Stochastic heat equations with values in a Riemannian manifold

Xiangchan Zhu

Bielefeld University

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The invariant measure $\mu = N(0,(-\Delta_D)^{-1})$ = law of Brownian bridge. When Δ_D is replaced by $\Delta_{D,N}$, i.e. Laplacian with boundary condition $h(0)=0, h'(1)=0$, the invariant measure = law of Brownian motion.

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[Funaki92/ Hairer16]: Formally write the equation in the local coordinates:

$$\dot{u}^\alpha = \underbrace{\partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\partial_x u^\beta \partial_x u^\gamma}_{-\nabla E,\text{Eells-Sampson harmonic map}} + \underbrace{\sigma_i^\alpha(u)\xi_i}_{\text{white noise on loop space}},$$

which is multi-component version of the KPZ equation. By regularity structure theory, local well-posedness has been obtained, (see also [Bruned, Hairer, Zambotti16, Chandra, Hairer16])

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Is it possible to construct stochastic heat equation on Riemannian manifold by Dirichlet form?

$M = \mathbb{R}^d$

Drichlet form associated with stochastic heat equation:
$E = L^2([0,1];\mathbb{R}^d)$, cylinder function, $F(\gamma) = f(\langle e_1, \gamma\rangle_{L^2}, ..., \langle e_k, \gamma\rangle_{L^2}), \gamma \in E$

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DF: L^2 derivative $= \sum D_{e_k} F e_k$, $\{e_k\}$ is a basis in $L^2([0,1];\mathbb{R}^d)$.

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DF: L^2 derivative $= \sum D_{e_k}Fe_k$, $\{e_k\} \subset H_0^{1,2}$ is a basis in $L^2([0,1];\mathbb{R}^d)$.

$$D_{e_k}\left(\int_0^1 g_j(s,\gamma_s)ds\right) = \int_0^1 \langle \nabla g_j(s,\gamma_s), U_s e_k(s)\rangle_{T_{\gamma_s}M} ds,$$

where $U.$ is the stochastic parallel translation along γ.

Dirichlet form

By Driver's integration by parts formula for μ, $h \in H_0^{1,2}$

$$\int D_h F d\mu = \int F \beta_h d\mu$$

with:

$$L^2(E, \mu) \ni \beta_h := \int_0^1 \langle h_s' + \frac{1}{2} Ric_{U_s} h_s, dB_s \rangle.$$

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is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;\mu)$.

Martingale solution

Theorem([Röckner, Wu, Zhu,Z.17])There exists a (Markov) diffusion process $M = (\Omega, \mathcal{F}, \mathcal{M}_t, (X(t))_{t\geq 0}, (P^z)_{z\in E})$ on E *properly associated with* $(\mathcal{E}, D(\mathcal{E}))$ having μ as an invariant measure.

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For $u \in D(\mathcal{E})$

$$u(X_t) - u(X_0) = M_t^u + N_t^u \quad P^z - a.s.,$$

for q.e. z,

M^u: a martingale with quadratic variation process given by $\int_0^t |Du(X_s)|_{L^2}^2 ds$

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Remark: All these results also hold in the free loop and free path case.

Infinite volume case

- $\mu =$ law of Brownian motion on $C([0, \infty); M)$ (space variable: half line)
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$\Rightarrow$ Existence of Martingale solution

Ergodicity in the Finite volume case

Theorem([Röckner, Wu, Zhu,Z.17])
$\mathrm{Ric} \geq -K$ with $K \in \mathbb{R}$. Log-Sobolev inequality holds for any $G \in D(\mathcal{E})$

$$\int G^2 \log G^2 d\mu \leq 2C(K)\mathcal{E}(G,G) + \int G^2 d\mu \log \int G^2 d\mu.$$

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$\Leftrightarrow$ Exponential ergodicity $\|P_t f - \int f d\mu\|_{L^2}^2 \leq e^{-t/C(K)}\|f\|_{L^2}^2$.

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  Ex: hyperbolic space

Approximation

Consider the piecewise geodesics space from $[0,1]$ to M and the measure

$$\mu_\varepsilon := \frac{1}{Z_\varepsilon}\exp(-\frac{1}{2}E)\mathrm{Vol}_{G_\varepsilon}.$$

with $E(\gamma)=\frac{1}{2}\int_0^1\langle\gamma'(s),\gamma'(s)\rangle ds$.

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$$dx^\varepsilon_{t,s} = \frac{1}{\sqrt{\varepsilon}}\sigma(x^\varepsilon_t)\circ dW_t + \beta^\varepsilon_s(x^\varepsilon_t)dt,$$

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$$\beta^\varepsilon_{s_j}(\gamma) = \underbrace{\frac{1}{2\varepsilon^2}//_{s_j}(\gamma)(b(s_{j-1})+b(s_{j+1})-2b(s_j))}_{\nabla E} - \underbrace{\frac{1}{4}\mathrm{Ric}(\gamma'(s_j))+\mathrm{Err}}_{\mathrm{Vol}_{G_\varepsilon}},$$

with b being anti-development of γ.

Stochastic heat equation

Since

$$\beta^{\varepsilon} \to \frac{1}{2}\frac{\nabla}{ds}\partial_s\gamma - \frac{1}{4}\mathrm{Ric}(\partial_s\gamma),$$

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which formally leaves μ invariant.

Thank you