb{R}_+ \times [0,1]$ s.t.

$$\int_{\mathbb{R}_+ \times [0,1]} u \, \mathrm{d}\eta = 0.$$

This is the Nualart-Pardoux equation, whose invariant measure is the 3-Bessel bridge.

The equation corresponding to the δ-Bessel bridge for $\delta > 3$ is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{(\delta-1)(\delta-3)}{8u^3} + \xi$$

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What about $\delta < 3$? This question has been open since 2001.

Bessel-like SPDEs with $\delta < 3$

We concentrate on the drift

$$\kappa(\delta) \int_0^t \frac{1}{u^3(s,x)} \, \mathrm{d}s, \qquad \kappa(\delta) := \frac{(\delta-1)(\delta-3)}{8}.$$

We introduce diffusion local times

$$\int_0^t \varphi(u(s,x)) \, \mathrm{d}s = \int_0^\infty \varphi(a) \, \ell_{t,x}^a \, a^{\delta-1} \, \mathrm{d}a$$

and we can write

$$\kappa(\delta) \int_0^t \frac{1}{u^3(s,x)} \, \mathrm{d}s = \kappa(\delta) \int_0^\infty \frac{1}{a^3} \, \ell_{t,x}^a \, a^{\delta-1} \, \mathrm{d}a,$$

which however diverges for $\delta \leq 3$ if $\ell_{t,x}^0 > 0$.

Bessel-like SPDEs with $1 < \delta < 3$

Then, in analogy with Bessel processes, we write a renormalised version of the drift

$$\kappa(\delta) \int_0^\infty \frac{1}{a^3} \left(\ell^a_{t,x} - \ell^0_{t,x}\right) a^{\delta-1} \, \mathrm{d}a,$$

which may work as long as $\delta > 2$.

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however it turns out that

$$\frac{\partial}{\partial a} \ell^a_{t,x} \Big|_{a=0} = 0$$

so that the same expression is valid for $\delta \in]1,3[$.

Bessel-like SPDEs with $\delta = 1$

The most important and interesting case is $\delta = 1$, which, together with $\delta = 3$, is a critical case.

As $\delta \downarrow 1$, the previous expression can be seen to converge to

$$-\frac{1}{8}\,\frac{\partial^2}{\partial a^2}\ell^a_{t,x}\Big|_{a=0}.$$

We write therefore the SPDE for $\delta = 1$:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{8}\,\frac{\partial}{\partial t}\frac{\partial^2}{\partial a^2}\ell^a_{t,x}\Big|_{a=0} + \xi.$$

Motivated by scaling limits of dynamical critical pinning models.

Bessel-like SPDEs with $0 < \delta < 1$

For $\delta \in]0, 1[$ one expects

$$\kappa(\delta) \int_0^\infty \frac{1}{a^3} \left(\ell^a_{t,x} - \ell^0_{t,x} - \frac{a^2}{2} \frac{\partial^2}{\partial a^2} \ell^a_{t,x} \Big|_{a=0} \right) a^{\delta-1} \, \mathrm{d}a$$

with a Taylor expansion of order 2 of $a \mapsto \ell^a_{t,x}$.

A general formula

For $\alpha > 0$ we define the measure on $\mathbb{R}_+$

$$\mu_\alpha(\mathrm{d}x) := \frac{x^{\alpha-1}}{\Gamma(\alpha)} \mathbb{1}_{(x>0)}\, \mathrm{d}x.$$

For $\alpha \le 0$ we define the Schwartz distribution on $\mathbb{R}_+$

$$\mu_\alpha(\varphi) := (-1)^{-\alpha} \varphi^{(-\alpha)}(0)$$

if $-\alpha \in \mathbb{N}$, and

$$\mu_\alpha(\varphi) := \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \left(\varphi(x) - \sum_{0 \le i \le -\alpha} \frac{x^i}{i!} \varphi^{(i)}(0) \right) \mathrm{d}x$$

otherwise.

A general formula

Then we can write the above family of SPDEs in a unified way for all $\delta > 0$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \Gamma(\delta - 3)\,\kappa(\delta)\,\mu_{\delta-3}(\ell_{t,x}^{(\cdot)}) + \xi$$

Results

Most of the above is conjectural.

We do have integration by parts formulae on the law of δ-Bessel processes for $\delta < 3$, see Henri's talk, which give the form of the equation.

By Dirichlet forms methods, at least in the cases $\delta = 1, 2$ we can construct (stationary) solutions to the SPDE.

Major open questions

- pathwise uniqueness ????
- local times for SPDEs ???
- the Strong Feller property ??
  (Henri proved it for Bessel processes uniformly in δ)
- the associated Dirichlet forms ? (for $\delta \neq 1, 2$)

Pathwise uniqueness

Recall that for Bessel processes and $\delta < 1$, we have pathwise uniqueness by setting $Y_t := X_t^2$ and applying

- the Itô formula in order to compute the SDE solved by Y,
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For space-time white noise driven SPDEs, Itô calculus is notoriously difficult. Carlo Bellingeri is investigating this with regularity structures.

Pathwise uniqueness

Since the drift is proportional to u^{-3}, it seems reasonable to set $v := u^4$ and study pathwise uniqueness for v. The equation is

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{3}{2} : \left(\frac{\partial \sqrt{v}}{\partial x}\right)^2 : + \frac{(\delta-1)(\delta-3)}{2} + 4v^{\frac{3}{4}}\,\xi$$

This equation is impossible to solve today, but the exponent $\frac{3}{4}$ has been shown by Mueller-Mytnik-Perkins to be critical for pathwise uniqueness, as for Yamada-Watanabe in one-dimensional diffusions.

Hitting

Theorem (Dalang, Mueller, Z. (2006))

Let $\delta \geq 3$ *and* $k \in \mathbb{N}$ *such that*

$$k > \frac{4}{\delta - 2}.$$

Then $\mathbb{P}(\exists t > 0,\, x_1, \ldots, x_k \in]0,1[:\ u(t, x_i) = 0) = 0.$

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What about $\delta < 2$, in particular $\delta = 1$?